CONTACT AND SYMPLECTIC HOMOLOGY OF MANIFOLDS WITH NAMES

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Abstract. This note contains examples of symplectic and contact homology computations. All results are completely standard, but usually they are not written down. This note will expand as time goes on. The sections called "Results" contain the raw data; other sections are just fluff containing arguments and proofs.

1. Basic tools

I will try to state most results in the form of equivariant symplectic homology, since contact homology is still not very solid. This means that there will be degree shifts and sign changes here and there. I will use $\mathbb{Q}$-coefficients unless stated otherwise. It is possible to use $\mathbb{Z}$-coefficients, but then several complications arise and the relationship with contact homology is unclear.

The following section is directly taken from [KvK]. I am using the conventions from Bourgeois-Oancea for symplectic homology, [BO3].

1.1. **Black boxes: Morse-Bott spectral sequences.** In this section we will describe a Morse-Bott spectral sequence for (equivariant) symplectic homology: this is a very useful tool, and applies due to the S^1-symmetry that Brieskorn manifolds have. Other versions of this spectral sequence with different conventions appeared in [S, Formula (3.2) and (8.9)].

Consider a simply-connected Liouville manifold $(W^{2n}, d\lambda)$ with $c_1(W) = 0$. Suppose that W has contact type boundary Σ with a periodic Reeb flow. We do not require all orbits to have the same period. Define Σ_T as the Morse-Bott submanifold in Σ consisting of all periodic Reeb orbits of period T,

$$\Sigma_T = \{p \in \Sigma \mid Fl^R_T(p) = p\}.$$

We can associate a Maslov index, often referred to as Robbin-Salamon index, with each connected component of a Morse-Bott submanifold Σ_T. We are using the so-called transverse Maslov index, and we will add correction terms when necessary.

In the following spectral sequences we will use $\mathbb{Q}$-coefficients as this will simplify equivariant homology.

Proposition 1.1. *There is a spectral sequence converging to $SH_*(W)$ whose E^1-page is given by*

$$E^1_{pq}(SH) = \left\{ \begin{array}{ll} H_{2n-q}(W;\mathbb{Q}) & \text{if } p = -n \\ 0 & \text{otherwise} \end{array} \right\} \oplus \bigoplus_{\substack{T \text{ such that} \\ \mu(\Sigma_T) - \frac{1}{2}\dim(\Sigma_T/S^1) = p}} H_q(\Sigma_T;\mathbb{Q}). \tag{1.1}$$

The $+$-part of S^1-equivariant symplectic homology has E^1-page given by

$$E^1_{pq}(SH^{S^1,+}) = \bigoplus_{\substack{T \text{ such that} \\ \mu(\Sigma_T) - \frac{1}{2}\dim(\Sigma_T/S^1) = p}} H^{S^1}_q(\Sigma_T;\mathcal{L}). \tag{1.2}$$

Here $\mathcal{L}$ is a twisting/orientation bundle.

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Note the shift by $\frac{1}{2}\dim(\Sigma_T/S^1)$. This is precisely the contribution of the degenerate endpoint of the symplectic path to the crossing formula if the contact form is of Morse-Bott type. Put into words, the above formula means the following. We fill the E^1-page with copies of the (equivariant) homology groups of submanifolds consisting of periodic orbits. Note also that above direct sum, is just a finite sum for fixed p, provided the Maslov index of a principal orbit is not equal to zero (in other words, if the so-called mean index is not equal to zero).

2. Unit cotangent bundles of manifolds whose geodesics are all periodic

Suppose that (M,g) is a Riemannian manifold whose geodesics are all periodic with period 2π. Then ST^*M carries an S^1-action induced by the Hamiltonian $H=\frac{1}{2}g^*(p,p)$. Symplectic reduction gives a symplectic manifold $Q:=ST^*M/S^1$.

2.1. **$\mathbb{K}\mathbb{P}^n$'s and spheres.** With the Gysin sequence we can compute the homology of ST^*M as a $\dim M-1$-dimensional sphere bundle over M with Euler class $\chi(M)PD(pt)$. The homology of Q can also computed with a Gysin sequence since ST^*M is a principal S^1-bundle over Q.

Before we give the results for the $\mathbb{K}\mathbb{P}^n$'s and spheres, we recall some definitions. By an SC_k-manifold we mean a Riemannian manifold (M,g) whose geodesics are all periodic with the same period (simple closed) and whose (Morse) index equals k. (Note that this is a Morse-Bott setup). For the $\mathbb{K}\mathbb{P}^n$'s this index can be found in [Be, Theorem 255]. The result is originally due to Bott, and was obtained by counting conjugate points.

From [FS, Proposition 2.10] we deduce the following lemma.

Lemma 2.1 (Mean index)**.** *The mean index of a dimensional SC_k-manifold is $k+d-1$.*

2.2. **Betti numbers.** These Betti numbers can also be found in Ziller [Z, page 141]. We will give the computation because an intermediate stage is useful to determine symplectic homology.

Write $d:=\dim\mathbb{K}$.

Lemma 2.2 (Homology of $ST^*\mathbb{K}\mathbb{P}^n$)**.** *The homology groups are*

$$H_k(ST^*\mathbb{K}\mathbb{P}^n;\mathbb{Z})\cong\begin{cases}\mathbb{Z} & k=0,d,\dots,d(n-1),d(n+1)-1,d(n+2)-1,\dots d(2n)-1,\\ \mathbb{Z}_{n+1} & k=d\cdot n-1,\\ 0 & \textit{otherwise.}\end{cases}$$

Lemma 2.3 (Betti numbers of $ST^*\mathbb{K}\mathbb{P}^n/S^1$)**.** *The Betti numbers are given by the following formula.*

$$\begin{cases}\mathbb{C}\mathbb{P}^n & b_0=1,b_2=2,b_4=3,\dots,b_{2n-2}=n=b_{2n},b_{2n+2}=n-1,\dots,b_{4n-2}=1\\ \mathbb{H}\mathbb{P}^n & b_0=b_2=1,\ b_4=b_6=2,\dots,b_{4n-4}=b_{4n-2}=n=b_{4n}=b_{4n+2},\dots,b_{8n-4}=b_{8n-2}=1\\ Ca\mathbb{P}^2 & b_0=b_2=b_4=b_6=1,\ b_8=b_{10}=\dots=b_{22}=2,\ b_{24}=\dots=b_{30}=1\end{cases}$$

Proof. We will do all cases at once by working with the "imaginary space" $\mathbb{K}\mathbb{P}^n$, where $\mathbb{K}$ is a division algebra of real dimension $2d$.

The first lemma follows from the Gysin sequence for a sphere bundle. Use that $\langle e(T\mathbb{K}\mathbb{P}^n),[\mathbb{K}\mathbb{P}^n]\rangle=\chi(\mathbb{K}\mathbb{P}^n)=n+1$.

The second lemma follows from inspecting the Gysin sequence for a circle bundle. This is a somewhat time consuming investigation of the Gysin sequence. □

2.3. **Common parts of the argument.** This is a direct application of the Morse-Bott spectral sequence. After the 0-th page of the spectral sequence all total degree differences are even, so the spectral sequence already abuts at the first page.

In fact, by using suitable Finsler metrics as in [Z], one can find a non-degenerate contact form for which the differential for contact homology vanishes. This is a perfect contact form in the sense of [Gu]. In symplectic homology terms, one still has to use the Morse-Bott spectral sequence, but the spaces of periodic orbits will just be 1-dimensional if one uses a Hamiltonian that corresponds to a perfect contact form.

$*$						
$4n-2$		$\mathbb{Q}^1$			$\mathbb{Q}^1$	
$*$		$\bullet$			$\bullet$	
$2n$		$\mathbb{Q}^n$			$\mathbb{Q}^n$	
$2n-2$		$\mathbb{Q}^n$			$\mathbb{Q}^n$	
$*$		$\bullet$			$\bullet$	
0		$\mathbb{Q}^1$			$\mathbb{Q}^1$	
	0	1	$*$	$*$	$1+2n$	$*$

Figure 1. E^1-page of Morse-Bott spectral sequence for $T^*\mathbb{CP}^n$

2.4. $\mathbb{CP}^n$**.** The mean index for $T^*\mathbb{CP}^n$ equals $2n$, so the Morse-Bott spectral sequence looks like in Figure 1. All total degree differences are even, so the differential is trivial. Columns are located at $p = 1, 1+2n, 1+4n, \dots$, and are filled with the groups from Lemma 2.3. Vertically, each column runs from 0 to $4n-2 = \dim Q$. We see that there is some overlap between the column at $p = 1 + 2n \cdot k$ and the column at $p = 1 + 2n \cdot (k+1)$. This occurs in total degree $1 + 2n \cdot k + 2\ell = 1 + 2n \cdot (k+1) + 2(\ell - n)$ for $\ell = n, \dots, 2n-1$ where the total rank equals $n+1$. There is no overlap in total degree $1 + 2n, \dots, 1 + 2n + 2n - 2$, since $k \in \mathbb{Z}_{\geq 0}$. Hence we find the result given in (2.1).

2.5. $\mathbb{HP}^n$**.** The same argument applies, but the mean index for $T^*\mathbb{HP}^n$ equals $3 + 4n - 1 = 4n - 2$. The Morse-Bott spectral sequence needs a lot of space, so we will drop the figure. The idea is the same, of course.

The formula for the degree shift is

$$S_N = N\Delta - \frac{1}{2}\dim ST^*\mathbb{HP}^n = 3N - 4n + 1,$$

so columns are located at $p = 3, 3 + 4n - 2, 3 + 8n - 4, \dots$, and are filled with the groups from Lemma 2.3. Vertically, each column runs from 0 to $8n - 2$. The picture is similar to that of Figure 1. There is no overlap up to total degree $4n + 3$. In total degree $4n + 5$ there are n generators coming from column degree 3, and 1 generator from the next column, whose column degree is $4n + 5$, hence the number of generators with total degree $4n + 5$ equals $n + 1$. In degree $4n+7$, we have $n-1$ generators coming from column degree 3 and 1 generator from column degree $4n + 5$. Hence n generators.

This game repeats until the total degree $8n + 1$, which has $n + 1$ generators. In total degree $8n+3$, column degree 3 no longer contributes, column degree $4n+5$ contributes n generators and column degree $8n + 7$ does not yet contribute. The same holds in total degree $8n + 5$.

This story starts all over again at total degree $8n + 7$: column degree $4n + 5$ contributes n generators, and column degree $8n + 7$ contributes 1: we have again $n + 1$ generators as in total degree $4n + 5$. We have hence periodic repetition of the above groups with period $4n + 2$.

2.6. $Ca\mathbb{P}^2$**.** Again, the same argument applies, but the mean index for $T^*Ca\mathbb{P}^2$ equals 22, so the Morse-Bott spectral sequence looks similar to the one for $\mathbb{CP}^n$ in Figure 1. Columns are located at $p = 7, 7 + 22, 7 + 2 \cdot 22, \dots$, and are filled with the groups from Lemma 2.3. Vertically, each column runs from 0 to $16 \cdot 2 - 2$. Overlap is now even more complicated than before, but this is the last case, so with some staring we arrive at the result.

2.7. **Results.**

2.7.1. *Spheres.* If n is odd, then

$$SH_k^{S^1,+}(T^*S^n, \lambda_{can}) \cong \begin{cases} \mathbb{Q}^2 & \text{if } k = n - 1 + (n-1) \cdot \mathbb{Z}_{\geq 1} \\ 0 & \text{if } k \text{ is odd or } k < n - 1 \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

If n is even, then

$$SH_k^{S^1,+}(T^*S^n,\lambda_{can}) \cong \begin{cases} \mathbb{Q}^2 & \text{if } k = n-1+2(n-1)\cdot\mathbb{Z}_{\geq 1} \\ 0 & \text{if } k \text{ is even or } k < n-1 \\ \mathbb{Q} & \text{otherwise.} \end{cases}$$

2.7.2. *Complex projective space.*

$$SH_k^{S^1,+}(T^*\mathbb{CP}^n;\mathbb{Q}) \cong \begin{cases} \mathbb{Q}^{\ell+1} & \text{if } k = 1+2\ell \text{ with } \ell = 0,\dots,n-1, \\ \mathbb{Q}^{n+1} & k \in 2n-1+2\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

So the ranks follow the pattern (only writing the ranks in odd degrees from 1 on)

$$1,2,\dots,n,\mathbf{n+1},\dots$$

The periodic part is printed in bold-face (period is 2).

The mean Euler characteristic equals

$$\chi_m(T^*\mathbb{CP}^n) = -\frac{n+1}{2}.$$

2.7.3. *Quaternionic projective space.*

$$SH_k^{S^1,+}(T^*\mathbb{HP}^n;\mathbb{Q}) \cong \begin{cases} \mathbb{Q}^{\lfloor \frac{\ell}{2} \rfloor+1} & k = 3+2\ell,\ \ell = 0,\dots,2n-1 \\ \mathbb{Q}^n & k = 4n+3+2\cdot\{0,2,4,\dots,2n-2,2n\}+(4n+2)\cdot\mathbb{Z}_{>0} \\ \mathbb{Q}^{n+1} & k = 4n+3+2\cdot\{1,3,5,\dots,2n-1\}+(4n+2)\cdot\mathbb{Z}_{>0} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

So the ranks follow the pattern (only writing the ranks in odd degrees from 3 on)

$$1,1,2,2,\dots,n-1,n-1,n,n,\mathbf{n},\mathbf{n+1},\mathbf{n},\mathbf{n+1},\mathbf{n},\dots,\mathbf{n+1},\mathbf{n},\dots$$

The periodic part is printed in bold-face (period is $4n+2$).

The mean Euler characteristic equals

$$\chi_m(T^*\mathbb{HP}^n) = -\frac{n(n+1)}{2n+1}.$$

2.7.4. *Cayley plane.*

$$SH_k^{S^1,+}(T^*Ca\mathbb{P}^2;\mathbb{Q}) \cong \begin{cases} 1 & k = 7,9,11,13 \\ 3 & k \in 22\mathbb{N}+7 \cup 22\mathbb{N}+15 \\ 0 & k \text{ even, or } k < 7 \\ 2 & \text{otherwise.} \end{cases} \tag{2.3}$$

So the ranks follow the pattern (only writing the ranks in odd degrees from 7 on)

$$1,1,1,1,2,\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{3},\dots$$

The periodic part is printed in bold-face (period is 22). The mean Euler characteristic equals

$$\chi_m(T^*Ca\mathbb{P}^2) = -\frac{24}{22} = -12/11.$$

3. Contact homology of cotangent bundles of spheres

Here are the homology groups using contact homology conventions.

Let me first list the methods that are convenient to use.

1. Realize ST^*S^n as the Brieskorn manifold $\Sigma(2,\dots,2)$: use S^1-symmetry or use explicit perturbations (also doable).
2. Observe that ST^*S^n is a prequantization bundle over the quadric in projective space.
3. Use minimal model to compute equivariant homology of the loop space.
4. Morse-Bott spectral sequence for equivariant symplectic homology.

3.1. ST^*S^2. This is a special case, since $\pi_1(ST^*S^2) \cong \mathbb{Z}_2 \neq 0$. Also, in this case there is another method to compute the contact homology groups: Observe that $(ST^*S^2, \lambda_{can}) \cong (S^3, \lambda_0)/\mathbb{Z}_2$. You can then use the ellipsoid model to finish the computations; take care about the contractible and non-contractible loops.

I will give two answers using two possible different conventions. There is another convention in SFT which in general assigns a rational degree to orbits with a torsion homology class.

3.1.1. *Convention 1: grade using trivializations that extend over the filling T^*S^2.* For the contractible orbits, we can just grade using a trivialization that extends over a disk, but for the non-contractible orbits, some choice has to be made. Fortunately, the contact structure is trivial as a symplectic vector bundle, so we can just take a global trivialization. This trivialization also extends over the filling (important for symplectic homology) The result is then

$$HC_k = \begin{cases} \mathbb{Q} & k = 0 \\ \mathbb{Q}^2 & k \text{ is even, with } k \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

3.1.2. *Convention 2: filter contact homology by homotopy class.* The contact homology groups consisting of contractible orbits are

$$HC_k^0 = \begin{cases} \mathbb{Q} & k \text{ is even, with } k \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

For the non-contractible orbits, the contact homology groups are

$$HC_k^1 = \begin{cases} \mathbb{Q} & k \text{ is even, with } k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

3.2. **Higher dimensional unit cotangent bundles.** The unit cotangent bundle ST^*S^n is a prequantization bundle over the quadric Q^{2n-2} (real dimension). For the Maslov index of a principal orbit is

$$\mu := \mu(ST^*S^n) = 2(n-1).$$

Here are some ways to see this,

- direct computation with Brieskorn manifolds
- compute the Chern class of the quadric (see McDuff-Salamon)
- Use that the index of a geodesic is $n-1$. Since there is also an $n-1$-dimensional family of directions, we get the above Maslov index.

The homology of the quadric can be computed using the Gysin sequence, or with the argument in McDuff-Salamon. The result is

$$H_k(Q^{2n-2}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k \text{ is even, with } 0 \leq k \leq 2n-2 \\ 0 & \text{otherwise} \end{cases} \oplus \begin{cases} \mathbb{Z} & \text{if } k = n-1 \text{ with } n-1 \text{ even} \\ 0 & \textit{otherwise.} \end{cases}$$

The answer for contact homology is then

$$\begin{aligned} HC_k(ST^*S^n, \lambda_{can}; \mathbb{Q}) &= \bigoplus_{d=1}^{\infty} H_{k-d\mu+\frac{1}{2}\dim(ST^*S^n/S^1)-(n-3)}(Q^{2n-2}; \mathbb{Q}) \\ &= \bigoplus_{d=1}^{\infty} H_{k-d\mu+2}(Q^{2n-2}; \mathbb{Q}) \end{aligned}$$

Plugging in the above formula for the homology of a quadric gives the following results.

3.3. Results.

Proposition 3.1. *If n is odd, then*

$$HC_k(ST^*S^n, \lambda_{can}) \cong \begin{cases} \mathbb{Q}^2 & \text{if } k = 2n-4+d(n-1) \text{ with } d \in \mathbb{Z}_{\geq 1} \\ \mathbb{Q} & \text{if } k \text{ satisfies } k \text{ even, } k \geq 2n-4 \text{ and } k \neq 2n-4+d(n-1) \text{ with } d \in \mathbb{Z}_{\geq 1} \\ 0 & \text{otherwise} \end{cases}$$

If n is even, then

$$HC_k(ST^*S^n, \lambda_{can}) \cong \begin{cases} \mathbb{Q}^2 & \text{if } k = 2n-4+2d(n-1) \text{ with } d \in \mathbb{Z}_{\geq 1} \\ \mathbb{Q} & \text{if } k \text{ satisfies } k \text{ even, } k \geq 2n-4 \text{ and } k \neq 2n-4+2d(n-1) \text{ with } d \in \mathbb{Z}_{\geq 1} \\ 0 & \text{otherwise} \end{cases}$$

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