MATHEMATICAL STATISTICS **QUALIFYING EXAM**
Spring 2013

NAME:______________________________ ID:___________________________________

Instructions: There are 4 problems – you are required to solve them all. Please show detailed work for full credits. This is a close book exam. Please do NOT use calculator or cell phone during the exam.

1. A bivariate population of (X, Y) is sampled independently on three occasions. On the first, a random sample of size n_0 is taken and only $T = \min(X, Y)$ is observed for each pair. On the second, a random sample of size n_1 is taken, and only the X-marginal is observed for each pair. Finally, a random sample of size n_2 is taken, and only the Y-marginal is observed for each pair. Therefore, the combined set of observations is of the form $(\boldsymbol{T}, \boldsymbol{X}, \boldsymbol{Y})$, where $\boldsymbol{T} = (T_1, \dots, T_{n_0})$, $\boldsymbol{X} = (X_{11}, \dots, X_{1n_1})$ and $\boldsymbol{Y} = (Y_{21}, \dots, Y_{2n_2})$. Assume the following two-parameter probability model for (X, Y):
$$P(X > x, Y > y) = \exp\left[-\frac{1}{\theta}\left(x^{\frac{1}{\delta}} + y^{\frac{1}{\delta}}\right)^{\delta}\right],$$
$x > 0, y > 0, \theta > 0, 0 < \delta \le 1$ with unknown parameters θ and δ.
   (a) Find the joint pdf of $(\boldsymbol{T}, \boldsymbol{X}, \boldsymbol{Y})$.
   (b) Identify the distributions of T_1, X_{11} and Y_{21}.

2. Suppose $X_1, \dots, X_n$are iid Possion(μ).
   (a) Find the Cramér-Rao lower bound for the variance of an unbiased estimator of μ.
   (b) Find the ML estimator $(\hat{\mu})$ of μ
   (c) Find the mean and variance of $\hat{\mu}$. What can you conclude from this?
   (d) Find the ML estimator $(\hat{h})$ of $h(\mu) = \mu^2 e^{-\mu}$.
   (e) Prove or disprove the following statement: $\hat{h}$ is an unbiased estimator of $h(\mu)$.
   (f) Find the best unbiased estimator of $h(\mu)$.

3. Suppose we have two independent random samples from two normal populations: $X_1, X_2, \cdots, X_{n_1} \sim N(\mu_1, 9\sigma^2)$, and $Y_1, Y_2, \cdots, Y_{n_2} \sim N(\mu_2, \sigma^2)$.
(a) At the significance level α, please construct a test using the pivotal quantity approach to test whether $\mu_1 = 2\mu_2$ or not. (*Please include the derivation of the pivotal quantity, the proof of its distribution, and the derivation of the rejection region for full credit.)
(b) At the significance level α, please derive the likelihood ratio test for testing whether $\mu_1 = 2\mu_2$ or not.
(c) Subsequently, please show whether this LR test derived in part (b) is equivalent to the one derived in part (a).

4. Let X and Y be random variables with joint pdf

$f_{X,Y}(x, y) =$

$$\frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2-2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\},$$

where $-\infty < x < \infty,\ -\infty < y < \infty$. Then X and Y are said to have a *bivariate normal distribution*. The joint moment generating function for X and Y is

$$M(t_1, t_2) = \exp\left[t_1\mu_X + t_2\mu_Y + \frac{1}{2}\left(t_1^2\sigma_X^2 + 2\rho t_1 t_2\sigma_X\sigma_Y + t_2^2\sigma_Y^2\right)\right].$$

(a) Please derive the conditional pdf of $Y|X = x$;

(b) Furthermore, suppose it is known that $\sigma_X^2 = \sigma_Y^2$, please construct an exact test to test whether $\rho = 0$ or not at the significance level α. Please include all key steps for full credits.