Transformations

General Form of a Transformed Function

$$y = af[b(x-h)]+k$$

Math 30-1 AP
Mrs. D. Atkinson

TRANSFORMATIONS

Horizontal and Vertical Translations of Functions

A **transformation** is an operation that moves (or maps) a figure from an original position to a new position. Transformations that we will consider are **translations**, **reflections**, **stretches**.

Translations:

A **translation** is a transformation that slides each point of a figure the same distance in the same direction.

In general, if $y-k=f(x-h)$ OR $y=f(x-h)+k$

- $h>0$ there is a horizontal translations of h units right
- $h<0$ there is a horizontal translations of h units left
- $k>0$ there is a vertical translation of k units up
- $k<0$ there is a vertical translation of k units down

MAPPING and REPLACEMENT NOTATION

If the graph of $y=f(x)$ is transformed to the graph of $y-k=f(x-h)$, then the point $(x,\ y)$ is translated to the point $(x+h,\ y+k)$. This is called **mapping notation**!

If the graph of $y=f(x)$ is transformed to the graph of $y-k=f(x-h)$, then we say that $x\to x-h$ and $y\to y-k$. This is called **replacement notation**!

Example:
The graph of $y=f(x)$ is transformed to the graph of $y-2=f(x-3)$

The replacements for x and y are $x\to x-3$ AND $y\to y-2$
All points will move 3 units right and 2 units up
The mapping for co-ordinates $(x,y)\to(x+3,y+2)$

So if the point (4, 6) was on the graph of $y=f(x)$ then the point would become (7, 8)

Ex. 1 Describe how the graphs of the following functions relate to the graph of $y = f(x)$.

a) $y = f(x-3)$

b) $y = f(x)+4$

c) $y = f(x)-8$

d) $y = f(x+7)+4$

Ex. 2 What happens to the graph of the function $y = f(x)$ if the following changes are made to its equation?

a) Replace x with x + 2

b) replace y with y - 8

Ex.3 The point (2, -3) lies on the graph of $y = f(x)$. State the coordinates of the image of this point under the following transformations

a) $y+8 = f(x)$

b) $y = f(x-7)+5$

Ex. 4 Write the equation of the image of $y = f(x)$ after each transformation.

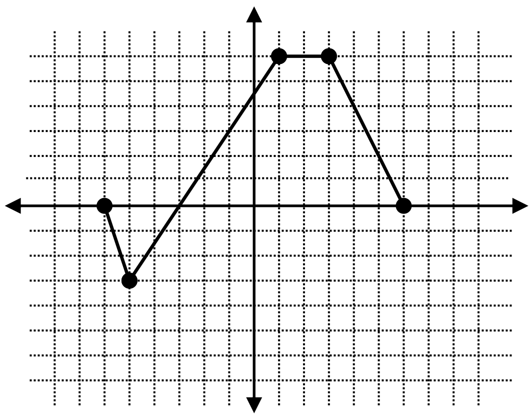
a) A horizontal translation of 5 units left

b) A vertical translation of 3 units up

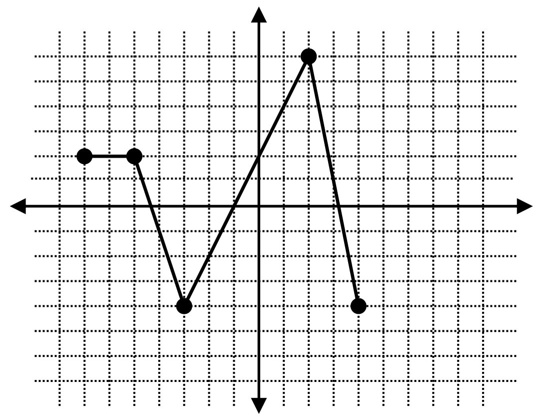
c) A horizontal translation of m units right and a vertical translation of p units down.

Ex. 5 In each case the graph of $y = f(x)$ is shown. Sketch the following transformations for each.

a) $y = f(x-2)$



b) $y+2 = f(x+3)$



Ex.6 State the coordinates of the image of the point (-3,5) under the transformation described by

a) $(x, y) \to (x-7, y+4)$

b) $(x, y) \to (x-6, y+1)$

Ex. 7 Describe how the graph of the second function compares to the graph of the first function.

a) $y = x^4$
$y = x^4 + 3$

b) $y = 6x - 3$
$y = 6(x-1) - 3$

c) $y = |x|$
$y = |x-6| + 2$

d) $y = \frac{1}{\sqrt{x}}$
$y = \frac{1}{\sqrt{x+1}} + 3$

Ex. 8 Write the equation of the image of:

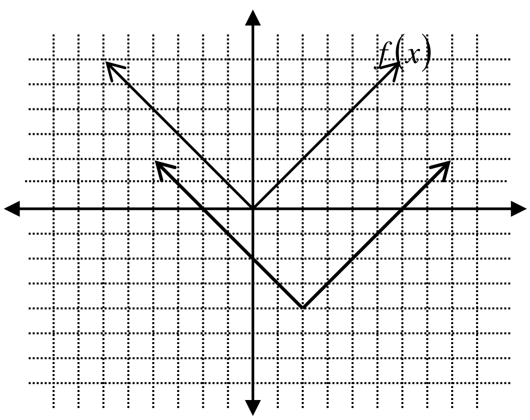
a) $y = x^2$ after a horizontal translation of 3 units to the right.

b) $y = 10^x$ after a vertical translation of 2 units up.

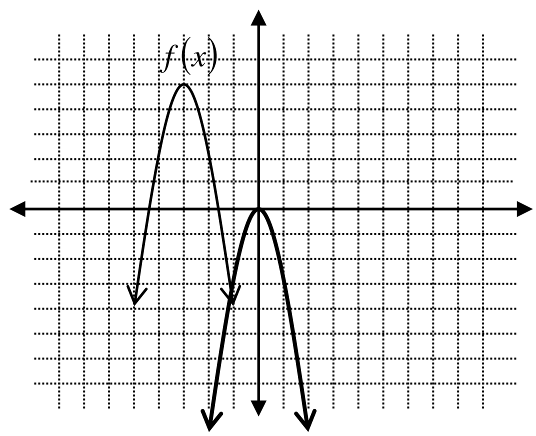
c) $y = \sqrt{x}$ after a horizontal translation of 4 units to the left and a vertical translation of 3 units down.

Ex. 9 The function represented by the thick line is a transformation of the function represented by the thin line. Write an equation for each function represented by the thick line.

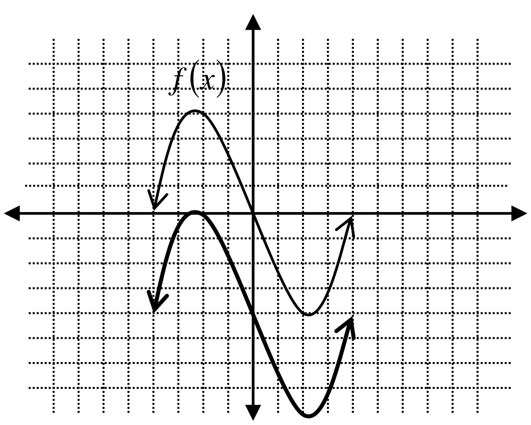
a)



b)



c)



Ex. 10 $y = \sqrt{x}$ is a radical function.

a) What vertical translation would be applied to $y = \sqrt{x}$ so that the translation image passes through $(16,\ 7)$?

b) What horizontal translation would be applied to $y = \sqrt{x}$ so that the translation image passes through $(17,\ 8)$?

Assignment Page 12 #1ace,2ac,3ac,4ac,5ac,6,7,8,9,11,17,18ac

Reflections

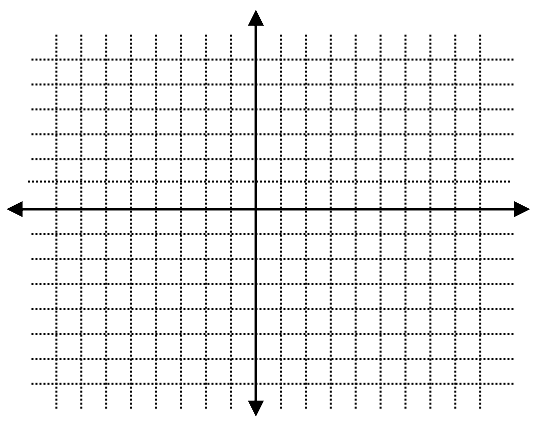
A **Reflection** is a mirror image about a line or axis. **Invariant Points** are points that do not move after a transformation.

Reflections in the x-axis Comparing $y = f(x)$ and $y = -f(x)$

Ex. 1 The graph of $y = f(x) = x^2 - 4x + 4$ is shown.

a) Write the equation that represents $y = -f(x)$.

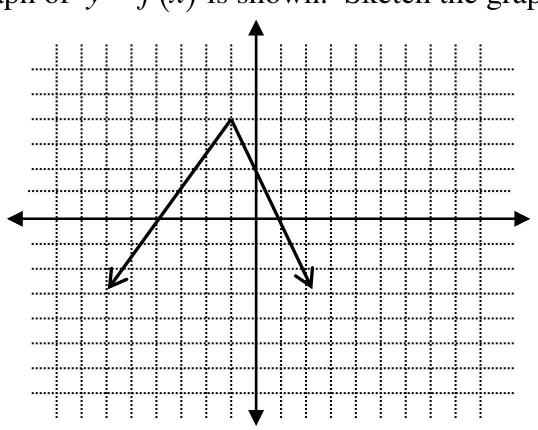
b) Use the graphing calculator to sketch $y = f(x)$ **and** $y = -f(x)$ and show the graphs on the grid.



c) State the coordinates of the invariant point(s).

d) How does the graph of $y = -f(x)$ compare with the graph of $y = f(x)$**.**

Ex. 2 The graph of $y = f(x)$ is shown. Sketch the graph of $y = -f(x)$.



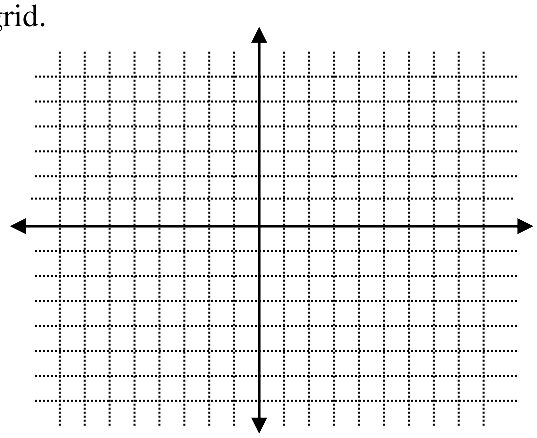
<u>Reflections in the y-axis</u>

Comparing $y = f(x)$ **and** $y = f(-x)$

<u>Ex. 3</u> The graph of $y = f(x) = x^2 - 4x + 4$ is shown.

a) Write the equation that represents $y = f(-x)$.

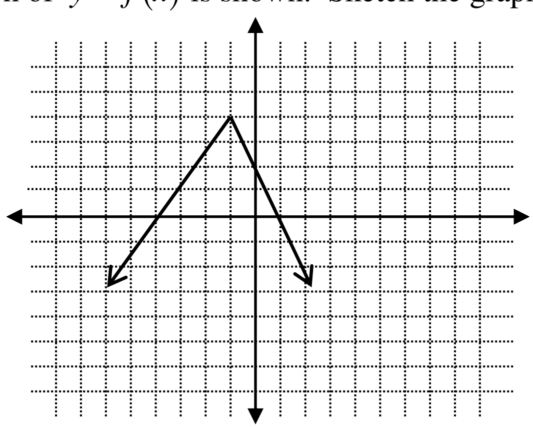
b) Use the graphing calculator to sketch $y = f(x)$ **and** $y = f(-x)$ and show the graphs on the grid.



c) State the coordinates of the invariant point(s).

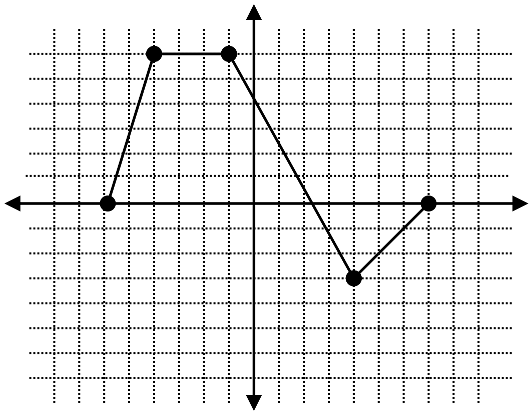
d) How does the graph of $y = f(-x)$ compare with the graph of $y = f(x)$.

<u>Ex. 4</u> The graph of $y = f(x)$ is shown. Sketch the graph of $y = f(-x)$.



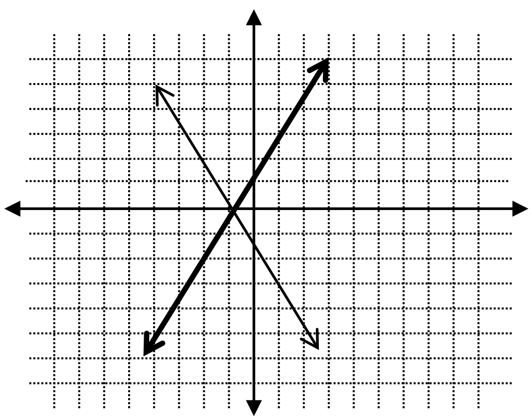
Ex.5 The graph of $y = f(x)$ is shown. Sketch the following reflections on the same grid.

a) $y = -f(x)$ b) $y = f(-x)$

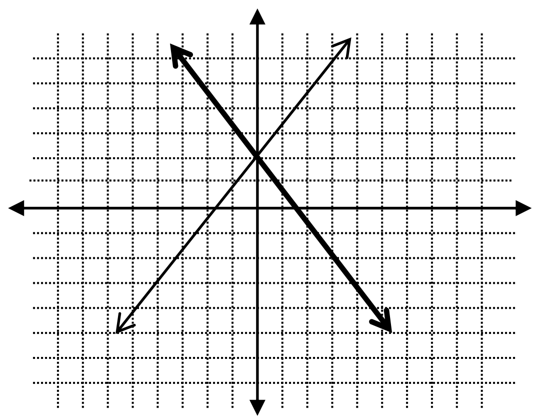


Ex.6 The graph drawn in the thick line is a reflection of the graph drawn in the thin line. Write the equation for each graph drawn in the thick line.

a)



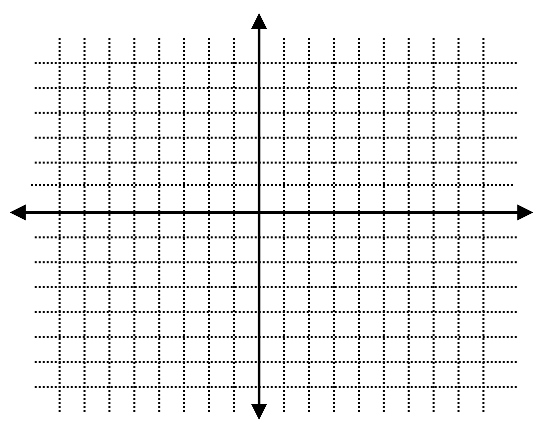
b)



Ex. 7 Write the equation of the image of :

a) $y = 10^x$ after a reflection in the y-axis

b) $y = \sqrt{x}$ after a reflection in the x-axis



Ex. 8

a) Sketch the graph of $y = \frac{6}{x^2 + 3}$

b) Write the equation and sketch the graph for

i) $y = -f(x)$ ii) $y = f(-x)$

c) State whether the following are functions

Ex. 9 Given $y = \sqrt{x-2}$, find:

a) $y = -f(x)$ b) $y = f(-x)$

Ex. 10 Given $y = x^2 + 1$, find:

a) The equation of a reflection in the y-axis.

b) The equation of a reflection in the x-axis.

Assignment Page 28 #1row 1and 4:abcd,3a,4b,5cd

Stretches

Stretches of Functions

A stretch is a transformation that changes the size and shape of the graph; the original and the new graph are not congruent.

There are two types of stretches:

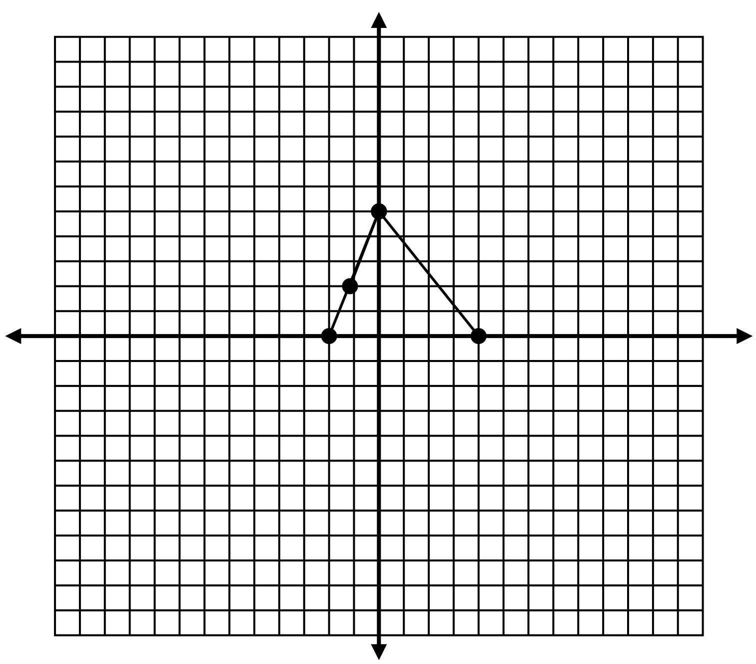
1. Vertical
2. Horizontal

Vertical Stretches

- **Vertical stretches are in the form $y = af(x)$, where a stretches the graph vertically by a factor of a.**
- The point $(x, \ y)$ becomes the point $(x, \ ay)$.
- The points on the x-axis are invariant.
- If the a has a negative sign in front of it, the graph is reflected in the x-axis.

Ex. 1 Describe the transformations that occurred if the original graph of $y = -4f(x)$ was $y = f(x)$.

Ex. 2 Given the graph of $y = f(x)$, use a table of values to sketch $y = 2f(x)$.



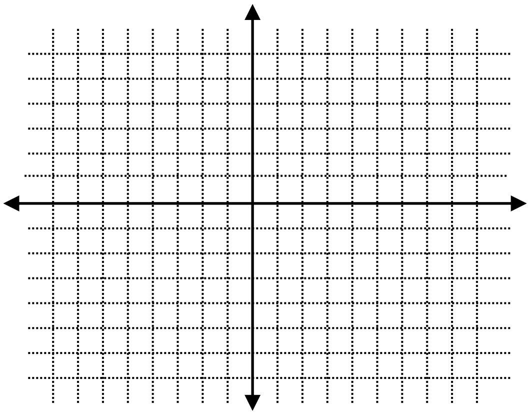
$y = f(x)$

x	y
-2	**0**
-1	**2**
0	**5**
4	**0**

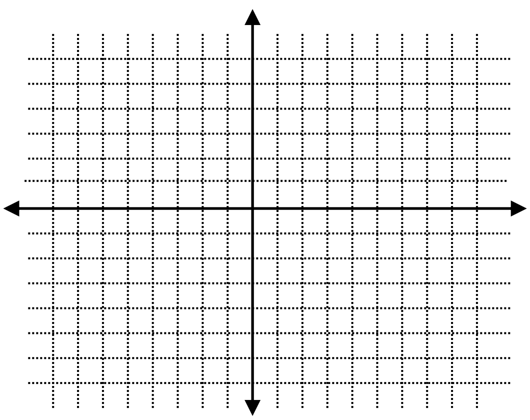
$y = 2f(x)$

x	y

Ex. 3 Graph $y = x^2$, then replace y with $3y$ and graph it on the same axes and describe the transformations.



Ex. 4 Graph $y = \sqrt{x}$, then replace y with $\frac{1}{2}y$ and graph it on the same axes and describe the transformations.

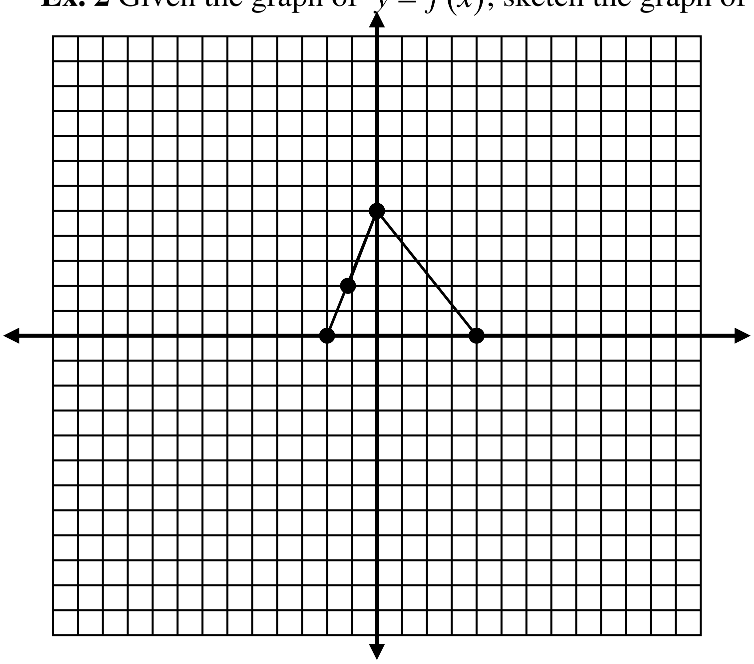


<u>**Horizontal Stretches**</u>

- Horizontal stretches are in the form $y = f(bx)$, where b stretches the graph by a factor of $\frac{1}{b}$.
- The point $(x,\ y)$ becomes the point $\left(\frac{x}{b},\ y\right)$
- Points on the y-axis are invariant.
- If b has a negative sign in front of it, the graph is reflected in the y-axis.

Ex. 1 Describe the transformations that occurred if the original graph of $y = f(5x)$ was $y = f(x)$.

Ex. 2 Given the graph of $y = f(x)$, sketch the graph of $y = f(2x)$.



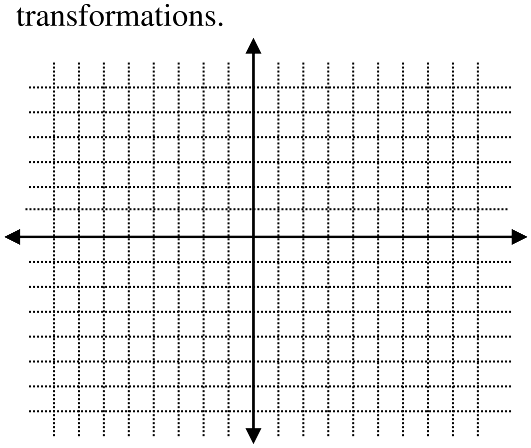
$y = f(x)$

x	y
-2	**0**
-1	**2**
0	**5**
4	**0**

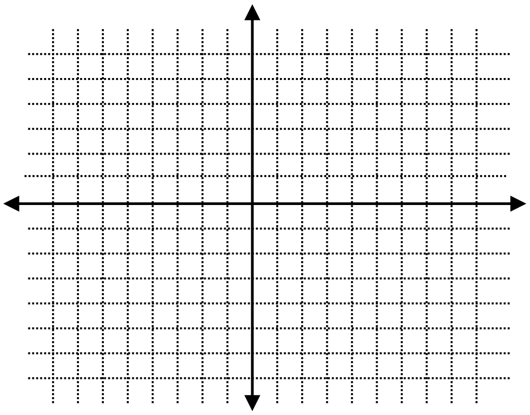
$y = f(2x)$

x	y

Ex. 3 Graph $y = x^2$ then replace x with $2x$ and graph it on the same axes and describe the transformations.



Ex. 4 Graph $y=\sqrt{x}$ then replace x with $\frac{1}{3}x$ and graph it on the same axes and describe the transformations.



Ex. 5 Describe the transformations that occurred to each if the original function is $y=4-x^3$

a) $y=3(4-x^3)$

b) $y=4-(2x)^3$

Ex. 6 Describe the transformations that occurred if the original function was $y=x^2$ and the new function is;

a) $y=-2x^2+3$

b) $y=(-2x+4)^2$

c) $y=(x-2)^2+1$

Ex. 7 Describe the transformations on the function $y=f(x)$.

a) $y=2f(x+1)$

b) $y=-f(x)-b$

c) $y=f(-x+3)$

Ex. 8 The polynomial $P(x)=(x+1)(x-2)(x+4)$ has three zeros at -1, 2, and -4. Use transformations to determine the zeros of each of the following:

a) $y=-2P(x)$

b) $y=P\left(\frac{1}{3}x\right)$

Ex.9

Write the replacement for x or y and write the equation of the image of $y=f(x)$ after each transformation.

a) A horizontal stretch by a factor of 6 about the y-axis.

b) A vertical stretch by a factor of $\frac{1}{5}$ about the x-axis.

c) A reflection in the x-axis and a vertical stretch about the x-axis by a factor of 3.

d) A horizontal stretch about the y-axis by a factor of $\frac{1}{2}$ and a vertical stretch about the x-axis by a factor of $\frac{1}{4}$.

Ex.10

How does the graph of $3y=f(x)$ compare with the graph of $y=f(x)$?

What happens to the graph of the function $y=f(x)$ if you make these changes?

a) Replace x with $4x$.

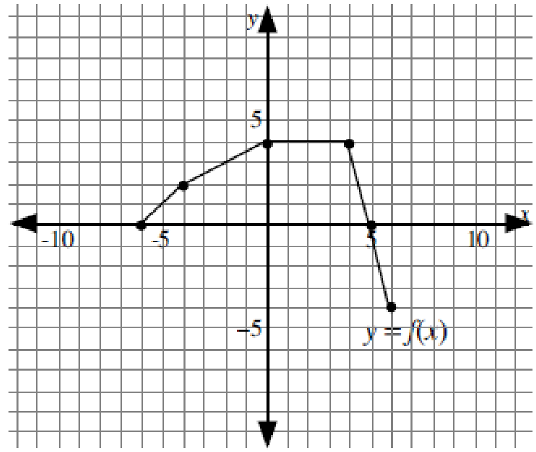
b) Replace y with $\frac{1}{3}y$.

c) Replace y with $6y$ and x with $\frac{1}{3}x$.

Ex.11

The graph of $y = f(x)$ is shown.

Sketch $y = f(-2x)$.



Ex.12

Write the equation of the image of:

a) $y = x^2$ after a horizontal stretch about the y-axis by a factor of $\frac{3}{4}$.

b) $y = \sqrt{x} - 3$ after a horizontal stretch by a factor of 4 about the y-axis and a vertical stretch by a factor of 2 about the x-axis.

c) $y = 3x + 7$ after a vertical stretch about the x-axis by a factor of $\frac{1}{3}$ and a reflection in the x-axis.

Ex.13

Describe how the graph of the second function compares to the graph of the first function.

a) $y = f(x)$

$y = f\left(\frac{1}{2}x\right)$

b) $y = 2^x$

$y = 2^{3x}$

c) $y = |x|$

$y = -2|x|$

d) $y = |x|$

$y = |-2x|$

e) $y = |x|$

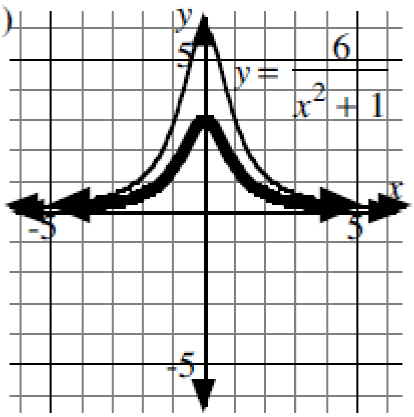
$y = 2\left|\frac{1}{3}x\right|$

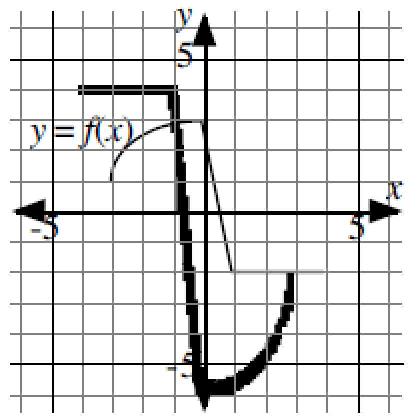
f) $y = x^3$

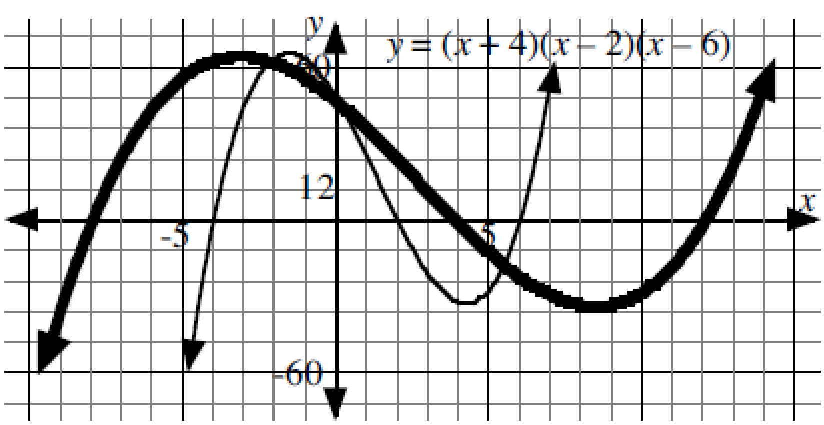
$3y = x^3$

Ex.14

The function represented by the thick line is a stretch of the function represented by the line. Write an equation for each function represented by the thick line.

a) 

b) 

c) 

Example 15
Given the domain of the function $y = f(x)$ is $\{x | -6 \le x \le 3, \quad x \in R\}$, determine the domain of the function $y = f\left(\frac{1}{3}x\right)$.

Example 16
Given the range of the function $y = f(x)$ is $[-4, 5]$, determine the range of the function $y = \frac{1}{4}f(x)$.

Assignment Page 28 #2 row 1 and 4abcd, 5ab, 6, 7, 9, 14

Combinations of Transformations **S R T**

When drawing transformations, use the following order:

1. **S**tretches
2. **R**eflections
3. **T**ranslations

$y = af[b(x-h)]+k$

$a =$ a vertical stretch by a factor of a. If a is negative then it is also a reflection in the x-axis. X-intercepts are invariant.

$b =$ a horizontal stretch by a factor of $\frac{1}{b}$. If b is negative then it is also a reflection in the y-axis. Y-intercepts are invariant.

$h =$ horizontal translation
- $h > 0$ horizontal translation h units right
- $h < 0$ horizontal translation h units left

$k =$ vertical translation
- $k > 0$ vertical translation k units up
- $k < 0$ vertical translation k units down

When graphing:

1. Create a table of values for original function and graph.
2. Create a second table showing changes using the mapping $(x, y) \rightarrow \left(\frac{x}{b}+h,\ ay+k\right)$

Note

Many students have difficulties seeing where these parameters fit into the specific functions we deal with in this course.

Quadratic: $y = x^2$ becomes $y = a\left[b(x-h)\right]^2 + k$

Cubic: $y = x^3$ becomes $y = a\left[b(x-h)\right]^3 + k$

Absolute Value: $y = |x|$ becomes $y = a\left|b(x-h)\right| + k$

Square Root: $y = \sqrt{x}$ becomes $y = a\sqrt{b(x-h)} + k$

Example 1

Suppose you have the function $y = f(x)$. If you were to perfom a horizontal stretch by a factor of $\frac{1}{2}$ and a vertical translation of 4 down, does the order matter? Write new equations for both scenarios.

Example 2

Suppose you have the function $y = f(x)$. If you were to perfom a horizontal stretch by a factor of $\frac{3}{2}$, and a horizontal translation of 2 units left, does the order matter? Write new equations for both scenarios.

Note:

- To sketch functions with a combination of transformations from an equation, we follow "**ORDER OF OPERATIONS**" rules (UNLESS TOLD TO DO IN A PARTICULAR ORDER)
    1. Deal with stretches and reflections FIRST (stretches and refelctions both involve "multiplication")
    2. Deal with translations LAST (which involve addition and subtraction)

- Always express your function in factored form $y = af\left[b\left(x-h\right)\right]+k$.

Example 3

Describe how the graph of the function $y = 3\sqrt{2x-6}+5$ relates to the graph of $y = \sqrt{x}$

Example 4

The following transformations are applied to the function $y = x^2 + 1$ **IN THE ORDER GIVEN:**

- A horizontal translation left 2 units
- A reflection in the x-axis
- A vertical stretch about the x-axis by a factor of ¼
- A vertical translation of 3 units down

Write the equation which represents the final position of the graph.

Example 5

The graph of $y = f(x)$ where $f(x) = x^2$ is transformed and the resulting function is represented by the equation $g(x) = 2f\left(\frac{1}{2}(x+5)\right) - 8$. Describe the transformations that occurred to $y = f(x)$ to get $g(x)$ and determine the new equation.

Example 6
Write the equation of the transformed functions in each of the following scenarios:

a) The graph of $y = f(x)$ is horizontally stretched by a factor of $\frac{1}{4}$ about the y-axis and is vertically translated 5 units down.

b) The graph of $y = f(x)$ is vertically stretched by a factor of $\frac{3}{5}$ about the x-axis, is reflected in the y-axis and is horizontally translated 2 units left.

Example 7
Given that the point P(6, –8) is on the graph of *y = f(x),* state the **coordinates** of the corresponding image point on the graph of

a) $y = 2f(x+3)$ ________

b) $y+4 = f(x-2)$ _________

c) $y = -f(3x)$ ________

d) $3y+6 = f(-x)$ ________

e) $y = 4f(-2x+10)$ _______________

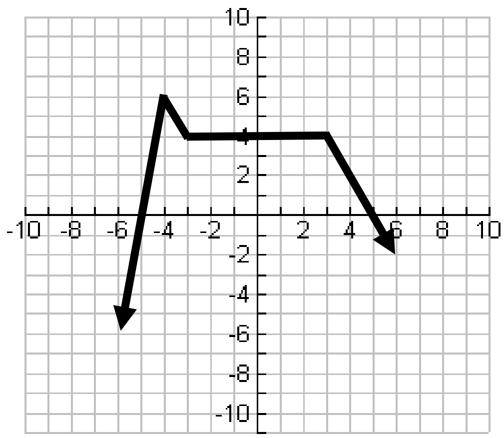
Example 8
A function $g(x) = x^3$ is transformed into a new function $P(x)$. To form the new function $P(x)$, $g(x)$ is stretched vertically about the x-axis by a factor of $\frac{1}{4}$, reflected in the y-axis and translated 3 units to the right. Write the equation of the new function $P(x)$.

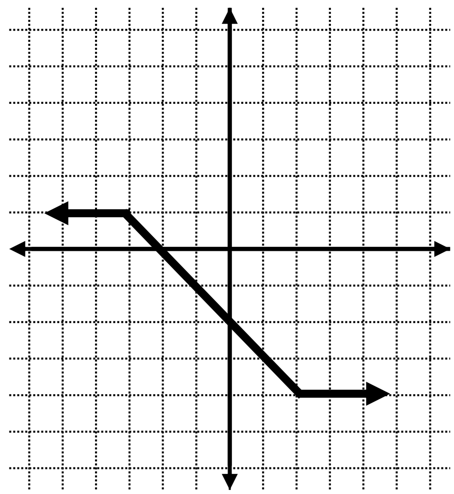
Example 9

The graph of $y = f(x)$ is shown. Sketch the graph of

a) $y = -2f(2x) - 3$

b) $2y = f(\frac{1}{2}x + 1) + 1$



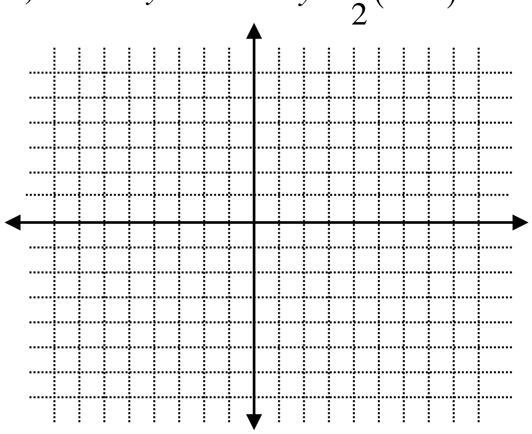


Example 10 The point $(m,\ 3n)$ is on the curve $y = f(x)$. Describe how the transformations affect this ordered pair if $y = 2f(-x+3) - 1$.

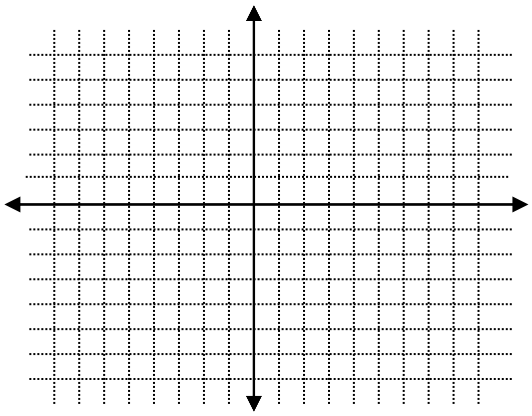
Example 11 The graph of $y = x^3$ is stretched vertically by a factor of 3, reflected in y-axis, and translated 2 units left and 4 units up. Write the equation of the transformed graph $g(x)$.

Example 12 Sketch each set of functions on the same set of axes.

a) Sketch $y = x^2$ and $y = \frac{1}{2}(x+4)^2 - 5$



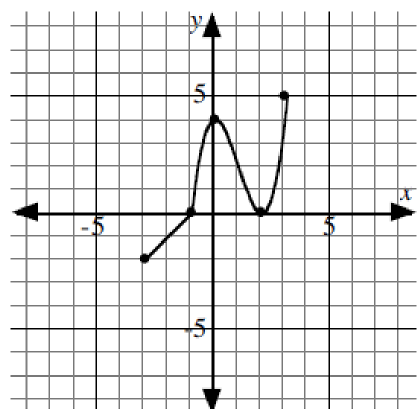
b) Given $f(x)=\sqrt{x}$ and $y=2\sqrt{(-x-3)}+4$



Example 13

The graph of $y = f(x)$ is shown.

Sketch the graph of $y = -2f(x-3)+1$.



Example 14

Consider the function $y = f(x)$. In each case determine:

- the replacements for x and y which would result in the following combinations of transformations
- the equation of the transformed function in the form $y = af[b(x-h)]+k$

a) a horizontal stretch by a factor of $\frac{1}{4}$ about the y-axis and a vertical translation of 5 units down.

b) a vertical stretch by a factor of $\frac{3}{5}$ about the x-axis, a reflection in the y-axis, and a horizontal translation 2 units left.

Example 15

The function $f(x) = \sqrt{x}$ has been transformed into the function $g(x) = -2\sqrt{3x-12} + 5$.
Complete the following statement.

" The function $f(x)$ has been transformed to the function $g(x)$ by stretching horizontally about the y-axis by a factor of ___________, stretching vertically about the x-axis by a factor of ___________, reflecting in the ___________, translating ___________units up and ___________ units horizontally to the ___________ ."

Assignment Page 38 #1,#2, #3 row 1 and 3, 4 5b, 6ace, 7ace, 8, 9cde, 10bc, 11ab, 12, 15ab

Inverses

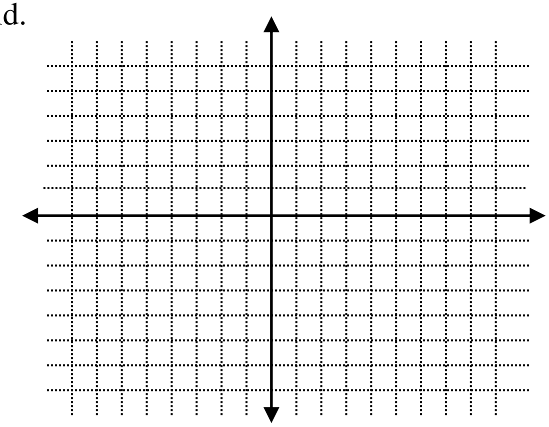
Reflections in the line $y = x$

Comparing $y = f(x)$ **and** $y = f^{-1}(x)$ **OR** $x = f(y)$

Ex. 1 The graph of $y = f(x) = (x-5)^2$ is shown.

a) Write an equation that represents $y = f^{-1}(x)$.

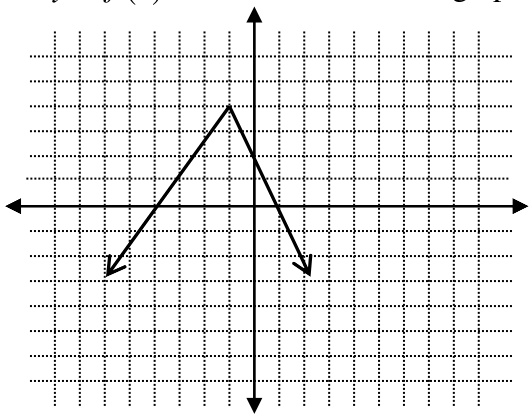
b) Use the graphing calculator to sketch $y = f(x)$ **and** $y = f^{-1}(x)$ and show the graphs on the grid.



c) Mark the invariant point(s) on the grid.

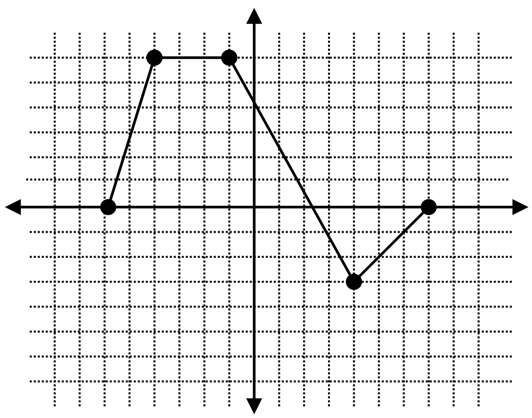
d) How does the graph of $y = f^{-1}(x)$ compare to the graph of $y = f(x)$.

Ex. 2 The graph of $y = f(x)$ is shown. Sketch the graph of $y = f^{-1}(x)$.



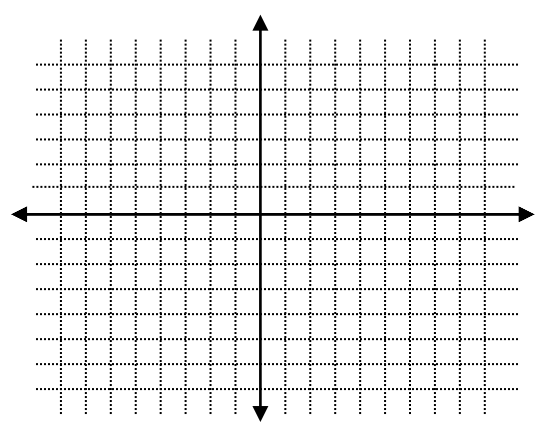
Ex.3 The graph of $y = f(x)$ is shown. Sketch the following reflections on the same grid.

a) $y = f^{-1}(x)$



Ex.4 Given $y = \sqrt{x-2}$, find:

a) $y = f^{-1}(x)$ OR $x = f(y)$ and sketch each

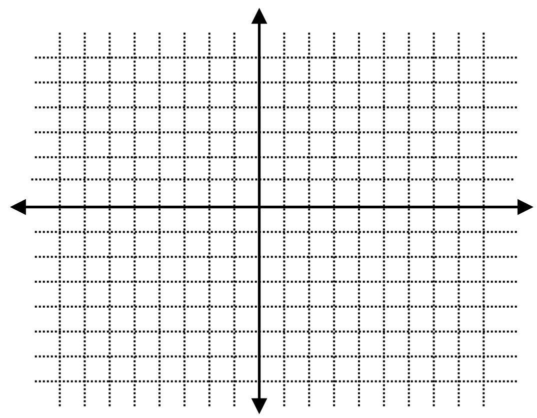


Ex.5 Given $f(x) = x^2 + 1$, find:

a) The equation of a reflection in the line $y = x$

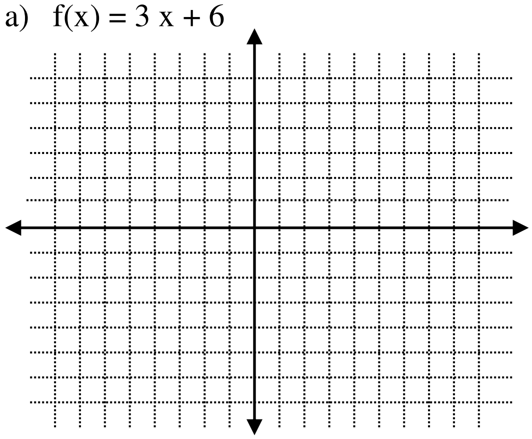
b) Graph and state the domain and range

c) Describe how the domain of f(x) could be restricted so that the inverse of f(x) is a function



Ex. 6 Determine the inverse and verify graphically

a) $f(x) = 3x + 6$



Assignment Page 51#1b,2b,3a,4a,d,5ace,6,7a,8ab,9ab,11,12ad,13bd,15ab
Review Page 56
Major Quiz