

Aspects of the planetary Birkhoff normal form*

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This paper is dedicated to Professor Alain Chenciner on his 70th birthday

Abstract

The discovery in [37], [16] of the Birkhoff normal form for the planetary many-body problem opened new insights and hopes for the comprehension of the dynamics of this problem. Remarkably, it allowed to give a *direct* proof of the celebrated Arnold’s Theorem [5] on the stability of planetary motions. In this paper, using a “ad hoc” set of symplectic variables, we develop an asymptotic formula for this normal form that may turn to be useful in applications. As an example, we provide two very simple applications to the three-body problem: we prove a conjecture by V. I. Arnold [5] on the *Kolmogorov set* of this problem and, using Nehorošev Theory [32], we prove, in the planar case, stability of *all* planetary actions over exponentially-long times, provided mean-motion resonances are excluded. We also briefly discuss perspectives and problems for full generalization of the results in the paper.

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1 Introduction and results

1.1 The planetary many–body problem consists in determining the dynamics of $(1 + n)$ masses undergoing Newtonian attraction. The term “planetary” is reserved to the case when one mass, the “sun”, or “star”, denoted with \bar{m}_0 , is taken to be much greater than the others, $\mu\bar{m}_1, \dots, \mu\bar{m}_n$, which are called “planets”. Here $\mu \ll 1$ is a small number. After the “heliocentric¹ reduction” of invariance by translations, this dynamical system is governed by the $3n$ degrees of freedom Hamiltonian

$$H_{\text{plt}} = \sum_{i=1}^n \left(\frac{|y^{(i)}|^2}{2m_i} - \frac{m_i M_i}{|x^{(i)}|} \right) + \mu \sum_{1 \leq i < j \leq n} \left(\frac{y^{(i)} \cdot y^{(j)}}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|x^{(i)} - x^{(j)}|} \right) \quad (1)$$

on the phase space

$$(y, x) = (y^{(1)}, \dots, y^{(n)}, x^{(1)}, \dots, x^{(n)}) \in (\mathbb{R}^3)^{2n} : \quad x^{(i)} \neq 0, \quad x^{(i)} \neq x^{(j)}$$

endowed with the standard 2– form

$$\Omega := dy \wedge dx := \sum_{i=1}^n \sum_{j=1}^3 dy_j^{(i)} \wedge dx_j^{(i)}$$

where $y^{(i)} = (y_1^{(i)}, y_2^{(i)}, y_3^{(i)})$, $x^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$. Here, m_i, M_i are suitable auxiliary masses related to \bar{m}_i and μ via

$$M_i = \bar{m}_0 + \mu\bar{m}_i \quad m_i = \frac{\bar{m}_0 \bar{m}_i}{\bar{m}_0 + \mu\bar{m}_i} .$$

A procedure commonly followed in the past [5], [18], [21], [32] to regard the system as a “close to integrable”, was to use a symplectic set of variables, usually called “Poincaré variables”. These variables, that we denote

$$(\Lambda_i, \lambda_i, \eta_i, \xi_i, p_i, q_i) \quad 1 \leq i \leq n ,$$

are “six per planet”. They were introduced by H. Poincaré by modifying another set of “action–angle” variables $(\Lambda_i, \Gamma_i, \Theta_i, \ell_i, g_i, \theta_i) \in \mathbb{R}^3 \times \mathbb{T}^3$ (where $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$), having the Λ_i ’s in common, called “Delaunay variables”. Delaunay variables are “natural”, “action–angle” variables related

¹See, e.g., [39].

to the ‘‘Cartesian variables’’ $(y^{(i)}, x^{(i)})$ in (1) via the integration of each of the ‘‘two-body’’ Hamiltonians

$$\frac{|y^{(i)}|^2}{2m_i} - \frac{m_i M_i}{|x^{(i)}|}.$$

The Poincaré variables are in part ‘‘action–angle’’ (i.e., $(\Lambda_i, \lambda_i) \in \mathbb{R} \times \mathbb{T}$), in part ‘‘rectangular’’ (i.e., $(\eta_i, \xi_i, p_i, q_i) \in \mathbb{R}^4$). The definition of Delaunay and Poincaré variables may be found, *e.g.*, in [15]. In Delaunay–Poincaré variables, any of the two-body Hamiltonian above takes the ‘‘Kepler form’’

$$h_{\text{Kep}}^{(i)}(\Lambda_i) = -\frac{M_i^2 m_i^3}{2\Lambda_i^2}.$$

It is ‘‘properly degenerate’’: two degrees of freedom disappear, as it is well known. This proper degeneracy naturally reflects on the system (1), which in fact takes the form

$$\mathcal{H}_P(\Lambda, \lambda, z) = h_{\text{Kep}}(\Lambda) + \mu f_P(\Lambda, \lambda, z) \quad (2)$$

where $h_{\text{Kep}}(\Lambda)$ is the n degrees of freedom ‘‘unperturbed’’ part $-\sum_{i=1}^n \frac{M_i^2 m_i^3}{2\Lambda_i^2}$, while $f_P(\Lambda, \lambda, z)$ is the $3n$ degrees of freedom ‘‘perturbation’’

$$\sum_{1 \leq i < j \leq n} \left(\frac{y^{(i)} \cdot y^{(j)}}{\bar{m}_0} - \frac{\bar{m}_i \bar{m}_j}{|x^{(i)} - x^{(j)}|} \right) \quad (3)$$

in (1), expressed in Poincaré variables. Here, we have denoted as (Λ, λ, z) the $3n$ –dimensional collection of

$$\Lambda = (\Lambda_1, \dots, \Lambda_n), \quad \lambda = (\lambda_1, \dots, \lambda_n), \quad z = (z_1, \dots, z_n) \quad (4)$$

with $z_i := (\eta_i, \xi_i, p_i, q_i)$.

A long outstanding problem lasted about fifty years concerned the existence of a *Birkhoff normal form* for the system (2).

Namely, if it were possible to conjugate the Hamiltonian (2) to an analogue one,

$$\mathcal{H}_{\text{bnf}} = h_{\text{Kep}} + \mu f_{\text{bnf}},$$

whose average (‘‘secular’’) perturbing function

$$(f_{\text{bnf}})_{\text{av}} := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_{\text{bnf}}$$

were in Birkhoff normal form of some order (see [22], [3], [43] for information on Birkhoff theory). The claim is perfectly natural, since in fact the average $(f_P)_{\text{av}}$ of f_P in (2) turns to have an elliptic equilibrium point in $\{z = 0\}$, for any choice of Λ . We recall that, physically, $\{z = 0\}$, the ‘‘secular origin’’, corresponds to circular and co-inclined unperturbed motions, a configuration with relevant physical meaning, being commonly observed in nature in many-body systems.

The problem was settled by V. I. Arnold, who, in the 60’s announced (1962’s International Congress for Mathematicians; Stockholm, [23]) and next (1963) published his more than celebrated ‘‘theorem on the stability of planetary motions’’; or ‘‘The Planetary Theorem’’, for short.

Theorem 1.1 (V. I. Arnold, [5, p. 127]) *In the n –body problem there exists a set of initial conditions having positive Lebesgue measure and such that, if the initial positions and velocities belong to this set, the distances of the bodies from each other will remain perpetually bounded.*

Arnold gave the details of the proof of the Planetary Theorem the case of three bodies constrained on a plane: the “first” non trivial case. He was aware that, to extend the result to the general problem, some extra–difficulty related to the “rotation invariance” of the system (1) was to be overcome. Namely, the invariance by the two–parameter group of (non–commuting) transformations

$$(y^{(i)}, x^{(i)}) \rightarrow (\mathcal{R}y^{(i)}, \mathcal{R}x^{(i)}) , \quad \mathcal{R} \in \text{SO}(3) . \quad (5)$$

From a dynamical point of view, rotation invariance is caused by the conservation, along the H_{plt} –trajectories, of the three components, C_1 , C_2 and C_3 , of the “angular momentum”

$$C = \sum_{i=1}^n x^{(i)} \times y^{(i)} , \quad (6)$$

where “ \times ” denotes skew–product.

To prove his Planetary Theorem, Arnold proved an abstract theorem (that he called “The Fundamental Theorem”, see Appendix A; Theorem A.1) on the conservation of quasi–periodic motions precisely suited for properly–degenerate systems. For such systems, indeed, “standard” non–degeneracy assumptions as the ones appeared in [25], [30] or [2] are strongly violated. The non–degeneracy condition of the Fundamental Theorem is a “strong” non–linearity condition we shall refer to as “full torsion”. It requires (besides the non–degeneracy of the unperturbed part) the existence of the Birkhoff normal form and the invertibility of the matrix of the coefficients of the second–order term (second–order “Birkhoff invariants”): see conditions (ii) and (iii) in Theorem A.1.

In the case of the problem in the space, the integral (6) causes another strong degeneracy in the perturbation: one of the first order Birkhoff invariants associated to $(f_{\text{P}})_{\text{av}}$, $\Omega_{2n}(\Lambda)$, vanishes identically. This “resonance” is apparently a problem for the construction of the Birkhoff normal form. See, *e.g.*, [22]. It is worth to remark that, moreover, another resonance is to be taken in account, which, even though not mentioned in [5], was later pointed out by M. Robert Herman: the sum of the remaining first invariants $\Omega_1(\Lambda), \dots, \Omega_{2n-1}(\Lambda)$, vanishes identically (see [1] for a study on Herman resonance). Such two resonances,

$$\Omega_{2n}(\Lambda) \equiv 0 , \quad \sum_{i=1}^{2n-1} \Omega_i(\Lambda) \equiv 0 ,$$

are usually referred to, respectively, as “rotational”, “Herman” resonance or, jointly, “secular resonances”.

To overcome the problem of the secular resonances (or, at least, of the rotational one), Arnold proposed, in [5], a sketchy program of which he did not give the complete details. Such details revealed to be not trivial at all. The spatial three–body case later was proved in the PhD dissertation by P. Robutel [39] (see also [26]), on the basis of a rigorous development of the ideas in [5].

The first complete proof of Arnold’s Planetary Theorem in the general case appeared in [18], including efforts by M. Robert Herman. This important and beautiful result was reached with a different KAM technique, avoiding Birkhoff normal form: only the properties of the first order invariants are exploited in [18]. The underlying elegant, KAM Theory in [18] (for “smooth” systems) is different from the one in [5]; it goes back to [40] (analytic) and exploits non–degeneracy conditions previously studied in the 80’s by Arnold, Piartly, Parasjuk, Sprinzuk and others; see [18] and references therein, for more information. Moreover, the problem of secular resonances is solved in [18] via arguments of abstract reductions, [4].

The complete achievement of Arnold’s program for $n \geq 3$ was reached in the PhD thesis [37] (next, published in [13], [14], [16]).

Switching from $n = 2$ (spatial) to $n \geq 3$ (spatial) required new ideas. Indeed, as Arnold pointed out in [5, Chapter III, §5, 4–5], while in the spatial three–body case a classical tool for reducing the integral (6), the so–called “Jacobi reduction of the nodes” [24] was available and in fact used in [39], this tool was instead lacking for the general spatial problem with more than two planets. By this reason, Arnold suggested a qualitatively different strategy to handle this latter case. He conjectured [5, Chapter III, §5, 5] it were possible to reduce only two (out of three) non–commuting components of C (or functions of them) and, simultaneously, keep the (*regular*) structure of the Hamiltonian (2) in Poincaré variables. He believed this should let the system free of the vanishing eigenvalue. Note that this seems in contrast with the strategy [5, Chapter III, §5, 4] for three bodies (based on Jacobi reduction), which reduces *all* the integrals and the reduction is *singular* for co–planar motions. It turns out that such two apparently different programs are both realizable general and, besides, intimately related. Indeed, they have been realized in [37].

The starting point in [37] was the construction of a set of “action–angle” variables, i.e., taking values in $\mathbb{R}^{3n} \times \mathbb{T}^{3n}$ and denoted as $(\Lambda, \Gamma, \Psi, \lambda, \gamma, \psi)$, that should extend to the case of $n \geq 3$ planets Jacobi’s reduction of the nodes. Such variables actually already existed: they had been considered, in a slightly different form, in the 80’s by F. Boigey [8] for $n = 3$ and A. Deprit [17] for $n \geq 4$. Next, Boigey–Deprit variables were rediscovered in “planetary” form² by the author (who was strongly motivated by the present application to the Planetary Theorem) during her PhD, in the first months of 2008. Incidentally, the author would be grateful to anyone who let her know of applications of Boigey–Deprit variables to physical systems with $n \geq 3$ particles (the first not known case, after Jacobi), before [37].

Next, a new set of *regular* variables, named “Regular”, “Planetary” and “Symplectic” – RPS – and denoted with analogue symbols as Poincaré variables,

$$\begin{aligned} \Lambda &= (\Lambda_1, \dots, \Lambda_n), & \lambda &= (\lambda_1, \dots, \lambda_n) \\ z &= (\eta_1, \dots, \eta_n, \xi_1, \dots, \xi_n, p_1, \dots, p_n, q_1, \dots, q_n), \end{aligned} \quad (7)$$

having all the properties conjectured by Arnold for the many–body case was determined, in [37]. Such variables were not discussed by Boigey and Deprit. They were obtained by applying to the $(\Lambda, \Gamma, \Psi, \lambda, \gamma, \psi)$ ’s a regularization similar to Poincaré’s regularization of Delaunay variables. Though being qualitatively similar to Poincaré variables, at contrast with them, RPS variables are not “six per planet” (the coordinates the i^{th} planet are determined by the variables λ_i and $(\Lambda_j, \eta_j, \xi_j, p_j, q_j)$ with $i \leq j \leq n$, because of a certain hierarchical structure in their definition, actually inherited by the $(\Lambda, \Gamma, \Psi, \lambda, \gamma, \psi)$ ’s). Moreover, RPS variables are better fitted to rotation invariance of the problem, since they exhibit a cyclic couple (p_n, q_n) of conjugated variables (integrals of motion). The disappearing of this latter couple of variables from the Hamiltonian implies that the number of degrees of freedom is reduced of one unit (it is $(3n - 1)$, one over the minimum) *and*, moreover, the system is let free of two (out of three³) non–commuting integrals, as Arnold claimed.

²The rediscovered variables $(\Lambda, \Gamma, \Psi, \lambda, \gamma, \psi)$ are different from the ones in [8]–[17]. They correspond to be the “planetary version” of Deprit’s variables. They are defined only for negative unperturbed energies, so are less general, but turn to be better fitted to the planetary problem, since involve the elliptic elements of the planets. Also the proof of their symplectic character is different from [8]–[17]: for $n = 2$ the $(\Lambda, \Gamma, \Psi, \lambda, \gamma, \psi)$ were obtained in [37] constructively (via generating function, starting with Delaunay variables). This proof was however never published, after realizing the partial coincidence with the variables of [17]. For $n \geq 2$, the proof in [37] is by induction. Part of this proof was later published in [14] as a “new proof of Deprit variables”. In [14] the relation between the two sets is also clarified.

³For this reason, following [29], the reduction performed by the RPS variables is sometimes called “partial reduction”, at contrast with the “full reduction”, also discussed in [37], that reduces the system to the minimum number $(3n - 2)$, of degrees of freedom. Pay attention not to confuse, however, the *regular* “partial reduction” performed by RPS variables with the elementary (but *singular*) reduction that can be obtained reducing the integral C_3 in Poincaré variables. This latter one does not exhibit a cyclic couple and has nothing to do with the aforementioned Arnold’s claim in [5, Ch. 3, §5, 5].

In place of the ‘‘Poincaré Hamiltonian’’ (2), we consider the ‘‘RPS Hamiltonian’’

$$\mathcal{H}_{\text{rps}} = h_{\text{Kep}}(\Lambda) + \mu f_{\text{rps}}(\Lambda, \lambda, \bar{z}) \quad (8)$$

with

$$\bar{z} = (\eta, \xi, \bar{p}, \bar{q}) , \quad (9)$$

where $\eta = (\eta_1, \dots, \eta_n)$, $\bar{p} = (p_1, \dots, p_{n-1})$ and so on.

Fixing the value of (p_n, q_n) corresponds to fix one of the ∞^2 invariant manifolds that foliate the phase space; letting the other $2(3n - 1)$ vary gives a symplectic chart on any of such manifolds. On any of such invariant manifolds, the Birkhoff normal form has been proved to exist (with the properties described at the beginning of the paragraph, but with $(3n - 1)$ degrees of freedom, instead of $3n$). Moreover, this normal form satisfies the non-degeneracy condition required by the Fundamental Theorem and the direct proof of Arnold’s Planetary Theorem follows.

It has also been proved [15] that this construction is necessary. Namely that the unreduced system in Poincaré variables (2) would admit a Birkhoff normal form (we remark, despite of the secular resonances), but this normal form would be degenerate *at any order*: the lowest order of it corresponding to the rotational resonance. At the fourth order, the system would exhibit an identically vanishing torsion (given by the torsion of the partially reduced system, bordered with a row and a column of zeroes) and so on. In particular, no KAM theory might be *directly* applied to the unreduced system (2).

We refer to [11] for more information on this topic. Other reviews appeared in [19], [10].

1.2 This paper is concerned with a more detailed study of the normal form constructed in [37], [16]. Before describing it, we anticipate two applications.

a) A ‘‘uniform’’ theorem on quasi-periodic motions The former result of this paper is an improvement of the statements of the Planetary Theorem found in [39] and [37]–[16], in the case of the spatial three-body problem. In such papers, a positive measure set of quasi-periodic motions has been obtained, provided eccentricities and the mutual inclination among the planets are suitably small. Moreover, the *Kolmogorov set* (the union of quasi-periodic motions) *depends* strictly on eccentricities and the inclination, in the sense that its density tends to one as eccentricities and the inclination go to zero. In fact, the proofs in such papers are based on the application of the Fundamental Theorem (or improved formulations of it, [13]), where this assumption is essential: compare the first inequality in (86) and the measure of $\mathcal{K}_{\mu, \epsilon}$ below.

In the case of the *planar* three-body problem this assumption can be relaxed. In Arnold’s words: [5, p. 128] ‘‘*In the case of three bodies [on a plane] we can obtain stronger results (...). It turns out that it is not necessary to require the eccentricities to be small; all that is necessary is that they should be small enough to exclude the possibility of collision.*’’

And in fact, he stated (we refer to Appendix A for notations)

Theorem 1.2 (V. I. Arnold, [5, p. 128]) *In the case of the planar three-body problem, it is possible to find $\mu_* > 0$, $a_* > 0$ such that if*

$$|\mu| < \mu_* \quad (10)$$

an invariant set $\mathcal{K}_\mu \subset \mathcal{P}_{\epsilon_0}$, with

$$\text{meas } \mathcal{K}_\mu \geq (1 - \mu^{a_*}) \text{meas } \mathcal{P}_{\epsilon_0}$$

formed by the union of invariant four-dimensional tori, on which the motion is analytically conjugated to linear Diophantine quasi-periodic motions.

He then *conjectured* the same should hold also for the *spatial* problem:

Conjecture 1.1 (V. I. Arnold, [5, p. 129]) *An analogous [to Theorem 1.2] theorem is valid for the space three-body problem. In this case, one has to add to condition (10) a smallness condition for inclinations.*

In [5], Arnold gave some hints to prove Conjecture 1.1. In the 90's M. Robert Herman pointed out a serious gap in such indications. Since then, this stronger case of the Planetary Theorem remained unproved.

We shall prove the following

Theorem A *In the spatial three-body problem, there exist numbers α_* , μ_* , ϵ_* , $c_* < C_*$ and β_* such that, if the numbers α and μ (where μ is the masses ratio) verify*

$$0 < \mu < \mu_* , \quad 0 < \alpha < \alpha_* , \quad \mu < c_* \log(\alpha^{-1})^{-4\beta_*}$$

in the domain \mathcal{D}_α where semi-axes a_1, a_2 , eccentricities e_1, e_2 and mutual inclination ι verify

$$\mathcal{D}_\alpha : \quad a_- \leq a_1 < \alpha a_2 , \quad |(e_1, e_2, \iota)| < \epsilon_*$$

a set $\mathcal{K}_{\mu, \alpha} \subset \mathcal{D}_\alpha$ may be found, formed by the union of invariant 5-dimensional tori, on which the motion is analytically conjugated to linear Diophantine quasi-periodic motions. The set $\mathcal{K}_{\mu, \alpha}$ is of positive Liouville–Lebesgue measure and satisfies, uniformly in ϵ ,

$$\text{meas } \mathcal{K}_{\mu, \alpha} > \left(1 - C_*(\sqrt[4]{\mu}(\log \alpha^{-1})^{\beta_*} + \sqrt{\alpha})\right) \text{meas } \mathcal{D}_\alpha .$$

The same assertion holds for the planar $(1+n)$ -body problem.

Note that the thesis of Theorem A is a bit weaker than the one of Theorem 1.2, since, in Theorem A this density is not uniform with respect to the semi-major axes ratio.

b) A “full” Nehorošev stability theorem The latter result of the paper is concerned with the stability for the planetary system. To introduce it, we recall the following fundamental result by N. N. Nehorošev⁴, mainly motivated by its application to the Hamiltonian (2).

Theorem 1.3 (N. N. Nehorošev, 1977, [32], [33]) *Let*

$$H(I, \varphi, p, q) = H_0(I) + \mu P(I, \varphi, p, q) , \quad (I, \varphi, p, q) \in \mathcal{P} \subset \mathbb{R}^{n_1} \times \mathbb{T}^{n_1} \times \mathbb{R}^{2n_2}$$

be of the form of (2), real-analytic. Assume that $H_0(I)$ is “steep”. Then, one can find $a, b > 0$, C and μ_0 such that, if $\mu < \mu_0$, any trajectory $t \rightarrow \gamma(t) = (I(t), \varphi(t), p(t), q(t))$ solution of H such that

$$(p(t), q(t)) \in \Pi_{(p, q)} \mathcal{P} , \quad \forall 0 \leq t \leq T_0 := \frac{1}{C\mu} e^{\frac{1}{c\mu^a}} \quad (11)$$

verifies

$$|I(t) - I(0)| \leq r_0 := \frac{C}{2} \mu^b \quad \forall 0 \leq t \leq T_0 .$$

As for the definition of “steepness”, we refer to the papers [32], [33] and [31]. See also [35] for an equivalent definition. We aim to point out that, despite of the almost 150–pages length of the proof of Theorem 1.3 and the complication of notion of steepness, in [32] Nehorošev easily⁵ applied

⁴A more technical statement of Theorem 1.3 is given in Appendix D: Compare Theorem D.1. Recall that other improved statements of Theorem 1.3 have later been found in particular cases: see, for example, [38], [9] and references therein.

⁵The only delicate point in the application of Theorem 1.3 to \mathcal{H}_P consisted in checking assumption (11), that Nehorošev accomplished using the conservation of the third component C_3 of the total angular momentum (6) along the \mathcal{H}_P -trajectories. Note that, in the non-degenerate case, i.e., when the variables (p, q) do not appear, this assumption is void.

Theorem 1.3 to the planetary Hamiltonian \mathcal{H}_P in (2) (with $I = \Lambda$, the actions related to the semi-axes, and $(p, q) = z$ in (4), the secular variables related to eccentricities and inclinations), since the unperturbed term $H_0 = h_{\text{Kep}}$ is *concave*, a special case of steepness. Nehorošev then obtained a spectacular result of stability for the planetary semi-axes (hence, absence of collisions) over exponentially-long times *for all initial data* in phase space (see also [34] for a different approach and improved estimates). Up to now, Nehorošev’s result is the only rigorous, global (i.e., valid on the whole phase space, or, possibly, on a very large open subset of it) stability result for the planetary problem. Indeed, there do exist in literature results involving also strong numerical efforts for physical systems (see, e.g., [41], [20] and references therein) true on Cantor sets (in general, they are obtained via KAM techniques).

A physically relevant and widely studied open problem is related to the study of the stability of the whole system; i.e., the study of the secular variation of eccentricities and inclinations of the planets’ instantaneous orbits, besides the ones of semi-axes. See, for example, [27] and references therein. Partial rigorous results in this direction have been obtained in [15], where it has been proved that, if eccentricities and inclinations are initially suitably small, they remain confined with respect to their initial values over *polynomially* long times, up to exclude the so-called⁶ “mean-motion resonances”. More precisely, the following result has been proved.

Theorem 1.4 ([15]) *Whatever is the number of planets, for any arbitrarily fixed $s \in \mathbb{N}$, with $s \geq 5$, one can find positive numbers $C, \underline{a}_j, \bar{a}_j, \underline{\epsilon}, \bar{\epsilon}$ with $\underline{a}_j < \bar{a}_j < \underline{a}_{j+1}$ and $\underline{\epsilon} < \bar{\epsilon}$ such that for any $\kappa > 0$, in the domain where semi-major axes a_i , eccentricities e_i and mutual inclinations ι_j verify*

$$\hat{\mathcal{D}}_{s,\epsilon} : \quad \underline{a}_j \leq a_j \leq \bar{a}_j \quad \underline{\epsilon} < \max_{i,j} \{e_i, \iota_j\} < \epsilon < \bar{\epsilon}$$

under suitable relations between μ and ϵ , one can find an open set $\hat{\mathcal{D}}_{s,\mu,\epsilon}$ such that, for all the motions starting in $\hat{\mathcal{D}}_{s,\mu,\epsilon}$, the displacement of eccentricities and inclinations with respect to their initial values is bounded by $\kappa \underline{\epsilon}$, for all

$$|t| \leq \frac{C\kappa}{\mu \underline{\epsilon}^s}.$$

The proof of Theorem 1.4 again relies with the Birkhoff normal form of the system: the time of stability is related in fact to the remainder of this normal form. No analysis of resonance zones, trapping arguments... is used for its proof. An undesirable aspect of Theorem 1.4, is that the size of $\hat{\mathcal{D}}_{s,\mu,\epsilon}$ decreases with with the time of stability.

In this paper, we prove a stronger result, at least for the planar three-body problem.

Theorem B *In the planar three-body problem, there exist numbers $\bar{a}_-, \bar{\alpha}, \bar{\epsilon}, \bar{a}, \bar{b}, \bar{c}, \bar{d}$ such that, in the domain*

$$\bar{\mathcal{D}}_\epsilon : \quad \bar{a}_- \leq a_1 < \bar{\alpha} a_2, \quad \underline{\epsilon} < |(e_1, e_2)| < \epsilon < \bar{\epsilon}$$

under suitable relations between μ and ϵ , one can find an open set $\bar{\mathcal{D}}_{\mu,\epsilon} \subset \bar{\mathcal{D}}_\epsilon$, defined by absence of mean-motion resonances up to a suitable order, such that, for all the motions with initial datum in $\bar{\mathcal{D}}_{\mu,\epsilon}$, one has

$$|a_i(t) - a_i(0)|, |e_i(t) - e_i(0)| \leq \bar{r} := \max\{\delta^{\bar{b}}, \mu^{1/12}, \epsilon\} \quad \forall 0 \leq t \leq \bar{T} = \frac{e^{\frac{1}{\delta}}}{\delta}$$

where $\bar{\delta} := \frac{\mu^{\bar{a}} \epsilon}{\bar{c}}$.

1.3 Let us sketch the proofs of Theorems A and B and make some comment.

⁶I. e., resonances of the Keplerian frequencies $\omega_{\text{Kep}} := \partial h_{\text{Kep}}$.

The proof of Theorem A is a remake of an idea by V. I. Arnold in [5]. His proof of Theorem 1.2 relies on the observation that the planar three-body system⁷ $\mathcal{H}_{\text{pl3b}} = h_{\text{Kep}}(\Lambda) + \mu f_{\text{pl3b}}(\Lambda, \lambda, \eta, \xi)$ enjoys the strong property that secular perturbation $(f_{\text{pl3b}})_{\text{av}}$ is *integrable*. It has two degrees of freedom (related to the secular variables (η_1, ξ_1) and (η_2, ξ_2)) and two commuting integrals: the third component of the angular momentum (6) (the only one non to vanish, since the problem is planar) and itself. Then the Birkhoff series of $(f_{\text{pl3b}})_{\text{av}}$ converges and Arnold can use a KAM theory (recalled in Appendix A, Theorem A.2) that is less general than the Fundamental Theorem but better fitted to this case. In the proof of Theorem A we use a similar idea. Let us denote as $f_{3\text{b}}$ the function f_{rps} for the three-body case; $(f_{3\text{b}})_{\text{av}}$, its the averaged value. We shall see below that a suitable approximation $(f_{3\text{b}})_{\text{av}}^{(2)}$ defined in Eq. (14) below, is *integrable*. This fact has been already used, in different settings, in [28], [44] and [36]. Moreover, the same property of integrability is proved to hold for the *planar* many-body problem; see below for more details on this assertion. Then, we apply Arnold’s argument, but working on $(f_{3\text{b}})_{\text{av}}^{(2)}$, $(f_{\text{pl}})_{\text{av}}^{(2)}$, respectively, simply suitably modifying Theorem A.2: see Theorem 3.1.

The proof of Theorem B is an application of the Nehorošev’s Theorem in the non-degenerate case. Essentially, it relies on checking “steepness” of some integrable truncation of the “Birkhoff-normalized” system

$$H_0 := h_{\text{Kep}} + \mu(f_{\text{bnf}})_{\text{av}}$$

in all of its degrees of freedom. Here the difficulty is that, at contrast with the application in [32] (where only the concavity of h_{Kep} is exploited), the “full torsion” of the system, given by the Hessian of h_{Kep} and the matrix β of the second-order Birkhoff invariants, is *not convex*, nor quasi-convex. Its eigenvalues are alternating in sign. Therefore, it is necessary to consider higher orders of Birkhoff normal form and apply more refined conditions for steepness. It is not clear (and actually an open question) what is the right order of the Birkhoff series to be involved for general n and, especially, how steepness can be checked for systems with many degrees of freedom (see [42] for progresses in this direction). For three-degrees of freedom systems Nehorošev proved that the “three-jet condition” (recalled in Appendix D) is “generic”. But the *planar* three-body problem, after reducing completely rotations, has three degrees of freedom, so it is not surprising that this problem satisfies three-jet. We do this check in §4.4.

Before passing to describe technical aspects, we provide a few comments.

- Theorem B is stated for the planar three-body problem. As previously outlined, the secular problem associated to it is *integrable*: its Birkhoff normal form converges. And in fact this circumstance allowed Arnold to obtain refined results for this case (see §1.2): the independence of the *Kolmogorov set* on the eccentricities. One might ask if such independence holds also in the statement of Theorem B. I. e., if the set $\bar{\mathcal{D}}_{\mu, \epsilon}$ may be chosen to be independent of ϵ . However, with our proof we are not able⁸ to refine the result in that direction. The reason is technical: instead of the (integrable) secular system $\mathcal{H}_{\text{pl3b}} := h_{\text{Kep}} + \mu(f_{\text{pl3b}})_{\text{av}}$ that would be more natural, during the proof we consider a *non integrable* system close⁹ to it, by performing not only one but *many* steps of averaging with respect to fast (mean motion) frequencies. Therefore, we need to *truncate* the Birkhoff series associated to this closely to integrable system and this is the reason we have the dependence of ϵ . In turn, the exigency of many¹⁰ steps comes succeeding in applying the theory developed in [32].
- In §4 we do more than we need for Theorem B. We compute the Birkhoff normal form of

⁷In “planar” Poincaré variables $(\Lambda_i, \lambda_i, \eta_i, \xi_i)$, $i = 1, 2$.

⁸The dependence of $\bar{\mathcal{D}}_{\mu, \epsilon}$ on ϵ may be read in inequality just before (66) and by the formula (66), that define this set.

⁹Compare the system $h_{\text{Kep}} + \mu(\hat{N} + \hat{N}_*)$ in (76).

¹⁰Compare Lemma 4.1.

the *spatial* three-body problem, which is¹¹, which is

$$\begin{aligned}
(f_{\text{bnf}})_{\text{av}} &= -\frac{\bar{m}_1\bar{m}_2}{a_2} - \bar{m}_1\bar{m}_2\frac{a_1^2}{4a_2^3}\left(\left(1+3\frac{t_1}{\Lambda_1}+3\frac{t_2}{\Lambda_2}-3\left(\frac{1}{\Lambda_1}+\frac{1}{\Lambda_2}\right)t_3\right)\right. \\
&\quad - \bar{m}_1\bar{m}_2\frac{a_1^2}{4a_2^3}\left(-\frac{3}{2}\frac{t_1^2}{\Lambda_1^2}+6\frac{t_2^2}{\Lambda_2^2}+\frac{3}{2}\frac{t_3^2}{\Lambda_1^2}+9\frac{t_1t_2}{\Lambda_1\Lambda_2}-12\frac{t_1t_3}{\Lambda_1^2}-9\frac{t_2t_3}{\Lambda_1\Lambda_2}\right. \\
&\quad + 10\frac{t_2^3}{\Lambda_2^3}-\frac{3}{2}\frac{t_3^3}{\Lambda_1^2\Lambda_2}-\frac{9}{2}\frac{t_1^2t_2}{\Lambda_1^2\Lambda_2}-\frac{105}{4}\frac{t_1^2t_3}{\Lambda_1^3}-18\frac{t_2^2t_3}{\Lambda_1\Lambda_2^2}+18\frac{t_1t_2^2}{\Lambda_1\Lambda_2^2} \\
&\quad \left.\left.+\frac{105}{4}\frac{t_1t_3^2}{\Lambda_1^3}+\frac{9}{2}\frac{t_2t_3^2}{\Lambda_1^2\Lambda_2}-36\frac{t_1t_2t_3}{\Lambda_1^2\Lambda_2}\right)(1+O\left(\frac{\Lambda_1}{\Lambda_2}\right))+\frac{a_1^2}{a_2^3}O(|t|^{7/2})+O\left(\frac{a_1^3}{a_2^4}\right)\right) \quad (12)
\end{aligned}$$

and then we reduce to the planar case setting $t_3 = 0$. However, we are not able to extend Theorem B to the spatial case, since we are not able to check steepness for this case. The three-jet condition might fail at least on manifolds of co-dimension one: see Remark 4.1.

- Besides the previous case, a possible extension of Theorem B to the general *planar* problem might be helped by the fact that, for this case we know a good approximation of $(f_{\text{bnf}})_{\text{av}}$, at any order. This result is a corollary of the analysis of §2. See also §1.4 below.
- In our strategy of proofs, the planetary Birkhoff normal form (hence, the system (8) in RPS variables) plays a central rôle. The author is not aware (and would be interesting to know) what kind of results could be obtained (and what would be the relative difficulty) via Herman–Féjóz’s normal form [18].

1.4 The main novelty of this paper (with respect to our previous ones on this subject) is a technical lemma of geometrical nature that helps in the analysis of the secular perturbing function of the system (8). This reflects on the computation of the Birkhoff invariants at higher orders.

Let us remark, at this respect that, in general, computing the Birkhoff invariants of the planetary problem is a huge work. See, for example the computations of the torsion in [5] ($n = 2$, planar), [39] ($n = 2$, spatial), [21] ($n \geq 2$, planar), [37]–[16] ($n \geq 2$, spatial). So, our main progress relies on an improvement of the technique of computation of such invariants, which is particularly desirable if one wants to extend Theorem B to the general problem.

Let us introduce it briefly, referring to the following section for details.

Consider the system (8) and, in particular, its secular perturbing function $(f_{\text{rps}})_{\text{av}}$. Since the indirect¹² part has zero λ -average, $(f_{\text{rps}})_{\text{av}}$ is given by

$$(f_{\text{rps}})_{\text{av}} = - \sum_{1 \leq i < j \leq n} \frac{\bar{m}_i\bar{m}_j}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|x^{(i)}(\Lambda, \lambda_i, \bar{z}) - x^{(j)}(\Lambda, \lambda_j, \bar{z})|}.$$

De-homogeneizing with respect to a_j , we expand each of the terms

$$(f_{\text{rps}}^{(ij)})_{\text{av}} := -\frac{\bar{m}_i\bar{m}_j}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|x^{(i)}(\Lambda, \lambda_i, \bar{z}) - x^{(j)}(\Lambda, \lambda_j, \bar{z})|}.$$

in powers of the ratio $\frac{a_i}{a_j}$, with a_j fixed:

$$(f_{\text{rps}}^{(ij)})_{\text{av}} = (f_{\text{rps}}^{(ij)})_{\text{av}}^{(0)} + (f_{\text{rps}}^{(ij)})_{\text{av}}^{(1)} + (f_{\text{rps}}^{(ij)})_{\text{av}}^{(2)} + \dots \quad (13)$$

¹¹In particular, truncating this formula to the fourth order we recover the formulae found in [37]–[16].

¹²The former term in (3) is often referred to as “indirect part”; the latter as “direct part”. As far as the author knows, this terminology has been introduced by the French school. The vanishing of the average of the indirect part, known Poincaré variables, holds also in RPS variables.

Clearly, to this expansion there corresponds an analogue expansion of

$$(f_{\text{rps}})_{\text{av}} = (f_{\text{rps}})_{\text{av}}^{(0)} + (f_{\text{rps}})^{(1)} + (f_{\text{rps}})_{\text{av}}^{(2)} + \dots \quad (14)$$

Analogously to what happens for the Poincaré Hamiltonian (2), one has that, in these expansions, the zeroth order terms $(f_{\text{rps}}^{(ij)})_{\text{av}}^{(0)}$ are independent¹³ of \bar{z} by well known properties of the two-body potential and that the linear terms $(f_{\text{rps}}^{(ij)})_{\text{av}}^{(1)}$ vanish by Fubini's and Newton equation¹⁴. The lowest order information on $(f_{\text{rps}})_{\text{av}}$ is then given by the second-order terms $(f_{\text{rps}})_{\text{av}}^{(2)}$.

By [37]–[16] $(f_{\text{rps}})_{\text{av}}^{(2)}$ may be splitted into a sum

$$(f_{\text{rps}})_{\text{av}}^{(2)} = (f_{\text{pl}})_{\text{av}}^{(2)} + (f_{\text{vert}})_{\text{av}}^{(2)} \quad (15)$$

of a “planar” and¹⁵ a “vertical” part, where $(f_{\text{pl}})_{\text{av}}^{(2)}$ corresponds to the term that we would have for the problem in the plane, while $(f_{\text{vert}})_{\text{av}}^{(2)}$ vanishes for $(\bar{p}, \bar{q}) = 0$ and is even in (\bar{p}, \bar{q}) . In §2 we prove that $(f_{\text{pl}})_{\text{av}}^{(2)}$, $(f_{\text{vert}})_{\text{av}}^{(2)}$ are given by, respectively,

$$\begin{aligned} (f_{\text{pl}})_{\text{av}}^{(2)} &= -\frac{1}{4} \sum_{1 \leq i < j \leq n} \bar{m}_i \bar{m}_j \frac{a_i^2}{a_j^3} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\zeta}{1 - e_j \cos \zeta} \left(1 + \frac{3}{2} e_i^2\right) \\ (f_{\text{vert}})_{\text{av}}^{(2)} &= +\frac{3}{4} \sum_{1 \leq i < j \leq n} \bar{m}_i \bar{m}_j \frac{a_i^2}{a_j^3} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\zeta}{1 - e_j \cos \zeta} \frac{1}{2\pi} \int_{\mathbb{T}} (\hat{x}^{(i)} \cdot \hat{C}^{(j)})^2 d\lambda_i, \end{aligned} \quad (16)$$

where e_i 's are the eccentricities, expressed in terms of Λ_i and $\frac{\eta_i^2 + \xi_i^2}{2}$; $\hat{C}^{(j)}$ are the planets' normalized angular momenta $\frac{\mathbf{C}^{(j)}}{|\mathbf{C}^{(j)}|}$ and $\hat{x}^{(i)} := \frac{x^{(i)}(\Lambda, \lambda_i, z)}{a_i}$.

The author is not aware if the formulae (16) had been already noticed before (they hold also in the case of the Poincaré system (2)). Such formulae are the thesis of Proposition 2.1, that we prove using a new set of symplectic variables, defined in (26), and tools of normal form theory. The variables (26) in a sense resemble the well known Adoyer–Deprit variables of the rigid body, with the difference that have six degrees of freedom instead of three. Also the thesis of Proposition 2.1 resembles certain formulae for the rigid body, as outlined in Remark 2.1.

In particular, inspecting (16), it is to be remarked that $(f_{\text{pl}})_{\text{av}}^{(2)}$ not only is *integrable*, but is in *Birkhoff normal form*. This fact implies the validity of Theorem A for the planar general problem and, especially, is of great help in the computation of its Birkhoff invariants *at any order*.

Secondly, formulae (16) imply that, in the three-body case ($n = 2$), $(f_{3\text{b}})_{\text{av}}^{(2)} := (f_{\text{rps}})_{\text{av}}^{(2)}|_{n=2}$ is independent of the argument of (η_2, ξ_2) , therefore, it is *integrable* (compare [28] for an analogue assertion in a different setting and [44] and [36] for applications). More in general, for $n \geq 2$, $(f_{\text{rps}})_{\text{av}}^{(2)}$ is independent on the argument of (η_n, ξ_n) . But while, for this general case, the expression of $(f_{\text{vert}})_{\text{av}}^{(2)}$ in terms of RPS variables is complicated, due to the factors $(\hat{x}^{(i)} \cdot \hat{C}^{(j)})^2$, it is not so for three bodies, where there is only one of such factors ($i = 1, j = 2$). The aspect of the corresponding vertical term is nice

$$(f_{3\text{bvert}})_{\text{av}}^{(2)} = \frac{3}{4} \bar{m}_1 \bar{m}_2 \frac{a_1^2}{a_2^3} \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\zeta}{1 - e_2 \cos \zeta} \left(\left(1 + \frac{3}{2} e_1^2\right) (ivv^*) + \frac{5}{2} ((u_1^*)^2 v^2 + (v^*)^2 u_1^2) \bar{e}_1^2 \right) \bar{\mathbf{s}}^2 \quad (17)$$

¹³They are given by given by $-\frac{\bar{m}_i \bar{m}_i}{a_j}$.

¹⁴I. e., by the vanishing of

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{x^{(j)}(\Lambda, \lambda_j, \bar{z})}{|x^{(j)}(\Lambda, \lambda_j, \bar{z})|^3} d\lambda_j = \frac{1}{T_j} \int_0^{T_j} \frac{d}{dt} y^{(j)}(\Lambda, \omega_j t, \bar{z}) dt$$

with some T_j and $\omega_j = \frac{2\pi}{T_j}$.

¹⁵We follow the terminology in [18].

where u_i, u_i^* are the Birkhoff variables associated to (η_i, ξ_i) ; (v, v^*) to (p_1, q_1) , \bar{e}_1 and \bar{s} are suitable functions in normal form. Since the first non-normal terms in this formula appear from the fourth order on, the computation of the sixth orders Birkhoff invariants for the three-body case is quickly done: it takes less than two pages (see §4.1) and gives (12).

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2 An asymptotic formula for the secular perturbation

Let, for fixed $1 \leq i < j \leq n$,

$$f_{ij}(\Lambda, \bar{z}) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|x^{(i)}(\Lambda, \lambda_i, \bar{z}) - x^{(j)}(\Lambda, \lambda_j, \bar{z})|}$$

so as to write

$$(f_{\text{rps}})_{\text{av}}(\Lambda, \bar{z}) = - \sum_{1 \leq i < j \leq n} \bar{m}_i \bar{m}_j f_{ij}(\Lambda, \bar{z}).$$

Here¹⁶ $(\Lambda, \lambda_i, \bar{z}) \rightarrow x^{(i)}(\Lambda, \lambda_i, \bar{z})$ denotes the $x^{(i)}$ -projection of the map

$$\phi_{\text{rps}}^{-1} : (\Lambda, \lambda, z) \rightarrow (y, x) \in \mathbb{R}^{3n} \times \mathbb{R}^{3n} \quad (18)$$

Consider the formal expansions

$$f_{ij} = f_{ij}^{(0)} + f_{ij}^{(2)} + \dots \quad (19)$$

in powers of the semi-major axes ratio $\alpha_{ij} := a_i/a_j$, with a_j fixed. Here,

$$f_{ij}^{(k)} := \frac{1}{k!} \frac{d^k}{d\varepsilon^k} \left[\frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \frac{d\lambda_i d\lambda_j}{|\varepsilon x^{(i)}(\Lambda, \lambda_i, \bar{z}) - x^{(j)}(\Lambda, \lambda_j, \bar{z})|} \right]_{\varepsilon=0}.$$

¹⁶Actually, the map (18) depends on z , rather than \bar{z} . However, by the independence of the Hamiltonian (8) of (p_n, q_n) , we may arbitrarily fix such couple of variables to some value, *e.g.*, $(0, 0)$. Abusively, *just in* (20) and similar formulae below, we denote again as $(\Lambda, \lambda, \bar{z}) \rightarrow (y(\Lambda, \lambda, \bar{z}), x(\Lambda, \lambda, \bar{z}))$ the map $\phi_{\text{rps}}^{-1}|_{(p_n, q_n)=(0,0)}$.

In particular, we focus on the second-order term of this expansion, given by

$$f_{ij}^{(2)} = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} d\lambda_i d\lambda_j \frac{3(x^{(i)}(\Lambda, \lambda_i, \bar{z}) \cdot x^{(j)}(\Lambda, \lambda_j, \bar{z}))^2 - |x^{(i)}(\Lambda, \lambda_i, \bar{z})|^2 |x^{(j)}(\Lambda, \lambda_j, \bar{z})|^2}{2|x^{(j)}(\Lambda, \lambda_j, \bar{z})|^5}. \quad (20)$$

Note that $(f_{\text{rps}})_{\text{av}}^{(2)}$ in (14) corresponds to

$$(f_{\text{rps}})_{\text{av}}^{(2)} = - \sum_{1 \leq i < j \leq n} \bar{m}_i \bar{m}_j f_{ij}^{(2)}. \quad (21)$$

Let $C^{(i)}(\Lambda, \bar{z}) := x^{(i)}(\Lambda, \lambda_i, \bar{z}) \times y^{(i)}(\Lambda, \lambda_i, \bar{z})$ (by definition of the map (18), $C^{(i)}(\Lambda, \bar{z})$ is independent of λ_i). We have the following identity

Proposition 2.1

$$f_{ij}^{(2)} = - \frac{M_j m_j^2}{4} \frac{\frac{1}{2\pi} \int_{\mathbb{T}} (3(C^{(j)} \cdot x^{(i)})^2 - |x^{(i)}|^2 |C^{(j)}|^2) d\lambda_i}{|C^{(j)}|^4} \left(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\lambda_j}{|x^{(j)}|^2} \right) \quad (22)$$

Note that Eqs. (21), (22) and the formulae of $|C^{(i)}|$, $|x^{(j)}|$ in terms of RPS variables (see [37], [16] and eventually Appendix B) imply (15)–(16).

We first discuss

2.1 The three-body case

Let

$$P^{(2)} := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} d\lambda_1 d\lambda_2 \frac{3(x^{(1)}(\Lambda, \lambda_1, z) \cdot x^{(2)}(\Lambda, \lambda_2, z))^2 - |x^{(1)}(\Lambda, \lambda_1, z)|^2 |x^{(2)}(\Lambda, \lambda_2, z)|^2}{2|x^{(2)}(\Lambda, \lambda_2, z)|^5}.$$

where, for $i = 1, 2$,

$$(\Lambda_1, \Lambda_2, \lambda_i, z) \in \mathcal{A}^2 \times \mathbb{T}^1 \times B^8 \rightarrow (y^{(i)}(\Lambda_1, \Lambda_2, \lambda_i, z), x^{(i)}(\Lambda_1, \Lambda_2, \lambda_i, z)) \in \mathbb{R}^3 \times \mathbb{R}^3$$

are two mappings such that

(A) The map $(\Lambda_1, \Lambda_2, \lambda_2, z) \rightarrow (y^{(2)}(\Lambda_1, \Lambda_2, \lambda_2, z), x^{(2)}(\Lambda_1, \Lambda_2, \lambda_2, z))$ solves the two-body problem ODE

$$\partial_t y^{(2)}(\Lambda_2, \omega_{\text{Kep}}^{(2)} t, z) = -m_2 M_2 \frac{x^{(2)}(\Lambda_2, \omega_{\text{Kep}}^{(2)} t, z)}{|x^{(2)}(\Lambda_2, \omega_{\text{Kep}}^{(2)} t, z)|^3} \quad (23)$$

where $\omega_{\text{Kep}}^{(2)} = \omega_{\text{Kep}}^{(2)}(\Lambda_2) = \frac{M_2^2 m_3^3}{\Lambda_2^3}$;

(B) The map

$$\bar{\phi}: (\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, z) \rightarrow (y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)}) \quad (24)$$

is symplectomorphism of $\mathcal{A}^2 \times \mathbb{T}^2 \times B^8$ into \mathbb{R}^{12} (where $\mathcal{A}^2 \subset \mathbb{R}^2$, $B^8 \subset \mathbb{R}^8$ are open and connected).

Proposition 2.2 *Under assumptions (A) and (B), the following identity holds*

$$P^{(2)} = -\frac{M_2 m_2^2}{4} \frac{\frac{1}{2\pi} \int_{\mathbb{T}} (3(C^{(2)} \cdot x^{(1)})^2 - |x^{(1)}|^2 |C^{(2)}|^2) d\lambda_1}{|C^{(2)}|^4} \left(\frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\lambda_2}{|x^{(2)}|^2} \right) \quad (25)$$

where $C^{(2)}(\Lambda_1, \Lambda_2, z) := x^{(2)}(\Lambda_1, \Lambda_2, \lambda_2, z) \times y^{(2)}(\Lambda_1, \Lambda_2, \lambda_2, z)$.

Remark 2.1

- Note that, in the case $n = 2$, the map (18) satisfies assumptions (A) and (B), hence Proposition 2.2 is just Proposition 2.1 in this particular case.
- We shall prove more than (25): letting $P^{(1)}(\Lambda, \lambda_1, z)$ as in (34) below, then $P^{(1)}$ satisfies an analogue identity as in (25), but neglecting the first average $\frac{1}{2\pi} \int_{\mathbb{T}} d\lambda_1$.
- The formula (25) resembles the expression of the averaged quartic term in the spin–orbit problem, using Andoyer–Deprit coordinates: see [6, Eq. (24)], in turn based on the expansions in [12, §12].

In the next sections, we prove Proposition 2.2. Next (in §2.5), we discuss the general case.

2.2 A six–degrees of freedom set of symplectic variables

The proof of Proposition 2.2 is based on the use of a “ad hoc” variables for the three–body problem. Let us introduce them.

Let $(k^{(1)}, k^{(2)}, k^{(3)})$ be a prefixed orthonormal frame in \mathbb{R}^3 and let

$$(y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)}) \in (\mathbb{R}^3)^4, \quad (y^{(i)}, x^{(i)}) = (y_1^{(i)}, y_2^{(i)}, y_3^{(i)}, x_1^{(i)}, x_2^{(i)}, x_3^{(i)})$$

be a system of “Cartesian coordinates” in the configuration space \mathbb{R}^3 , with respect to $(k^{(1)}, k^{(2)}, k^{(3)})$.

Denote as

$$C^{(i)} := x^{(i)} \times y^{(i)}$$

(with “ \times ” denoting skew product) the i^{th} angular momentum, and let $C := C^{(1)} + C^{(2)}$ the total angular momentum. For $u, v \in \mathbb{R}^3$ lying in the plane orthogonal to a vector w , let $\alpha_w(u, v)$ denote the positively oriented angle (mod 2π) between u and v (orientation follows the “right hand rule”). Define the “nodes”

$$\nu_1 := k^{(3)} \times C, \quad \nu_2 := C \times x^{(1)}, \quad \nu_3 := x^{(1)} \times C^{(2)}.$$

Let \mathcal{P}_\star^{12} denote the subset of $(\mathbb{R}^3)^4$ where $C, C_2, x^{(1)}, x^{(2)}, \nu_1, \nu_2$ and ν_3 simultaneously do not vanish. On \mathcal{P}_\star^{12} define a map

$$\phi^{-1} : (y^{(1)}, y^{(2)}, x^{(1)}, x^{(2)}) \rightarrow (C_3, G, R_1, \Theta, R_2, \Phi_2, \zeta, \mathfrak{g}, r_1, \vartheta, r_2, \varphi_2)$$

via the following formulae

$$\phi^{-1} : \begin{cases} C_3 := C \cdot k^{(3)} \\ G := |C| \\ R_1 := \frac{y^{(1)} \cdot x^{(1)}}{|x^{(1)}|} \\ \Theta := \frac{C^{(2)} \cdot x^{(1)}}{|x^{(1)}|} \\ R_2 := \frac{y^{(2)} \cdot x^{(2)}}{|x^{(2)}|} \\ \Phi_2 := |C^{(2)}| \end{cases} \quad \begin{cases} \zeta := \alpha_{k^{(3)}}(k^{(1)}, \nu_1) \\ \mathfrak{g} := \alpha_C(\nu_1, \nu_2) \\ r_1 := |x^{(1)}| \\ \vartheta := \alpha_{x^{(1)}}(\nu_2, \nu_3) \\ r_2 := |x^{(2)}| \\ \varphi_2 := \alpha_{C_2}(\nu_3, x^{(2)}) \end{cases} \quad (26)$$

Proposition 2.3 *The map ϕ^{-1} in (26) is invertible on \mathcal{P}_*^{12} and preserves the standard Liouville 1-form $\lambda = \sum_{i=1}^6 P_i dQ_i$.*

We denote as

$$\mathcal{R}_1(i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix}, \quad \mathcal{R}_3(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The invertibility is proven by exhibiting the inverse ϕ . Indeed, the definitions in (26) and elementary geometric considerations easily imply the following

Lemma 2.1 *On $\phi^{-1}(\mathcal{P}_*^{12})$, the inverse map of ϕ^{-1} in in (26), has the following analytical expression:*

$$\phi : \begin{cases} x^{(1)} = \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(\mathfrak{g})\mathcal{R}_1(i_1) \begin{pmatrix} 0 \\ 0 \\ r_1 \end{pmatrix} \\ y^{(1)} := \frac{R_1}{r_1}x^{(1)} + \frac{1}{r_1^2}C^{(1)} \times x^{(1)} \\ x^{(2)} = \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(\mathfrak{g})\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2) \begin{pmatrix} r_2 \cos \varphi_2 \\ r_2 \sin \varphi_2 \\ 0 \end{pmatrix} \\ y^{(2)} = \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(\mathfrak{g})\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2) \begin{pmatrix} R_2 \cos \varphi_2 - \frac{\Phi_2}{r_2} \sin \varphi_2 \\ R_2 \sin \varphi_2 + \frac{\Phi_2}{r_2} \cos \varphi_2 \\ 0 \end{pmatrix} \end{cases} \quad (27)$$

where, if $i, i_1, i_2 \in (0, \pi)$ are defined by

$$\cos i = \frac{C_3}{G}, \quad \cos i_1 = \frac{\Theta}{G}, \quad \cos i_2 = \frac{\Theta}{\Phi_2} \quad (28)$$

and $C, C^{(2)}$ by

$$\begin{aligned} C &:= \mathcal{R}_3(\zeta)\mathcal{R}_1(i) \begin{pmatrix} 0 \\ 0 \\ G \end{pmatrix} \\ C^{(2)} &:= \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(\mathfrak{g})\mathcal{R}_1(i_1)\mathcal{R}_3(\vartheta)\mathcal{R}_1(i_2) \begin{pmatrix} 0 \\ 0 \\ \Phi_2 \end{pmatrix} \end{aligned} \quad (29)$$

then

$$C^{(1)} := C - C^{(2)}. \quad (30)$$

To prove symplecticity we shall use the following easy

Lemma 2.2 ([14]) *Let*

$$x = \mathcal{R}_3(\theta)\mathcal{R}_1(i)\bar{x}, \quad y = \mathcal{R}_3(\theta)\mathcal{R}_1(i)\bar{y}, \quad C := x \times y, \quad \bar{C} := \bar{x} \times \bar{y},$$

with $x, \bar{x}, y, \bar{y} \in \mathbb{R}^3$. Then,

$$y \cdot dx = C \cdot k^{(3)}d\theta + \bar{C} \cdot k^{(1)}di + \bar{y} \cdot d\bar{x}.$$

Proof of Proposition 2.3. Let us preliminarily verify that, if $C^{(i)}$ are as in (29)–(30), and $y^{(i)}$, $x^{(i)}$ as in (27), then as expected,

$$x^{(i)} \times y^{(i)} = C^{(i)} . \quad (31)$$

Indeed, for $i = 2$, this identity follows trivially from the definitions. To check that it holds also for $i = 1$, one can do as follows: firstly, to check that $x^{(1)} \cdot C^{(1)} = 0$. This is an elementary consequence of (27) and, in particular, of (28). Next, using the rule of the double skew product, one has

$$\begin{aligned} x^{(1)} \times y^{(1)} &= x^{(1)} \times \left(\frac{R_1}{r_1} x^{(1)} + \frac{1}{r_1^2} C^{(1)} \times x^{(1)} \right) \\ &= 0 + \frac{1}{r_1^2} (r_1^2 C^{(1)} - (x^{(1)} \cdot C^{(1)}) x^{(1)}) = C^{(1)} . \end{aligned}$$

Define now

$$\begin{aligned} \bar{C}^{(1)} &:= \mathcal{R}_1(-i) \mathcal{R}_3(-\zeta) C^{(1)} \\ \bar{C}^{(2)} &:= \mathcal{R}_3(\mathfrak{g}) \mathcal{R}_1(i_1) \mathcal{R}_3(\vartheta) \mathcal{R}_1(i_2) \begin{pmatrix} 0 \\ 0 \\ \Phi_2 \end{pmatrix} \\ \bar{\bar{C}}^{(1)} &:= \mathcal{R}_1(-i_1) \mathcal{R}_3(-\mathfrak{g}) \begin{pmatrix} 0 \\ 0 \\ G \end{pmatrix} - \mathcal{R}_3(\vartheta) \mathcal{R}_1(i_2) \begin{pmatrix} 0 \\ 0 \\ \Phi_2 \end{pmatrix} \\ \bar{\bar{C}}^{(2)} &:= \mathcal{R}_3(\vartheta) \mathcal{R}_1(i_2) \begin{pmatrix} 0 \\ 0 \\ \Phi_2 \end{pmatrix} \\ \bar{\bar{\bar{C}}}^{(2)} &:= \begin{pmatrix} 0 \\ 0 \\ \Phi_2 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \bar{y}^{(1)} &:= \begin{pmatrix} 0 \\ 0 \\ R_1 \end{pmatrix} + \frac{1}{r_1^2} \bar{C}^{(1)} \times \begin{pmatrix} 0 \\ 0 \\ r_1 \end{pmatrix}, \quad \bar{x}^{(1)} := \begin{pmatrix} 0 \\ 0 \\ r_1 \end{pmatrix} \\ \bar{\bar{x}}^{(2)} &:= \begin{pmatrix} r_2 \cos \varphi_2 \\ r_2 \sin \varphi_2 \\ 0 \end{pmatrix}, \quad \bar{\bar{y}}^{(2)} := \begin{pmatrix} R_2 \cos \varphi_2 - \frac{\Phi_2}{r_2} \sin \varphi_2 \\ R_2 \sin \varphi_2 + \frac{\Phi_2}{r_2} \cos \varphi_2 \\ 0 \end{pmatrix} \end{aligned}$$

so as to write

$$\begin{aligned} y^{(1)} &= \mathcal{R}_3(\zeta) \mathcal{R}_1(i) \mathcal{R}_3(\mathfrak{g}) \mathcal{R}_1(i_1) \bar{y}^{(1)} \\ x^{(1)} &= \mathcal{R}_3(\zeta) \mathcal{R}_1(i) \mathcal{R}_3(\mathfrak{g}) \mathcal{R}_1(i_1) \bar{x}^{(1)} . \\ x^{(2)} &= \mathcal{R}_3(\zeta) \mathcal{R}_1(i) \mathcal{R}_3(\mathfrak{g}) \mathcal{R}_1(i_1) \mathcal{R}_3(\vartheta) \mathcal{R}_1(i_2) \bar{\bar{x}}^{(2)} \\ y^{(2)} &= \mathcal{R}_3(\zeta) \mathcal{R}_1(i) \mathcal{R}_3(\mathfrak{g}) \mathcal{R}_1(i_1) \mathcal{R}_3(\vartheta) \mathcal{R}_1(i_2) \bar{\bar{y}}^{(2)} \end{aligned}$$

Applying repeatedly Lemma 2.2, Eq. (31) and the rule

$$\mathcal{R}x \times \mathcal{R}y = \mathcal{R}(x \times y) \quad \text{for all } \mathcal{R} \in \text{SO}(3), x, y \in \mathbb{R}^3$$

gives

$$\begin{aligned}
y^{(1)} \cdot dx^{(1)} &= C^{(1)} \cdot k^{(3)} d\zeta + \bar{C}^{(1)} \cdot k^{(1)} di + \bar{C}^{(1)} \cdot k^{(3)} d\mathfrak{g} + \bar{\bar{C}}^{(1)} \cdot k^{(1)} di_1 + R_1 dr_1 \\
y^{(2)} \cdot dx^{(2)} &= C^{(2)} \cdot k^{(3)} d\zeta + \bar{C}^{(2)} \cdot k^{(1)} di + \bar{C}^{(2)} \cdot k^{(3)} d\mathfrak{g} + \bar{\bar{C}}^{(2)} \cdot k^{(1)} di_1 + \bar{\bar{C}}^{(2)} \cdot k^{(3)} d\vartheta \\
&+ \bar{\bar{C}}^{(2)} \cdot k^{(1)} di_2 + R_2 dr_2 + \Phi_2 d\varphi_2
\end{aligned}$$

Taking the sum of the two equations and recognizing that, if

$$e^{(i)} := \mathcal{R}_3(\zeta)\mathcal{R}_1(i)k^{(i)} \quad , \quad f^{(i)} := \mathcal{R}_3(\zeta)\mathcal{R}_1(i)\mathcal{R}_3(\mathfrak{g})\mathcal{R}_1(i_1)k^{(i)}$$

then

$$\begin{aligned}
(C^{(1)} + C^{(2)}) \cdot k^{(3)} &= C \cdot k^{(3)} = G \cos i = C_3 \\
(\bar{C}^{(1)} + \bar{C}^{(2)}) \cdot k^{(1)} &= C \cdot e^{(1)} = 0 \\
(\bar{C}^{(1)} + \bar{C}^{(2)}) \cdot k^{(3)} &= C \cdot e^{(3)} = G \\
(\bar{\bar{C}}^{(1)} + \bar{\bar{C}}^{(2)}) \cdot k^{(1)} &= C \cdot f^{(1)} = (Gk^{(3)}) \cdot (\mathcal{R}_3(\mathfrak{g})k^{(1)}) = 0 \\
\bar{\bar{C}}^{(2)} \cdot k^{(3)} &= \Phi_2 \cos i_2 = \Theta \\
\bar{\bar{C}}^{(2)} \cdot k^{(1)} &= 0
\end{aligned}$$

we have the thesis:

$$y^{(1)} \cdot dx^{(1)} + y^{(2)} \cdot dx^{(2)} = C_3 d\zeta + Gd\mathfrak{g} + \Theta d\vartheta + R_1 dr_1 + R_2 dr_2 + \Phi_2 d\varphi_2 \quad . \quad \blacksquare$$

2.3 Two–steps averaging for properly–degenerate systems

In this section we discuss a unicity argument for normal forms of degenerate systems.

Consider a real–analytic and properly–degenerate Hamiltonian

$$H(I, \varphi, u, v) = H_0(I) + \alpha P(I, \varphi, u, v) \quad , \quad 0 < \alpha < 1$$

defined on some phase $(n + m)$ –dimensional phase space of the form $V \times \mathbb{T}^{n_1} \times B^{2n_2}$, where V is an open, connected set of \mathbb{R}^{n_1} . Perturbation theory (*e.g.*, [5], [32], [38], [7], [13]) tells us that, under suitable assumptions of non resonance of the unperturbed frequency map $\omega := \partial_I H_0$ and of smallness of the perturbation αP , the system may be conjugated, at least formally, to a new system

$$H_p(I, \varphi, u, v) = H_0(I) + (\alpha \bar{P}_1(I, u, v) + \cdots + \alpha^p \bar{P}_p) + \alpha^{p+1} P_{p+1} \quad , \quad (P_1 \equiv P) \quad (32)$$

where the term inside parentheses (“ p –step normal form”) is of degree p and is *independent of* φ . Quantitative versions of this fact are well known in the literature since [5] and have been more and more refining themselves (depending on needs) both in the non–degenerate [38], [12] and degenerate case [5], [7], [32], [34]. Moreover, we know that, when the system is non–degenerate, *i.e.*, the variables (u, v) do not appear, the p –step normal form is uniquely determined (though the change of variables realizing it may be not). In general, when the system is degenerate, uniqueness does not hold. However, the following lemma is easily proved.

Lemma 2.3 *Let¹⁷ $n_1 = 1$ and H be a properly–degenerate system, such that*

$$P_{\text{av}} := \frac{1}{2\pi} \int_{\mathbb{T}^n} P(I, \varphi; u, v) d\varphi \equiv 0 \quad . \quad (33)$$

¹⁷We assume $n_1 = 1$ to avoid complications due to resonances of the frequency–map. This is enough for the purposes of the paper. Analogue statements for the case $n_1 \geq 1$ may be available.

Then, the two-step normal form

$$\tilde{H}(\tilde{I}, \tilde{\varphi}; \tilde{u}, \tilde{v}) = H_0(\tilde{I}) + (\alpha \bar{P}_1(\tilde{I}; \tilde{u}, \tilde{v}) + \alpha^2 \bar{P}_2(\tilde{I}; \tilde{u}, \tilde{v})) + O(\alpha^3)$$

is uniquely determined, up to real-analytic and symplectic changes $(\tilde{I}, \tilde{\varphi}; \tilde{u}, \tilde{v}) \in \tilde{V} \times \mathbb{T}^{n_1} \times \tilde{B}^{2n_2} \rightarrow (I, \varphi; u, v) \in V^n \times \mathbb{T}^n \times B^{2n_2}$, α -close to the identity.

Proof Let $p \geq 0$. Assuming to have reached the form in (32) (with the term inside parentheses identically vanishing for $p = 0$), the $(p + 1)^{\text{th}}$ Hamiltonian H_{p+1} is obtained applying to H_p any transformation in the class of *infinitesimal transformations* having as α^{p+1} germ the time-one flow of $\alpha^{p+1}\psi_{p+1}$, where

$$\psi_{p+1} := \sum_{k \neq 0} \frac{P_k^{(p+1)}(I; u, v)}{ik \cdot \omega(I)} e^{ik \cdot \varphi} + \bar{\psi}_p$$

if P_{p+1} has the Fourier expansion

$$P_{p+1} = \sum_{k \neq 0} P_k^{(p+1)}(I; u, v) e^{ik \cdot \varphi}$$

and $\bar{\psi}_p$ is any function independent of φ . Moreover, as it is known, \bar{P}_j 's and P_j 's are related by

$$\bar{P}_{p+1} = (P_{p+1})_{\text{av}} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} P_{p+1} d\varphi .$$

Therefore, if we perform two steps of the procedure, i.e., with $p = 0, 1$, we find the *two-step normal form* is defined by $\bar{P}_1 = P_{\text{av}} = 0$ and

$$\bar{P}_2 = \frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \{\psi_1, P\} d\varphi ,$$

where $\{\cdot, \cdot\}$ denotes Poisson parentheses with respect to all the variables. (The relative transformation will be given by $\phi_1 \circ \phi_2$, where ϕ_j is generated by $\alpha^j \psi_j$.) Therefore, to prove uniqueness, all we have to do is to check that, if we change $\psi_1 \rightarrow \psi_1 + \tilde{\psi}_1$, where $\tilde{\psi}_1$ is independent of φ , the function \bar{P}_2 does not change. And in fact this term changes by adding

$$\frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \{\tilde{\psi}_1, P\} d\varphi .$$

Since $\tilde{\psi}_1$ is independent of φ , Poisson parentheses and the integral may be exchanged and we see that this term vanishes

$$\frac{1}{2} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \{\tilde{\psi}_1, P\} d\varphi = \frac{1}{2} \frac{1}{(2\pi)^n} \left\{ \tilde{\psi}_1, \int_{\mathbb{T}^n} P d\varphi \right\} = \left\{ \tilde{\psi}_1, \frac{1}{2} P_{\text{av}} \right\} = 0$$

because of (33). \blacksquare

2.4 Proof of Proposition 2.2

To prove Proposition 2.2, we write $P^{(2)}$ as

$$P^{(2)} = \frac{1}{2\pi} \int_{\mathbb{T}} P^{(1)}(\Lambda, \lambda_1, z) d\lambda_1 \tag{34}$$

where

$$\begin{aligned} P^{(1)}(\Lambda, \lambda_1, z) &:= \frac{1}{2\pi} \int_{\mathbb{T}} d\lambda_2 \\ &\frac{3(x^{(1)}(\Lambda_1, \lambda_1, z) \cdot x^{(2)}(\Lambda_2, \lambda_2, z))^2 - |x^{(1)}(\Lambda_1, \lambda_1, z)|^2 |x^{(2)}(\Lambda_1, \lambda_1, z)|^2}{2|x^{(2)}(\Lambda_2, \lambda_2, z)|^5} \end{aligned} \quad (35)$$

Then we consider the auxiliary Hamiltonian

$$H_{\text{Dip}}(y^{(1)}, x^{(1)}, y^{(2)}, x^{(2)}) := \frac{|y^{(2)}|^2}{2m_2} - \frac{m_2 M_2}{|x^{(2)}|} - \alpha m_2 M_2 \frac{x^{(1)} \cdot x^{(2)}}{|x^{(2)}|^3}$$

on the phase space

$$\{(y^{(1)}, x^{(1)}, y^{(2)}, x^{(2)}) \in (\mathbb{R}^3)^4 : x^{(2)} \neq 0\}$$

endowed with the standard symplectic form

$$\omega := dy^{(1)} \wedge dx^{(1)} + dy^{(2)} \wedge dx^{(2)}$$

and $\alpha \ll 1$ a small positive parameter.

For $\alpha = 0$, H_{Dip} reduces to the two-body Hamiltonian

$$H_{2b} = \frac{|y^{(2)}|^2}{2m_2} - \frac{m_2 M_2}{|x^{(2)}|}.$$

Therefore, letting

$$(\Lambda_i, \lambda_i, z) \rightarrow (y^{(i)}(\Lambda_i, \lambda_i, z), x^{(i)}(\Lambda_i, \lambda_i, z))$$

the projection over the i^{th} planet of the map (24), in such variables, H_{Dip} takes the form

$$\begin{aligned} H(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, z) &= H_{\text{Kep}}(\Lambda_2) + \alpha P(\Lambda_1, \Lambda_2, \lambda_1, \lambda_2, z) \\ &= -\frac{M_2^2 m_2^3}{2\Lambda_2^3} - \alpha M_2 m_2 \frac{x^{(1)}(\Lambda_1, \lambda_1, z) \cdot x^{(2)}(\Lambda_2, \lambda_2, z)}{|x^{(2)}(\Lambda_2, \lambda_2, z)|^3}. \end{aligned} \quad (36)$$

Lemma 2.4 *Under assumptions of Proposition 2.2, the Hamiltonian in (36), endowed with the symplectic form*

$$\sum_{i=1}^2 d\Lambda_i \wedge d\lambda_i + \sum_{i=1}^4 du_i \wedge dv_i, \quad z = (u, v)$$

verifies the assumptions of Lemma 2.3, with to the “variables” $(I, \varphi) := (\Lambda_2, \lambda_2)$ and the “parameters” $(\Lambda_1, \lambda_1, z)$. Its (unique) two-step normal form is

$$\tilde{H}(\Lambda_1, \Lambda_2, \lambda_1, z) = H_{\text{Kep}}(\Lambda_2) + \alpha^2 M_2 m_2 P^{(1)}(\Lambda_1, \Lambda_2, \lambda_1, z) + O(\alpha^3)$$

with $P^{(1)}$ as in (35).

Proof We apply Lemma 2.3 to the Hamiltonian H in (36). Indeed, the assumption (23) implies that the zero-averaging (with respect to λ_2) assumption for P is satisfied:

$$\begin{aligned} \int_{\mathbb{T}} P d\lambda_2 &= \int_{\mathbb{T}} \left(-m_2 M_2 \frac{x^{(1)} \cdot x^{(2)}}{|x^{(2)}|^3} \right) d\lambda_2 = -m_2 M_2 x^{(1)} \cdot \int_{\mathbb{T}} \frac{x^{(2)}}{|x^{(2)}|^3} d\lambda_2 \\ &= m_2 M_2 \omega_{\text{Kep}}^{(2)} x^{(1)} \cdot \int_{\mathbb{T}} \partial_{\lambda_2} y^{(2)} = 0 \end{aligned}$$

Denote as

$$\tilde{H}(\Lambda_1, \Lambda_2, \lambda_1, z) = H_{\text{Kep}}(\Lambda_2) + \alpha^2 M_2 m_2 P^{(1)}(\Lambda_1, \Lambda_2; \lambda_1, z) + O(\alpha^3)$$

with

$$H_{\text{Kep}}(\Lambda_2) = -\frac{M_2^2 m_2^3}{2\Lambda_2^2}$$

the two-step normal form which is achieved via Lemma 2.3. Let ψ denote the symplectic, α -close-to-the-identity transformation realizing this normal form. Consider the auxiliary Hamiltonian

$$H^* = H + \alpha^2 M_2 m_2 Q, \quad (37)$$

where

$$Q := -\frac{3(x^{(1)}(\Lambda_1, \lambda_1, z) \cdot x^{(2)}(\Lambda_2, \lambda_2, z))^2 - |x^{(1)}(\Lambda_1, \lambda_1, z)|^2 |x^{(2)}(\Lambda_2, \lambda_2, z)|^2}{2|x^{(2)}(\Lambda_2, \lambda_2, z)|^5}.$$

Being α -close to the identity, ψ transforms H^* into

$$\tilde{H}^* = \tilde{H} + \alpha^2 M_2 m_2 Q + O(\alpha^3).$$

Hence, at expenses of a further λ_2 -averaging (α^2 -close to the identity), \tilde{H} can be let into

$$\begin{aligned} \tilde{H}(\Lambda_1, \Lambda_2, \lambda_1, z) &= \tilde{H} + \alpha^2 M_2 m_2 Q^{(1)}(\Lambda_1, \Lambda_2; \lambda_1, z) + O(\alpha^3) \\ &= H_{\text{Kep}}(\Lambda_2) + \alpha^2 M_2 m_2 P^{(1)}(\Lambda_1, \Lambda_2; \lambda_1, z) + \alpha^2 M_2 m_2 Q^{(1)}(\Lambda_1, \Lambda_2; \lambda_1, z) \\ &\quad + O(\alpha^3), \end{aligned}$$

with

$$Q^{(1)} := \frac{1}{2\pi} \int_{\mathbb{T}} Q d\lambda_2$$

On the other hand, one immediately sees that H^* in (37) may be written as

$$H^* = H_{2B}^* + O(\alpha^3)$$

H_{2B}^* is the well “familiar” one

$$H_{2B}^* := \frac{|y^{(2)}(\Lambda_2, \lambda_2, z)|^2}{2m_2} - \frac{m_2 M_2}{|x^{(2)}(\Lambda_2, \lambda_2, z) - \alpha x^{(1)}(\Lambda_1, \lambda_1, z)|}.$$

But (using, note, assumption (B)) H_{2B}^* may be symplectically conjugated, via an α -close to the identity map ψ' , to $H_{\text{Kep}} = -\frac{M_2^2 m_2^3}{2\Lambda_2^2}$. This implies that ψ' lets H^* into

$$\tilde{H}' = -\frac{M_2^2 m_2^3}{2\Lambda_2^2} + O(\alpha^3)$$

Uniqueness (claimed by Lemma 2.3) implies $\tilde{H} \equiv \tilde{H}' + O(\alpha^3)$, namely,

$$P^{(1)}(\Lambda_1, \Lambda_2; \lambda_1, z) = -Q^{(1)}(\Lambda_1, \Lambda_2; \lambda_1, z)$$

which is the thesis. \blacksquare

We are now ready for the

Proof of Proposition 2.2 For the purposes of this proof, if $f : x \in \mathbb{T} \rightarrow f(x) \in \mathbb{R}$ is continuous, we denote as $\langle f \rangle_x := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) dx$.

Consider the Hamiltonian H in (36); let $\bar{\phi}$ as in (24) and ϕ as in (27). Denote as $H_{\text{red}} := H \circ \bar{\phi}^{-1} \circ \phi$ the expression of H in the variables (26). This is

$$H_{\text{red}} = H \circ \bar{\phi}^{-1} \circ \phi = \frac{R_2^2}{2m_2} - \frac{M_2 m_2}{r_2} + \frac{\Phi_2^2}{2m_2 r_2^2} - M_2 m_2 \alpha \frac{r_1}{r_2^2} \sqrt{1 - \left(\frac{\Theta}{\Phi_2}\right)^2} \sin \varphi_2 . \quad (38)$$

Let us split H_{red} into two parts, a ‘‘radial’’ and a ‘‘tangential’’ one:

$$H_{\text{rad}} := \frac{R_2^2}{2m_2} - \frac{M_2 m_2}{r_2}$$

and

$$H_{\text{tan}} := \frac{\Phi_2^2}{2m_2 r_2^2} - M_2 m_2 \alpha \frac{r_1}{r_2^2} \sqrt{1 - \left(\frac{\Theta}{\Phi_2}\right)^2} \sin \varphi_2$$

and focus on H_{tan} . We shall eliminate the dependence from the angle φ_2 up to order α^3 . To this end, define h_0, P_0 via $H_{\text{tan}} =: h_0 + \alpha P_0$ and denote $\varpi := \partial_{\Phi_2} h = \frac{\Phi_2}{m_2 r_2^2}$. Since $\langle P_0 \rangle_{\varphi_2} = 0$, a Hamiltonian vector field the time-one flow of which eliminates the dependence on φ_2 up to $O(\alpha^2)$ has as Hamiltonian the function ψ_0 defined as a primitive

$$\psi_0 = \frac{1}{\varpi} \int^{\varphi_2} \alpha P_0 = M_2 m_2^2 \alpha \frac{r_1}{\Phi_2} \sqrt{1 - \left(\frac{\Theta}{\Phi_2}\right)^2} \cos \varphi_2$$

with $\langle \psi_0 \rangle_{\varphi_2} = 0$. It is a remarkable fact that r_2 is cancelled. Since ϕ_0 is also independent of R_1, R_2 and ϑ , this implies that its time-one flow, that we denote

$$\phi_0 : (\tilde{R}_1, \tilde{R}_2, \tilde{\Phi}_2, \tilde{\Theta}, \tilde{r}_2, \tilde{r}_2, \tilde{\varphi}_2, \tilde{\vartheta}) \rightarrow (R_1, R_2, \Phi_2, \Theta, r_2, r_2, \varphi_2, \vartheta) ,$$

leaves (R_2, r_2, Θ, r_1) unvaried. Using again $\langle P_0 \rangle_{\varphi_2} = 0$, we then have that H_0 is conjugated to

$$H_1 = H_{\text{tan}} \circ \phi_0 = h_0 + \alpha^2 P_1 + O(\alpha^3) ,$$

where

$$P_1 = \frac{1}{2} \{ \psi_0, P_0 \} = -\frac{M_2^2 m_2^3}{2} \frac{\tilde{r}_1^2}{\tilde{r}_2^2 \tilde{\Phi}_2^4} \left(\tilde{\Theta}^2 - \frac{1}{2} (\tilde{\Phi}_2^2 - \tilde{\Theta}^2) (1 + \cos 2\tilde{\varphi}_2) \right)$$

A further step of averaging defined by the time-one flow

$$\phi_1 : (\hat{R}_1, \hat{R}_2, \hat{\Phi}_2, \hat{\Theta}, \hat{r}_2, \hat{r}_2, \hat{\varphi}_2, \hat{\vartheta}) \rightarrow (\tilde{R}_1, \tilde{R}_2, \tilde{\Phi}_2, \tilde{\Theta}, \tilde{r}_2, \tilde{r}_2, \tilde{\varphi}_2, \tilde{\vartheta})$$

of

$$\begin{aligned} \psi_1 &= \frac{1}{\varpi} \int^{\varphi_2} \alpha^2 (P_1 - \langle P_1 \rangle) \\ &= +\frac{\alpha^2}{\varpi} \int^{\varphi_2} \frac{M_2^2 m_2^3}{4} \frac{r_1^2}{r_2^2 \Phi_2^4} (\Phi_2^2 - \Theta^2) \cos 2\varphi \\ &= +\alpha^2 \frac{M_2^2 m_2^4}{8} \frac{r_1^2}{\Phi_2^5} (\Phi_2^2 - \Theta^2) \sin 2\varphi_2 \end{aligned}$$

with $\langle \psi_1 \rangle_{\varphi_2} = 0$. As in the previous step, ψ_1 is independent of (R_1, R_2, ϑ) and, again r_2 , hence, ϕ_1 leaves (R_2, r_2, Θ, r_1) unvaried. Then H_1 is let into the form

$$H_2 = H_1 \circ \phi_1 = h_0 + \alpha^2 P_2 + O(\alpha^3) ,$$

where

$$P_2 = \langle P_1 \rangle_{\varphi_2} = -\frac{M_2^2 m_2^3}{4} \frac{\hat{r}_1^2}{\hat{r}_2^2 \hat{\Phi}_2^4} (3\hat{\Theta}^2 - \hat{\Phi}_2^2) .$$

Including also the term H_{rad} (left unvaried by this sequence of transformations) we finally have that the Hamiltonian H_{red} in (38) is transformed into

$$\begin{aligned} \hat{H} &:= H_{\text{red}} \circ \phi_0 \circ \phi_1 = \frac{\hat{R}_2^2}{2m_2} - \frac{M_2 m_2}{\hat{r}_2} + \frac{1}{2m_2 \hat{r}_2^2} \left(\hat{\Phi}_2^2 - \alpha^2 \frac{M_2^2 m_2^4}{2} \frac{\hat{r}_1^2}{\hat{\Phi}_2^4} (3\hat{\Theta}^2 - \hat{\Phi}_2^2) \right) \\ &+ O(\alpha^3) . \end{aligned} \quad (39)$$

Let now

$$(\hat{y}^{(1)}, \hat{y}^{(2)}, \hat{x}^{(1)}, \hat{x}^{(2)}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$$

be related to $(C_3, G, \hat{R}_1, \hat{R}_2, \hat{\Phi}_2, \hat{\Theta}, \zeta, \mathbf{g}, \hat{r}_1, \hat{r}_2, \hat{\varphi}_2, \hat{\vartheta})$ via relations analogue to (27)–(28), i.e., $(\hat{y}, \hat{x}) = \phi^{-1}(C_3, G, \hat{R}_1, \hat{R}_2, \hat{\Phi}_2, \hat{\Theta}, \zeta, \mathbf{g}, \hat{r}_1, \hat{r}_2, \hat{\varphi}_2, \hat{\vartheta})$, with ϕ^{-1} as in (26) and let $(\hat{\Lambda}, \hat{\lambda}, \hat{z})$ be defined via $(\hat{y}, \hat{x}) = \bar{\phi}(\hat{\Lambda}, \hat{\lambda}, \hat{z})$, with $\bar{\phi}$ as in (24). In the variables $(\hat{\Lambda}, \hat{\lambda}, \hat{z})$, the Hamiltonian (39) takes the form

$$\tilde{H} := \hat{H} \circ \phi^{-1} \circ \bar{\phi} = -\frac{M_2^2 m_2^3}{2\hat{\Lambda}_2^2} - \frac{\alpha^2}{2m_2 \hat{r}_2^2} \frac{M_2^2 m_2^4}{2} \frac{\hat{r}_1^2}{\hat{\Phi}_2^4} (3\hat{\Theta}^2 - \hat{\Phi}_2^2)$$

where $\hat{\Theta} = \hat{C}^{(2)} \cdot \hat{x}^{(1)}$, $\hat{r}_i = |\hat{x}^{(i)}|$, $\hat{\Phi}_2 = |\hat{C}^{(2)}|$, with $\hat{C}^{(2)} = \hat{x}^{(2)} \times \hat{y}^{(2)}$ have to be regarded as functions of $(\hat{\Lambda}, \hat{\lambda}, \hat{z})$. A further $\hat{\lambda}_2$ -averaging, α^2 -close to the identity

$$\hat{\phi}: \quad (\hat{\Lambda}, \hat{\lambda}, \hat{z}) \rightarrow (\hat{\Lambda}, \hat{\lambda}, \hat{z})$$

transforms \tilde{H} into

$$\hat{H} := \tilde{H} \circ \hat{\phi} = -\frac{M_2^2 m_2^3}{2\hat{\Lambda}_2^2} - \alpha^2 \hat{P}(\hat{\Lambda}, \hat{\lambda}_1, \hat{z}) , \quad \hat{P}(\hat{\Lambda}, \hat{\lambda}_1, \hat{z}) := \frac{M_2^2 m_2^3}{4} \frac{\hat{r}_1^2}{\hat{\Phi}_2^4} (3\hat{\Theta}^2 - \hat{\Phi}_2^2) \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\hat{\lambda}_2}{\hat{r}_2^2} , \quad (40)$$

where $\hat{\Theta} = \hat{C}^{(2)} \cdot \hat{x}^{(1)}$, $\hat{r}_i = |\hat{x}^{(i)}|$, $\hat{\Phi}_2 = |\hat{C}^{(2)}|$ have to be regarded as functions of $(\hat{\Lambda}, \hat{\lambda}, \hat{z})$. Note that we have used that $\hat{\Theta} = \hat{C}^{(2)} \cdot \hat{x}^{(1)}$, $\hat{r}_1 = |\hat{x}^{(1)}|$, $\hat{\Phi}_2 = |\hat{C}^{(2)}|$ are independent of $\hat{\lambda}_2$. By construction, the overall change

$$\bar{\phi}^{-1} \circ \phi \circ \phi_0 \circ \phi_1 \circ \phi^{-1} \circ \bar{\phi}: \quad (\hat{\Lambda}, \hat{\lambda}, \hat{z}) \rightarrow (\Lambda, \lambda, z)$$

is symplectic, α -close to the identity and puts the Hamiltonian H in (36) into the form claimed in Lemma 2.3. By the uniqueness claimed by this theorem, in comparison with the result of Lemma 2.4, we have that $\hat{P}(\hat{\Lambda}, \hat{\lambda}_1, \hat{z})$ in (40) satisfies

$$\alpha^2 \hat{P}(\hat{\Lambda}, \hat{\lambda}_1, \hat{z}) = -\alpha^2 \frac{M_2^2 m_2^3}{4} \frac{\hat{r}_1^2}{\hat{\Phi}_2^4} (3\hat{\Theta}^2 - \hat{\Phi}_2^2) \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\hat{\lambda}_2}{\hat{r}_2^2} \equiv \alpha^2 M_2 m_2 P^{(1)}(\hat{\Lambda}, \hat{\lambda}_1, \hat{z}) + O(\alpha^3) ,$$

where $P^{(1)}$ is as in (35) (and, as above, $\hat{\Theta}$, $\hat{\Phi}_2$, \hat{r}_1 and \hat{r}_1 are regarded as functions of $(\hat{\Lambda}_1, \hat{\Lambda}_2, \hat{\lambda}_1, \hat{\lambda}_2, \hat{z})$). Taking the average with respect to $\hat{\lambda}_1$, we have the thesis. \blacksquare

2.5 Proof of Proposition 2.1

We shall need definitions and a result from [15], to which paper we refer for notations and details.

Let, as in [15], $\mathcal{P}_P^{6n}, \mathcal{P}_{\text{rps}}^{6n} \subset \mathbb{R}^{3n} \times \mathbb{R}^{3n}$ denote the respective domains of the maps

$$\phi_P : (y, x) \in \mathcal{P}_P^{6n} \rightarrow (\Lambda, \lambda, z) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^{4n}, \quad \phi_{\text{rps}} : (y, x) \in \mathcal{P}_{\text{rps}}^{6n} \rightarrow (\Lambda, \lambda, z) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{R}^{4n}$$

between ‘‘Cartesian’’ and, respectively, Poincaré, RPS variables. Consider the common domain of ϕ_P and ϕ_{rps} , i.e. the set $\mathcal{P}_{\text{rps}}^{6n} \cap \mathcal{P}_P^{6n}$. On the ϕ_{rps} -image of such domain consider the symplectic map

$$\phi_P^{\text{rps}} : (\Lambda, \lambda, z) \rightarrow (\Lambda, \lambda, z) := \phi_P \circ \phi_{\text{rps}}^{-1} \quad (41)$$

which maps the RPS variables onto the Poincaré variables. Such a map has a particularly simple structure:

Theorem 2.1 ([15]) *The symplectic map ϕ_P^{rps} in (41) has the form*

$$\lambda = \lambda + \varphi(\Lambda, z) \quad z = \mathcal{Z}(\Lambda, z) \quad (42)$$

where $\varphi(\Lambda, 0) = 0$ and, for any fixed Λ , the map $\mathcal{Z}(\Lambda, \cdot)$ is 1:1, symplectic¹⁸ and its projections verify

$$\Pi_\eta \mathcal{Z} = \eta + O(|z|^3), \quad \Pi_\xi \mathcal{Z} = \xi + O(|z|^3), \quad \Pi_p \mathcal{Z} = \mathcal{V}p + O(|z|^3), \quad \Pi_q \mathcal{Z} = \mathcal{V}q + O(|z|^3)$$

for some $\mathcal{V} = \mathcal{V}(\Lambda) \in \text{SO}(n)$.

Now we proceed to prove Proposition 2.1. Consider the inverse maps

$$\phi_{\text{rps}}^{-1} : (\Lambda, \lambda, z) \in \mathcal{M}_{\text{rps}}^{6n} \rightarrow (y_{\text{rps}}(\Lambda, \lambda, z), x_{\text{rps}}(\Lambda, \lambda, z))$$

$$\phi_P^{-1} : (\Lambda, \lambda, z) \in \mathcal{M}_P^{6n} \rightarrow (y_P(\Lambda, \lambda, z), x_P(\Lambda, \lambda, z))$$

with $\mathcal{M}_{\text{rps}}^{6n} := \phi_{\text{rps}}(\mathcal{P}_{\text{rps}}^{6n})$, $\mathcal{M}_P^{6n} := \phi_P(\mathcal{P}_P^{6n})$. Let $y_{\text{rps}}^{(i)} \in \mathbb{R}^3, \dots$ be the i^{th} projection of y_{rps}, \dots ; i.e., to be defined by

$$y_{\text{rps}} = (y_{\text{rps}}^{(1)}, \dots, y_{\text{rps}}^{(n)}), \quad \dots$$

Let, finally,

$$\begin{aligned} \alpha_{ij}^2(f_{ij}^{(2)})_P &:= \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} d\lambda_i d\lambda_j \\ &\frac{3(x_P^{(i)}(\Lambda, \lambda_i, z) \cdot x_P^{(j)}(\Lambda, \lambda_j, z))^2 - |x_P^{(i)}(\Lambda, \lambda_i, z)|^2 |x_P^{(j)}(\Lambda, \lambda_j, z)|^2}{2|x_P^{(j)}(\Lambda, \lambda_j, z)|^5}. \end{aligned}$$

and¹⁹

$$\begin{aligned} \alpha_{ij}^2(f_{ij}^{(2)})_{\text{rps}} &:= \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} d\lambda_i d\lambda_j \\ &\frac{3(x_{\text{rps}}^{(i)}(\Lambda, \lambda_i, z) \cdot x_{\text{rps}}^{(j)}(\Lambda, \lambda_j, z))^2 - |x_{\text{rps}}^{(i)}(\Lambda, \lambda_i, z)|^2 |x_{\text{rps}}^{(j)}(\Lambda, \lambda_j, z)|^2}{2|x_{\text{rps}}^{(j)}(\Lambda, \lambda_j, z)|^5}. \end{aligned}$$

We shall use the following properties, easily deducible from [15]:

¹⁸I.e., it preserves the two form $d\eta \wedge d\xi + dp \wedge dq$.

¹⁹As observed in footnote 16, the map ϕ_{rps}^{-1} depends explicitly on (p_n, q_n) , while $\text{SO}(3)$ -invariant expressions, such as the right hand side of the formula below, do not.

- (i) For $1 \leq i \leq n$, $y_{\text{rps}}^{(i)}$, $x_{\text{rps}}^{(i)}$ depend on λ only via λ_i . Analogously, $y_{\text{P}}^{(i)}$, $x_{\text{P}}^{(i)}$ depend on λ only via λ_i . In particular $y_{\text{P}}^{(i)}$, $x_{\text{P}}^{(i)}$ depend on Λ only via Λ_i and depend on z only via z_i , but this will not be used.
- (ii) For any $1 \leq i < j \leq n$, the map

$$(\Lambda_i, \Lambda_j, \lambda_i, \lambda_j, z_i, z_j) \rightarrow (y_{\text{P}}^{(i)}(\Lambda, \lambda_i, z), y_{\text{P}}^{(j)}(\Lambda, \lambda_j, z), x_{\text{P}}^{(i)}(\Lambda, \lambda_i, z), x_{\text{P}}^{(j)}(\Lambda, \lambda_j, z)) \quad (43)$$

satisfies assumptions (A) and (B) of Proposition 2.2. Note that, unless we are in the case $n = 2$, this is not true for the map

$$(\Lambda, \lambda_i, \lambda_j, z) \rightarrow (y_{\text{rps}}^{(i)}(\Lambda, \lambda_i, z), y_{\text{rps}}^{(j)}(\Lambda, \lambda_j, z), x_{\text{rps}}^{(i)}(\Lambda, \lambda_i, z), x_{\text{rps}}^{(j)}(\Lambda, \lambda_j, z)) \quad (44)$$

In particular, both (43) and (44) satisfy assumption (A) (for any n and any $1 \leq i < j \leq n$), but assumption (B) fails for (44) (when $n > 2$).

- (iii) Letting $C_{\text{rps}}^{(i)} := x_{\text{rps}}^{(i)} \times y_{\text{rps}}^{(i)}$ and, analogously, $C_{\text{P}}^{(i)} := x_{\text{P}}^{(i)} \times y_{\text{P}}^{(i)}$, then, for any $1 \leq i \leq n$, $C_{\text{rps}}^{(i)}$ does not depend on λ_i and, analogously, $C_{\text{P}}^{(i)}$ does not depend on λ_i . This is because, as remarked in (ii), both (43) and (44) satisfy (A).

By the previous items, may apply Proposition 2.2 to the map (43). We find

$$\begin{aligned} \alpha_{ij}^2 (f_{ij}^{(2)})_{\text{P}}(\Lambda, z) &= -\frac{M_j m_j^2}{4} \\ &\times \frac{\frac{1}{2\pi} \int_{\mathbb{T}} (3(C_{\text{P}}^{(j)}(\Lambda, z) \cdot x_{\text{P}}^{(i)}(\Lambda, \lambda_i, z))^2 - |x_{\text{P}}^{(i)}(\Lambda, \lambda_i, z)|^2 |C_{\text{P}}^{(j)}(\Lambda, \lambda_j, z)|^2) d\lambda_i}{|C_{\text{P}}^{(j)}(\Lambda, \lambda_j, z)|^4} \\ &\times \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\lambda_j}{|x_{\text{P}}^{(j)}(\Lambda, \lambda_j, z)|^2} \end{aligned}$$

Letting now $z = \mathcal{Z}(\Lambda, z)$ and changing the integration variables $\lambda_i = \lambda_i + \varphi_i(\Lambda, z)$ with \mathcal{Z} , φ as in (42) we have the thesis. \blacksquare

3 Proof of Theorem A

In this section, we aim to prove Theorem A.

Remark 3.1 For definiteness, we prove Theorem A for the spatial three-body problem. In the case of the planar $(1+n)$ -body problem with²⁰ $n \geq 3$, assume the following asymptotic of semi-axes

$$\underline{a}_j \leq a_j \leq \bar{a}_j \quad (45)$$

where

$$\underline{a}_n := \underline{a}, \quad \bar{a}_n := \bar{a}, \quad \underline{a}_j := c \underline{\alpha}^{(\frac{3}{2})^{n-j}} \underline{a}_n, \quad \bar{a}_j := \underline{\alpha}^{(\frac{3}{2})^{n-j}} \underline{a}_n \quad (46)$$

where $0 < \underline{a} < \bar{a}$ and $0 < \underline{\alpha} < c < 1$ are fixed. With this assumption, the rest of the proof of this case is similar to the one of the spatial three-body case presented below. Let us sketch it briefly. An analogue splitting as in (49) below is available, with N , \tilde{N} replaced by

$$N' := - \sum_{1 \leq i < j \leq n} \bar{m}_i \bar{m}_j \sum_{k \in \{0, 2\}} f_{ij}^{(k)}|_{\text{pl}}, \quad \tilde{N}' := - \sum_{1 \leq i < j \leq n} \bar{m}_i \bar{m}_j \sum_{k=3}^{\infty} f_{ij}^{(k)}|_{\text{pl}}.$$

²⁰For $n = 2$ there is the stronger result of Theorem 1.2.

The integrability of N' has been discussed in the Introduction. Moreover, the functions in $f_{ij}^{(2)}|_{\text{pl}}$ have all the same strength, $\frac{1}{a_2}(\underline{\alpha})^{\left(\frac{3}{2}\right)^{n-2}}$, while the ones with $k \geq 3$ (hence, the remainder \tilde{N}) are of order $\frac{\alpha^{3/4}}{a_2}(\underline{\alpha})^{\left(\frac{3}{2}\right)^{n-2}}$. The remaining details are left to the reader. \blacksquare

Let us consider the spatial three-body Hamiltonian

$$\mathcal{H}_{3\text{b}} := h_{\text{Kep}}(\Lambda) + \mu f_{3\text{b}}(\Lambda, \lambda, \bar{z}) \quad (47)$$

namely, the Hamiltonian \mathcal{H}_{rps} in (8) for $n = 2$. Let $f_{ij}^{(k)}$ be as in (19); define

$$N := -\bar{m}_1 \bar{m}_2 \sum_{j \in \{0,2\}} f_{12}^{(j)}, \quad \tilde{N} := -\bar{m}_1 \bar{m}_2 \sum_{j=3}^{\infty} f_{12}^{(j)} \quad (48)$$

so as to split

$$(f_{3\text{b}}(\Lambda, \lambda, \bar{z}))_{\text{av}} = N + \tilde{N}. \quad (49)$$

with N integrable and $|\tilde{N}| \leq \text{const } \alpha^3$. Integrability of N is known since [28] and will be discussed in this setting in Claim 3.2.

3.1 Symmetries of the partially reduced system

We recall some properties discussed in [16] and [15], to which we refer for more details.

The Hamiltonian (1) remains unvaried by reflections with respect to coordinate planes $\{x_1 = x_2\}$, $\{x_3 = 0\}$ or rotations, for example, around the $k^{(3)}$ -axis. These transformations are, respectively,

$$\begin{aligned} \mathcal{R}_{1 \leftrightarrow 2} : \quad & x^{(i)} \rightarrow (x_2^{(i)}, x_1^{(i)}, x_3^{(i)}), & y^{(i)} & \rightarrow (-y_2^{(i)}, -y_1^{(i)}, -y_3^{(i)}) \\ \mathcal{R}_3^- : \quad & x^{(i)} \rightarrow (x_1^{(i)}, x_2^{(i)}, -x_3^{(i)}), & y^{(i)} & \rightarrow (y_1^{(i)}, y_2^{(i)}, -y_3^{(i)}) \\ \mathcal{R}_g : \quad & x^{(i)} \rightarrow \text{R}_3(g) x^{(i)}, & y^{(i)} & \rightarrow \text{R}_3(g) y^{(i)} \end{aligned}$$

where $\text{R}_3(g)$ denotes the matrix

$$\text{R}_3(g) := \begin{pmatrix} \cos g & -\sin g & 0 \\ \sin g & \cos g & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g \in \mathbb{T}.$$

Note, in particular, that \mathcal{R}_3^- and \mathcal{R}_g are symplectic transformations, while $\mathcal{R}_{1 \leftrightarrow 2}$ is an involution. The expressions of $\mathcal{R}_{1 \leftrightarrow 2}$, \mathcal{R}_3^- and \mathcal{R}_g in terms of the variables (7) turn out to be the same²¹ as in Poincaré variables. They are

$$\begin{aligned} \mathcal{R}_{1 \leftrightarrow 2}(\Lambda, \lambda, z) &:= \left(\Lambda, \frac{\pi}{2} - \lambda, \mathcal{S}_{1 \leftrightarrow 2} z \right); & \mathcal{R}_3^-(\Lambda, \lambda, z) &= \left(\Lambda, \lambda, \mathcal{S}_{34}^- z \right) \\ \mathcal{R}_g(\Lambda, \lambda, z) &= \left(\Lambda, \lambda + g, \mathcal{S}_g z \right) \end{aligned} \quad (50)$$

where

$$\begin{cases} \mathcal{S}_{1 \leftrightarrow 2}(\eta, \xi, p, q) := (\xi, \eta, q, p) \\ \mathcal{S}_{34}^-(\eta, \xi, p, q) := (\eta, \xi, -p, -q) \\ \mathcal{S}_g : (\eta_j + i\xi_j, p_j + iq_j) \rightarrow (e^{-ig}(\eta_j + i\xi_j), e^{-ig}(p_j + iq_j)) \end{cases}$$

²¹See, for example [21].

with $i := \sqrt{-1}$.

Since also the Hamiltonian \mathcal{H}_{rps} (8) is independent of (p_n, q_n) , in the above transformations, we may neglect this latter couple of variables and replace²² z with \bar{z} in (50). In particular, the one-parameter group $\{\bar{\mathcal{R}}_g\}_{g \in \mathbb{T}}$ defined by

$$\bar{\mathcal{R}}_g : (\Lambda, \lambda, \bar{z}, p_n, q_n) \rightarrow (\Lambda, \lambda + g, \mathcal{S}_g \bar{z}, p_n, q_n) \quad g \in \mathbb{T} \quad (51)$$

leaves \mathcal{H}_{rps} unvaried. This group of transformations corresponds to be the time- g flow of

$$G = \sum_{i=1}^n \Lambda_i - \sum_{i=1}^n \frac{\eta_i^2 + \xi_i^2}{2} - \sum_{i=1}^{n-1} \frac{p_i^2 + q_i^2}{2} \quad (52)$$

which is the Euclidean length of the angular momentum (6): $G = |\mathbf{C}|$, expressed in the variables (7). Therefore, $\bar{\mathcal{R}}_g$ may be identified to be the group g -rotations about the \mathbf{C} -axis.

In view of such relations, amusing symmetries (discussed²³ in [16]) appear among the Taylor coefficients of the expansion of the perturbation f_{rps} and hence also of its averaged value $(f_{\text{rps}})_{\text{av}}$. These symmetries are often referred to (for the classical Poincaré system (2)) as *D' Alembert rules*. To describe such relations, we switch²⁴ to “Birkhoff coordinates”

$$w_i = \frac{\eta_i - i\xi_i}{\sqrt{2}}, \quad w_{n+j} = \frac{p_j - iq_j}{\sqrt{2}}, \quad w_i^* = \frac{\eta_i + i\xi_i}{i\sqrt{2}}, \quad w_{n+j}^* = \frac{p_j + iq_j}{i\sqrt{2}} \quad (53)$$

with $1 \leq i \leq n$ and $1 \leq j \leq n-1$ and we regard (abusively) f_{rps} and $(f_{\text{rps}})_{\text{av}}$ as functions of $(\Lambda, \lambda, w, w^*)$.

Claim 3.1 ([16], [15])

- (i) \mathcal{R}_3 -invariance implies that f_{rps} is even in $(w_{n+1}, \dots, w_{2n-1}, w_{n+1}^*, \dots, w_{2n-1}^*)$ (equivalently, it is even in (\bar{p}, \bar{q}));
- (ii) $\bar{\mathcal{R}}_g$ -invariance implies that, the only non-vanishing monomials appearing in the Taylor expansion of $(f_{\text{rps}})_{\text{av}}$ in powers $\{w_i, w_i^*\}_{1 \leq i \leq 2n-1}$ are those with literal part $w^\alpha w^{*\alpha^*}$ for which

$$\sum_{i=1}^{2n-1} (\alpha_i - \alpha_i^*) = 0. \quad (54)$$

Claim 3.1 and the independence of $f_{12}^{(2)}$ on the argument of (η_2, ξ_2) (see the Introduction) have the following corollary. Let $\mathcal{A} = \mathcal{A}(\alpha)$ denote a set of the form

$$\mathcal{A} := (\Lambda_1, \Lambda_2) : a_- < \frac{1}{M_1} \left(\frac{\Lambda_1}{m_1} \right)^2 < \frac{\alpha}{M_2} \left(\frac{\Lambda_2}{m_2} \right)^2 < a_+ \quad (55)$$

(with $a_- < a_+$, $\alpha \in (0, 1)$) and let $\mathcal{M}_{\epsilon_0}^{10} := \mathcal{A} \times \mathbb{T}^2 \times B_{\epsilon_0}^6$.

Claim 3.2 N (namely²⁵, $f_{12}^{(2)}$) is integrable. More precisely: (i) it depends on (η_2, ξ_2) only via $\frac{\eta_2^2 + \xi_2^2}{2}$; (ii) one can find $\epsilon_0 > 0$ and a symplectic change of variables

$$(\Lambda, \check{\lambda}, \check{z}) \rightarrow (\Lambda, \lambda, \bar{z})$$

²²Recall the definitions in (7)–(9).

²³In [16], $\bar{\mathcal{R}}_g$ -invariance is called “rotation invariance”. Here, to avoid confusions, we reserve this name only to the transformations (5).

²⁴ $d\eta_i \wedge d\xi_i = dw_i \wedge dw_i^*$ and $dp_j \wedge dq_j = dw_{j+n} \wedge dw_{j+n}^*$

²⁵Recall that $f_{12}^{(0)}$ is independent of \bar{z} .

defined on the phase space $\mathcal{M}_{\epsilon_0}^{10} := \mathcal{A} \times \mathbb{T}^2 \times B_{\epsilon_0}^6$ of the form

$$\check{\phi}: \quad \Lambda = \check{\Lambda}, \quad \lambda = \check{\lambda} + \varphi(\Lambda, \check{z}), \quad \bar{z} = \check{Z}(\Lambda, \check{z}) \quad (56)$$

defined for $|\check{z}| < \epsilon_0$ which transforms N into a new function $\check{N}(\Lambda, \check{z})$ depending only on $\frac{\check{\eta}_1^2 + \check{\xi}_1^2}{2}$, $\frac{\check{\eta}_2^2 + \check{\xi}_2^2}{2}$ and $\frac{\check{p}_1^2 + \check{q}_1^2}{2}$. In particular, $\check{\psi}$ preserves $\frac{\check{\eta}_2^2 + \check{\xi}_2^2}{2}$ and $\frac{\check{\eta}_1^2 + \check{\xi}_1^2}{2} + \frac{\check{p}_1^2 + \check{q}_1^2}{2}$.

Proof Since $f_{12}^{(2)}$ is even in (p_1, q_1) and has only monomials with $\alpha_2 = \alpha_2^*$, Equation (54) with $n = 2$ implies that $f_{12}^{(2)}$ is even in (η_1, ξ_1) , (η_2, ξ_2) and (p, q) separately. Moreover, $f_{12}^{(2)}$ is integrable²⁶. Let $\bar{z} = \check{Z}(\Lambda, \check{z})$ the transformation (parametrized by Λ) verifying

$$\sum_{i=1}^2 d\eta_i \wedge d\xi_i + dp_1 \wedge dq_1 = \sum_{i=1}^2 d\check{\eta}_i \wedge d\check{\xi}_i + d\check{p}_1 \wedge d\check{q}_1$$

such that $\check{N}(\Lambda, \check{z}) := \bar{N} \circ \check{Z}$ has the claimed properties. Then, it is standard to prove that $\check{z} \rightarrow \check{Z}(\Lambda, \check{z})$ may be lifted to a transformation as in (56) (compare, for example, [16, Proposition 7.3]). ■

3.2 KAM Theory

In this section we complete the proof of Theorem A.

Let ϵ_0 , $\check{\phi}$ is as in Claim 3.2. For $(\Lambda, \check{\lambda}, \check{z}) \in \mathcal{M}^{10} := \mathcal{A} \times \mathbb{T}^2 \times B_{\epsilon_0}^6$, define

$$\begin{aligned} \check{\mathcal{H}}_{3b}(\Lambda, \check{\lambda}, \check{z}) &:= \mathcal{H}_{3b} \circ \check{\phi}(\Lambda, \check{\lambda}, \check{z}) \\ &= h_{\text{Kep}}(\Lambda) + \mu \check{f}_{3b}(\Lambda, \check{\lambda}, \check{z}) \end{aligned} \quad (57)$$

where $\check{\phi}$ is as in Claim 3.2. By Claim 3.2

$$(\check{f}_{3b})_{\text{av}} = \check{N} + \tilde{N} \quad (58)$$

where \check{N} depends only on $\frac{\check{\eta}_1^2 + \check{\xi}_1^2}{2}$, $\frac{\check{\eta}_2^2 + \check{\xi}_2^2}{2}$, $\frac{\check{p}_1^2 + \check{q}_1^2}{2}$ and

$$|\tilde{N}| \leq \text{const } \alpha^3. \quad (59)$$

To the system (57) we shall apply an abstract result (Theorem 3.1 below) that refines and generalizes Theorem A.2; see Remark 3.2. This is as follows.

Let $n_1, n_2 \in \mathbb{N}$, $B_\epsilon^{2n_2} = \{y \in \mathbb{R}^{2n_2} : |y| < \epsilon\}$ denote the $2n_2$ -ball of radius ϵ and let

$$\mathcal{P}_{\epsilon_0} := V \times \mathbb{T}^{n_1} \times B_{\epsilon_0}^{2n_2} \quad (60)$$

where V is a open, connected set of \mathbb{R}^{n_1} . Let

$$H(I, \varphi, p, q; \mu) := H_0(I) + \mu P(I, \varphi, p, q; \mu), \quad (61)$$

be real-analytic on \mathcal{P}_{ϵ_0} and such that

²⁶To integrate $f_{12}^{(2)}$, one can first reduce the integral $G_0 := \frac{\check{\eta}_1^2 + \check{\xi}_1^2}{2} + \frac{\check{p}_1^2 + \check{q}_1^2}{2}$ via the change of variables

$$\eta_1 + i\xi_1 = (\check{\eta}_1 + i\check{\xi}_1)e^{ig_0}, \quad p_1 + iq_1 = \sqrt{2(G_0 - \frac{\check{\eta}_1^2 + \check{\xi}_1^2}{2})}e^{ig_0}$$

with g_0 cyclic in $f_{12}^{(2)}$ (but not in f_{3b}). Note that this reduction does not cause singularities in f_{3b} , since f_{3b} is even in (p_1, q_1) . Next, once $f_{12}^{(2)}$ is reduced to one degree of freedom, its integration is trivial.

- (i) $\omega_0 := \partial H_0$ is a real-analytic diffeomorphism of V ;
- (ii) the average $P_{\text{av}}(I, p, q; \mu) = \frac{1}{(2\pi)^{n_1}} \int_{\mathbb{T}^{n_1}} P(I, \varphi, p, q; \mu) d\varphi$ has the form

$$P_{\text{av}}(I, p, q; \mu, \alpha) = N(I, J; \mu) + \tilde{N}(I, p, q; \mu),$$
 where

$$J = \left(\frac{p_1^2 + q_1^2}{2}, \dots, \frac{p_{n_2}^2 + q_{n_2}^2}{2} \right)$$
 and $\sup_{V \times B^{2n_2}} |\tilde{N}| \leq \kappa$;
- (iii) the Hessians $\partial_I^2 H_0$ $\partial_{I,J}^2 N(I, J; \mu)$ do not vanish, respectively, on V , $V \times B_{\epsilon_0}^{2n_2}$.

Theorem 3.1 *Under the previous assumptions, one can find positive numbers C_* , μ_* , κ_* , $\epsilon_1 < \epsilon_0$ depending only on H and ϵ_0 and an integer β depending only on n_1 , n_2 , such that, for*

$$|\mu| < \mu_* , \quad |\kappa| < \kappa_* , \quad |\mu| < (\log \kappa^{-1})^{-2\beta} \quad (62)$$

a set $\mathcal{K} \subset \mathcal{P}_{\epsilon_1}$ exists, formed by the union of H -invariant n -dimensional tori, on which the H -motion is analytically conjugated to linear Diophantine quasi-periodic motions. The set \mathcal{K} is of positive Liouville–Lebesgue measure and satisfies

$$\text{meas } \mathcal{K} > \left(1 - C_* (\sqrt[4]{\mu} (\log \kappa^{-1})^\beta + \sqrt{\kappa}) \right) \text{meas } \mathcal{P}_{\epsilon_1} . \quad (63)$$

Remark 3.2 Theorem 3.1 generalizes and refines Theorem A.2: to obtain Theorem A.2 from Theorem 3.1 it is sufficient to take $\kappa = \mu$. In this case condition (62) becomes just a smallness condition on μ (as in Theorem A.2) and, by (63), \mathcal{K} fills \mathcal{P}_{ϵ_1} up to a set of density $(1 - \tilde{C}\mu^a)$ with any $0 < a < \frac{1}{4}$. This should be compared with the measure estimate given in Theorem A.2, where $a \sim \frac{1}{n}$.

The proof of Theorem 3.1 is sketched in Appendix C.

We are now ready to complete the

Proof of Theorem A Apply Theorem 3.1 to $\check{\mathcal{H}}_{3b} := \mathcal{H}_{3b} \circ \check{\phi}$ (where $\check{\phi}$ is as in (56)), hence, with

$$n_1 = 2 , \quad n_2 = 3 , \quad V = \mathcal{A} , \quad \kappa = \text{const } \alpha^3 , \quad N = \check{N}$$

\check{N} as in (58) and ϵ_0 as in Claim 3.2. \blacksquare

4 Proof of Theorem B

In this section, we shall prove the following theorem, which is a more detailed statement of Theorem B. Let

$$\mathcal{H}_{\text{pl3b}} = h_{\text{Kep}} + \mu f_{\text{pl3b}} := \mathcal{H}_{3b}|_{p_1=q_1=0} \quad (64)$$

and denote as

$$\mathcal{M}_{\text{pl3b}}^8 : \quad \frac{\Lambda_1^2}{M_1 m_1^2} \geq \underline{a} , \quad \frac{M_2 m_2^2 \Lambda_1^2}{M_1 m_1^2 \Lambda_2^2} \leq \alpha , \quad \underline{\epsilon} < |z_{\text{pl}}| \leq \epsilon \leq \bar{\epsilon} , \quad \lambda_1, \lambda_2 \in \mathbb{T} \quad (65)$$

its eight-dimensional phase space. Here, \mathcal{H}_{3b} is as in (47) and $z_{\text{pl}} := (\eta_1, \eta_2, \xi_1, \xi_2)$.

Theorem 4.1 *There exists positive numbers $\bar{\epsilon}$, $\bar{\alpha}$, $\bar{\mu}$, $\bar{\beta}$, τ , \bar{K}_* , \bar{a} , \bar{b} , \bar{c} , \bar{d} such that, if*

$$0 < \alpha < \bar{\alpha} , \quad 0 < \mu < \bar{\mu} , \quad \mu < \bar{c} (\log \epsilon^{-1})^{-\bar{\beta}}$$

one can find an open set $\bar{\mathcal{M}}_{\text{pl3b}}^8 \subset \mathcal{M}_{\text{pl3b}}^8$ defined by the following inequalities for the Keplerian frequencies $\omega_{\text{Kep}} := \partial_{\bar{\Lambda}} h_{\text{Kep}}$

$$|\omega_{\text{Kep}} \cdot k| \geq \frac{\sqrt[4]{\mu}}{c\bar{K}} \quad \forall k : 0 < |k|_1 \leq \bar{K}$$

with

$$\bar{K} = \bar{K}_* \log(\epsilon^{-1}) \quad (66)$$

such that for the $\mathcal{H}_{\text{pl3b}}$ -flow starting from $\bar{\mathcal{M}}_{\text{pl3b}}^8$ the following holds. This flow is symplectically conjugated, via a $\{\mu^{1/12}, \epsilon^2\}$ -close to the identity transformation ϕ to a flow

$$t \rightarrow (\tilde{\Lambda}_1(t), \tilde{\Lambda}_2(t), \tilde{\eta}_1(t), \tilde{\eta}_2(t), \tilde{\xi}_1(t), \tilde{\xi}_2(t))$$

such that, letting $\tilde{t}_i(t) := \frac{\tilde{\eta}_i^2 + \tilde{\xi}_i^2}{2}$, then, for $i = 1, 2$,

$$|\tilde{\Lambda}_i(t) - \tilde{\Lambda}_i(0)| \leq \delta^{\bar{b}}, \quad |\tilde{t}_i(t) - \tilde{t}_i(0)| \leq \delta^{\bar{b}} \quad \forall 0 \leq t \leq \frac{e^{\frac{1}{\delta^{\bar{a}}}}}{\delta},$$

with $\delta := \mu^{\bar{d}} \epsilon$.

For part of the proof, we shall deal with the system $\mathcal{H}_{3\text{b}}$ in (47), which, reduces to the system $\mathcal{H}_{\text{pl3b}}$ in (64) when $p_1 = q_1 = 0$.

Proof Step 0. Let us denote again as $\check{\phi}$ a suitable symplectic transformation, whose existence is guaranteed by [37]–[16], that conjugates $\mathcal{H}_{3\text{b}}$ to a Hamiltonian $\check{\mathcal{H}}_{3\text{b}}$ having the same form as the one in (57)–(59), but with $\check{N} + \check{N}$ in Birkhoff normal form up to order $2m$, with possibly smaller \mathcal{A} of the form of (55), ϵ_0 . In the domain (65), $\check{\phi}$ is ϵ^2 -close to the identity.

4.1 Step 1: The Birkhoff normal form of order six

In this section, we aim to compute the Birkhoff normal form of order six if the three-body problem (planar and spatial).

Let

$$\check{u}_i := \frac{\check{\eta}_i - i\check{\xi}_i}{\sqrt{2}}, \quad \check{u}_i^* := \frac{\check{\eta}_i + i\check{\xi}_i}{\sqrt{2}i}, \quad \check{v} := \frac{\check{p}_1 - i\check{q}_1}{\sqrt{2}}, \quad \check{v}^* := \frac{\check{p}_1 + i\check{q}_1}{\sqrt{2}i}. \quad (67)$$

We shall show that, if $t_1 := i\check{u}_1\check{u}_1^*$, $t_2 := i\check{u}_2\check{u}_2^*$, $t_3 := i\check{v}\check{v}^*$,

Claim 4.1 *The Birkhoff normal form of order six of $(f_{3\text{b}})_{\text{av}}$ is given by (12).*

Note that the $(1 + O(\frac{\Lambda_1}{\Lambda_2}))$ -factor in (12) has not been written for simplicity (it is available from below).

Proof By Claim 3.2, the proof of (12) amounts to compute the Birkhoff normal form of order six of N in (48), up to an error of order $\frac{a_3^3}{a_2^4}$. The constant term $f_{12}^{(0)}$ in (48) contributes with $-\frac{\bar{m}_1\bar{m}_2}{a_2}$ to (12). We check that the Birkhoff normal form of $f_{12}^{(2)}$ is corresponds to what remains in (12). Recalling the definition of $f_{12}^{(2)}$ in (21) and the formulae in (15), (16) and (17), we have that the explicit formula of (22) in terms of RPS variables is

$$\begin{aligned} f_{12}^{(2)} &= \frac{a_1^2}{4a_2^3} \left(1 + 3iu_1u_1^*\bar{e}_1^2 - 3ivv^*\bar{s}^2 - 9(iu_1u_1^*)(ivv^*)\bar{s}^2\bar{e}_1^2 \right. \\ &\quad \left. - \frac{15}{2}((u_1^*)^2v^2 + (v^*)^2u_1^2)\bar{e}_1^2\bar{s}^2 \right) f \end{aligned} \quad (68)$$

where \bar{e}_1, \bar{s}, f are suitable functions of $iu_1u_1^*, iu_2u_2^*$ and ivv^* (see Appendix B for more details). Here we shall need only the first terms of their respective Taylor expansions, which are

$$\begin{aligned}
\bar{e}_1^2 &= \frac{1}{\Lambda_1} - \frac{iu_1u_1^*}{2\Lambda_1^2} \\
\bar{s}^2 &= \frac{1}{\Lambda_1} + \frac{1}{\Lambda_2} + \frac{iu_1u_1^*}{\Lambda_1^2} + \frac{iu_2u_2^*}{\Lambda_2^2} - \left(\frac{1}{4\Lambda_1^2} + \frac{1}{4\Lambda_2^2} + \frac{1}{\Lambda_1\Lambda_2}\right)ivv^* \\
&\quad + \frac{1}{\Lambda_1^3}(iu_1u_1^*)^2 + \frac{1}{\Lambda_2^3}(iu_2u_2^*)^2 - \left(\frac{1}{\Lambda_1^2\Lambda_2} + \frac{1}{2\Lambda_1^3}\right)(iu_1u_1^*)(ivv^*) \\
&\quad - \left(\frac{1}{\Lambda_1\Lambda_2^2} + \frac{1}{2\Lambda_2^3}\right)(iu_2u_2^*)(ivv^*) + \left(\frac{1}{4\Lambda_1\Lambda_2^2} + \frac{1}{4\Lambda_1^2\Lambda_2}\right)(ivv^*)^2 + \dots \\
f &= 1 + 3\frac{iu_2u_2^*}{\Lambda_2} + 6\left(\frac{iu_2u_2^*}{\Lambda_2}\right)^2 + 10\left(\frac{iu_2u_2^*}{\Lambda_2}\right)^3 + \dots
\end{aligned} \tag{69}$$

Since $f_{12}^{(2)}$ depends on (u_2, u_2^*) only via $iu_2u_2^*$, this ‘‘action’’ (besides being preserved by the transformation $\check{\psi}$ in (56)) is also preserved at any step of Birkhoff normalization. Since the factor f in (68) depends only on $iu_2u_2^*$ (see Appendix B), we may leave such factor aside and look separately at the term inside parentheses

$$F := 1 + 3iu_1u_1^*\bar{e}_1^2 - 3ivv^*\bar{s}^2 - 9(iu_1u_1^*)(ivv^*)\bar{s}^2\bar{e}_1^2 - \frac{15}{2}((u_1^*)^2v^2 + (v^*)^2u_1^2)\bar{e}_1^2\bar{s}^2.$$

Using this expression and (69), we see that the coefficients of $iu_1u_1^*$ and ivv^* (‘‘first order Birkhoff invariants’’), are, respectively, given by²⁷

$$\Omega_{u_1} = \frac{3}{\Lambda_1}, \quad \Omega_v = -3\left(\frac{1}{\Lambda_1} + \frac{1}{\Lambda_2}\right).$$

Letting

$$f := -\frac{15}{2}((u_1^*)^2v^2 + (v^*)^2u_1^2)\bar{e}_1^2\bar{s}^2, \quad \phi := -\frac{15}{2} \frac{1}{2i(\Omega_{u_1} - \Omega_v)}((u_1^*)^2v^2 - (v^*)^2u_1^2)\bar{e}_1^2\bar{s}^2,$$

one sees that the first step of Birkhoff normalization is obtained transforming F with the time–one flow of ϕ . Then F is transformed into

$$F_1 := 1 + 3iu_1u_1^*\bar{e}_1^2 - 3ivv^*\bar{s}^2 - 9(iu_1u_1^*)(ivv^*)\bar{s}^2\bar{e}_1^2 + \frac{1}{2}\{\phi, f\} + o(6).$$

where $o(6)$ stands for an expression starting with degree seven in (u_1, v, u_1^*, v^*) . The Birkhoff normal form of order six of F , obtained with a further step of Birkhoff normalization, is then

$$F_2 := 1 + 3iu_1u_1^*\bar{e}_1^2 - 3ivv^*\bar{s}^2 - 9(iu_1u_1^*)(ivv^*)\bar{s}^2\bar{e}_1^2 + \frac{1}{2}\Pi\{\phi, f\} + o(6). \tag{70}$$

where $\frac{1}{2}\Pi\{\phi, f\}$ is obtained picking up normal terms²⁸ of $\frac{1}{2}\{\phi, f\}$. But,

$$\frac{1}{2}\Pi\{\phi, f\} = \frac{225}{2} \frac{1}{(\Omega_{u_1} - \Omega_v)}((iu_1u_1^*)(ivv^*)^2 - (iu_1u_1^*)^2(ivv^*))\bar{s}^4\bar{e}_1^4 \tag{71}$$

where it is enough to replace \bar{s}, \bar{e}_1 with their respective lowest order terms in (69).

In view of (68), (69), (70) and (71), we have that (12) follows. \blacksquare

²⁷Note that we do not need to assume non–resonance of (Ω_{u_1}, Ω_v) since N in (58) is integrable.

²⁸Ie, monomials of the form $(iu_1u_1^*)^\alpha(ivv^*)^\beta$.

4.2 Step 2: Full reduction of the SO(3)–symmetry

The next step is to reduce completely the SO(3)–symmetry from the system $\check{\mathcal{H}}_{3b}$. Recall the definition of \mathcal{A} in (55), ϵ_0 as in Claim 3.2.

Since the procedure we follow is analogue²⁹ to the one in [16, §9], we shall skip some detail and refer to [16, §9] for complete information. We switch to a new set of symplectic variables $(\Lambda_1, \Lambda_2, G, \hat{u}_2, \hat{u}_2^*, \hat{\lambda}_1, \hat{\lambda}_2, \hat{g}, \hat{u}_2^*, \hat{u}_3^*)$ defined via³⁰

$$\hat{\phi} : \begin{cases} \Lambda_i = \Lambda_i & \begin{cases} \check{u}_2 = \hat{u}_2 e^{i\hat{g}} \\ \check{u}_2^* = \hat{u}_2^* e^{-i\hat{g}} \end{cases} & \begin{cases} \check{v}_2 = \hat{v}_2 e^{i\hat{g}} \\ \check{v}_2^* = \hat{v}_2^* e^{-i\hat{g}} \end{cases} \\ \check{\lambda}_i = \hat{\lambda}_i + \hat{g} \\ \check{u}_1 = \sqrt{\varrho^2/2 - \hat{t}_2 - \hat{t}_3 e^{i\hat{g}}} \\ \check{u}_1^* = -i\sqrt{\varrho^2/2 - \hat{t}_2 - \hat{t}_3 e^{-i\hat{g}}} \end{cases} \quad (72)$$

with $\check{u}_1, \check{u}_2, \check{v}, \check{u}_1^*, \check{u}_2^*, \check{v}^*$ defined as in (67), $\varrho^2/2 := \Lambda_1 + \Lambda_2 - G$, $\hat{t}_2 := i\hat{u}_2\hat{u}_2^*$, $\hat{t}_3 := i\hat{v}\hat{v}^*$. From the last couple of definitions, one sees that G is just the function in³¹ (52) (with $n = 2$) and hence its conjugated angle, \hat{g} , is cyclic in the system. Letting $(\hat{\eta}_2, \hat{\xi}_2), (\hat{p}_1, \hat{q}_1)$ the real variables associated, respectively, to $(u_2, u_2^*), (v, v^*)$ via (53) and $\hat{z} := (\hat{\eta}_2, \hat{p}_1, \hat{\xi}_2, \hat{q}_1)$. Fix $\varrho_* < \epsilon_0$. There follows from [16, Remark 9.1-(iv)] that $\hat{\phi}$ is well defined and symplectic in the domain defined by $(\lambda_1, \lambda_2, \hat{g}) \in \mathbb{T}^3$ and

$$G \in \mathbb{R}, (\Lambda_1, \Lambda_2) \in \mathcal{A}_G := \{(\Lambda_1, \Lambda_2) \in \mathcal{A} : 0 < \varrho_* \leq \varrho(\Lambda, G) < \epsilon_0\}, \quad |\hat{z}| < \varrho_*.$$

As usual, being \hat{g} cyclic, we regard G as an external fixed parameter so as to have a reduced (four–dimensional) phase space for the variables $(\Lambda, \hat{\lambda}, \hat{z})$.

Let

$$\hat{\mathcal{H}}_G := \check{\mathcal{H}}_{3b} \circ \hat{\phi} = h_{\text{Kep}} + \mu \hat{f}_G(\Lambda, \hat{\lambda}, \hat{z}) \quad (73)$$

denote the fully reduced system (where $\check{\mathcal{H}}_{3b}$ is as in Claim 3.2) on the phase space

$$\hat{\mathcal{M}}_G^8 := \mathcal{A}_G \times \mathbb{T}^2 \times B_{\varrho_*}^4 \quad (74)$$

We may assume that the function $\hat{N} + \check{N}$, where $\hat{N} := \check{N} \circ \hat{\phi}$ and $\check{N} := \tilde{N} \circ \hat{\phi}$, is again in Birkhoff normal form of order $2m$. If not, proceeding as in [15, Proof of Proposition 5.1], one can find a ϵ^{2m+1} –close to the identity symplectic transformation $\check{\phi}$ such that $\hat{N}' + \check{N} := (\hat{N} + \check{N}) \circ \check{\phi}$ is so. In the following statement, replace eventually $\hat{\phi}, \hat{N}$ and \check{N} with, respectively, $\hat{\phi}' := \hat{\phi} \circ \check{\phi}, \hat{N}', \check{N}'$.

Proposition 4.1 *The system (73)–(74) verifies*

$$(\hat{f}_G)_{\text{av}} = \hat{N} + \check{N}$$

²⁹The formulae in [16, §9] are a bit different (but obviously, equivalent) from (72), since in [16, §9] we reduce the last couple of variables, denoted as [16, $(\check{p}_{n-1}, \check{q}_{n-1})$] (corresponding to $(\check{p}_1, \check{q}_1)$ in our case), while in (72), we reduce the first couple. This different choice has two reasons: (i) it provides simultaneously reduction in the planar and the spatial problem and (ii) formulae are a bit simpler, since the term t_1^3 does not appear in (12).

³⁰Analogue transformations were considered in [29].

³¹As discussed in [16, Proposition 7.3] any step of Birkhoff normalization commutes with $\bar{\mathcal{R}}_g$ in (51), the the time– g flow of G in (52); equivalently, it preserves G .

where $\hat{N} + \tilde{N}$ is in Birkhoff normal form of order $2m$, $|\tilde{N}| \leq \text{const } \alpha^3$. Moreover, the first three orders of \hat{N} are given by

$$\begin{aligned} \hat{N} &:= -\frac{\bar{m}_1 \bar{m}_2}{a_2} - \bar{m}_1 \bar{m}_2 \frac{a_1^2}{4a_2^3} \left(\left(1 - 3\left(\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}\right)\hat{t}_2 - 3\left(\frac{2}{\Lambda_1} + \frac{1}{\Lambda_2}\right)\hat{t}_3\right) \right. \\ &\quad - \bar{m}_1 \bar{m}_2 \frac{a_1^2}{4a_2^3} \left(-\frac{3}{2} \frac{\hat{t}_2^2}{\Lambda_1^2} + 9 \frac{\hat{t}_2 \hat{t}_3}{\Lambda_1^2} + 12 \frac{\hat{t}_3^2}{\Lambda_1^2} \right. \\ &\quad - \frac{9}{2} \frac{\hat{t}_2^3}{\Lambda_1^2 \Lambda_2} - \frac{105}{4} \frac{\hat{t}_2^3 \hat{t}_3}{\Lambda_1^3} - \frac{315}{4} \frac{\hat{t}_2 \hat{t}_3^2}{\Lambda_1^3} - \frac{105}{2} \frac{\hat{t}_3^3}{\Lambda_1^3} \left. \right) \left(1 + \mathcal{O}\left(\frac{\Lambda_1}{\Lambda_2}\right) + \mathcal{O}(\varrho^2)\right) \\ &\quad \left. + \mathcal{O}(|t|^{7/2}) \right). \end{aligned} \tag{75}$$

Proof The term \hat{N} is easily computed from (12) and (72), which amounts to replace, in (12)

$$t_1 := \frac{\varrho^2}{2} - \hat{t}_2 - \hat{t}_3, \quad t_2 = \hat{t}_2, \quad t_3 = \hat{t}_3.$$

We then find (75). \blacksquare

4.3 Step 3: Averaging fast angles

In the next step we introduce, on a suitable phase space

$$\overline{\mathcal{M}}_G^8 := \bar{D} \times \mathbb{T}^2 \times B_{\epsilon_1/4}^4 \subset \hat{\mathcal{M}}_G^8, \tag{76}$$

(where $\hat{\mathcal{M}}_G^8$ is as in (74); $\epsilon_1 \leq \varrho_*$ will be arbitrary) a new system

$$\overline{\mathcal{H}}_G := h_{\text{Kep}}(\bar{\Lambda}) + \mu(\hat{N}(\bar{\Lambda}, \bar{z}) + \hat{N}_*(\bar{\Lambda}, \bar{z})) + \mu \overline{f}_G(\bar{\Lambda}, \bar{\lambda}, \bar{z}) \tag{77}$$

where \hat{N} is as in the previous sections, \hat{N}_* (as well as \hat{N}) depends only on $\bar{t}_1 = i\bar{u}_1 \bar{u}_1^*$, $\bar{t}_2 = i\bar{u}_2 \bar{u}_2^*$, $\bar{t}_3 = i\bar{v} \bar{v}^*$ and is suitably small and \overline{f}_G is suitably small.

Lemma 4.1 *There exist positive numbers $\bar{\rho}_0$, s_0 , depending only of h_{Kep} and f_{3b} in (8) such that, for any given $m \in \mathbb{N}$, one can find γ_* , α_* , μ_* , C (depending only on m , ϵ_0 , s_0) such that for any μ , α , $\bar{\gamma} > 0$, $\tau > 2$, $\bar{K} > \frac{6}{s_0}$, verifying $0 < \alpha < \alpha_*$, $0 < \mu < \mu_*$,*

$$\bar{\gamma} \geq \gamma_* \max\{\sqrt{\mu} \bar{K}^{\tau+1}, \sqrt[3]{\mu \epsilon_1} \bar{K}^{\tau+1}\}, \quad \bar{\rho} := \frac{\bar{\gamma}}{2M\bar{K}^{\tau+1}} \leq \rho_0, \tag{78}$$

an open set $\bar{D} \subset \mathcal{A}_G$ with

$$\text{meas}(\mathcal{A}_G \setminus \bar{D}) \leq C\bar{\gamma} \text{meas } \mathcal{A}_G$$

defined by the following inequalities for the Keplerian frequencies $\omega_{\text{Kep}} := \partial_{\bar{\Lambda}} h_{\text{Kep}}$

$$|\omega_{\text{Kep}} \cdot k| \geq \frac{\bar{\gamma}}{M\bar{K}^\tau} \quad \forall k: 0 < |k|_1 \leq \bar{K}$$

such that for any positive number $\epsilon_1 \leq \varrho_*$ a real-analytic transformation³²

$$\bar{\phi}: (\bar{\Lambda}, \bar{\lambda}, \bar{z}) \in \bar{D}_{\bar{\rho}/16} \times \mathbb{T}_{s_0/48}^2 \times B_{\epsilon_1/4}^4 \rightarrow (\Lambda, \hat{\lambda}, \hat{z}) \in (\mathcal{A}_G)_{\rho_0} \times \mathbb{T}_{s_0}^2 \times B_{\varrho_*}^4$$

³²We refer to [38] for (now, standard) notations of the kind \mathcal{A}_ρ , or \mathbb{T}_s^n , where \mathcal{A} is a subset of the reals and ρ , s are positive numbers.

exists, which is $\{\frac{\mu\bar{K}^{2(\tau+1)}}{\bar{\gamma}^2}, \frac{\mu\epsilon_1\bar{K}^{3(\tau+1)}}{\bar{\gamma}^3}\}$ -close to the identity and lets the Hamiltonian (73)–(74) into $\bar{\mathcal{H}}_G := \hat{\mathcal{H}}_G \circ \bar{\phi}$ as in (77) with \hat{N} as in Proposition 4.1, \hat{N}_* in Birkhoff normal form of order m , with Birkhoff invariants $\frac{\mu\bar{K}^{2\tau+1}}{\bar{\gamma}^2}$ -close to 0 and

$$|\bar{f}_G| \leq C\mu \max\{e^{-\bar{K}s_0/6}, \epsilon_1^{2m+1}\}. \quad (79)$$

The proof (sketched below) of Lemma 4.1 relies on Normal Form (Averaging³³) Theory for properly-degenerate systems and the classical Birkhoff theory (see, *e.g.*, [22]). As for Normal form theory, we refer to the theory developed in [7] (see also [13]), which, in turn, generalizes ideas and techniques of [38] to the degenerate case. For information on Normal Form theory, see [5], [32], [38], [7], [13] and references therein.

Sketch of proof of Lemma 4.1 We use analogue techniques as the ones in [13], therefore, we shall limit to describe the necessary changes. We refer, in particular, to [13, Steps 1–4 in the proof of Theorem 1.4]. First of all, choice, in [13, Steps 1–4 in the proof of Theorem 1.4],

$$\begin{aligned} n_1 &= 2, \quad n_2 = 2, \quad V = \mathcal{A}_G, \quad \kappa = \alpha^3, \quad \epsilon_0 := \varrho_*, \quad H := \hat{\mathcal{H}}_G \\ h &= h_{\text{Kep}}, \quad P_{\text{av}} = \hat{N} + \check{N}, \quad P := \mu\hat{f}_G, \\ I &= (\Lambda_1, \Lambda_2), \quad \varphi := (\hat{\lambda}_1, \hat{\lambda}_2), \quad p := (\hat{\eta}_2, \hat{p}_1), \quad q := (\hat{\xi}_2, \hat{q}_1) \\ \Omega &:= \frac{3}{4}\bar{m}_1\bar{m}_2\frac{a_1^2}{a_2^3\Lambda_1}\left(\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2}, \frac{2}{\Lambda_1} + \frac{1}{\Lambda_2}\right) + \mathcal{O}\left(\frac{a_1^3}{a_2^4}\right). \end{aligned}$$

Next, modify [13, Steps 1–4 in the proof of Theorem 1.4] as follows.

In [13, Step 1], neglect [13, Eq. (36)], so as to “leave \bar{K} free” and hence replace $\log \epsilon^{-1}$ with $\frac{\epsilon_0}{30}\bar{K}$ wherever it appears (i.e., [13, Eqs. (41), (42), (43)]). Neglect the second line in [13, Eq. (40)]. At the end of [13, Step 1, 2, 3, 4], in the definition of $\bar{H}, \check{H}, \hat{H}, \check{H}$, respectively, replace ϵ^5 with $e^{-\bar{K}s_0/6}$. At the beginning of [13, Step 2, 3, 4], in the definition of, respectively, $\check{v}, \hat{v}, \check{v}$, replace ϵ with $\epsilon_1 \leq \epsilon_0$. In [13, Step 2] replace “ \bar{N} also has a $\mu(\log \epsilon^{-1})^{2\tau+1}\bar{\gamma}^{-2}$ -close-to-0 elliptic equilibrium point” with “ \bar{N} also has a $\mu\bar{K}^{2\tau+1}\bar{\gamma}^{-2}$ -close-to-0 elliptic equilibrium point”. Replace³⁴ [13, Eqs. (43), (44), (45), (46)] with, respectively: (43): $|\bar{p} - \check{p}|, |\bar{q} - \check{q}| \leq C\frac{\mu\bar{K}^{2\tau+1}}{\bar{\gamma}^2}$, $|\bar{\varphi} - \check{\varphi}| \leq C\max\left\{\frac{\epsilon_1^2\bar{K}^{\tau+1}}{\bar{\gamma}}, \frac{\mu\epsilon_1\bar{K}^{3\tau+2}}{\bar{\gamma}^3}\right\}$; (44): $|\check{p} - \hat{p}|, |\check{q} - \hat{q}| \leq C\max\left\{\frac{\mu\epsilon_1\bar{K}^{2\tau+1}}{\bar{\gamma}^2}\right\}$, $|\check{\varphi} - \hat{\varphi}| \leq C\max\left\{\frac{\mu\epsilon_1^2\bar{K}^{3\tau+2}}{\bar{\gamma}^3}\right\}$; (45): $|\hat{\Omega} - \Omega|, |\hat{R}| \leq C\frac{\mu\bar{K}^{2\tau+1}}{\bar{\gamma}^2}$ and (46): $|\hat{p} - \check{p}|, |\hat{q} - \check{q}| \leq C\frac{\mu\epsilon_1^2\bar{K}^{2\tau+1}}{\bar{\gamma}^2}$, $|\hat{\varphi} - \check{\varphi}| \leq C\frac{\mu\epsilon_1^3\bar{K}^{3\tau+2}}{\bar{\gamma}^3}$, by suitably modifying the proofs below. Moreover replace Equation just before [13, Eq. 45] with³⁵ $\hat{N}(I, p, q) := \check{N} \circ \hat{\phi} = \hat{N} + \hat{R}$ (where \hat{N} is as in (75)) and replace the last line in [13, Step 4] with “where $\hat{N}(I, \check{r})$ is a polynomial of degree m in $(I_{n_1+1}, \dots, I_{n_2})$ ”. Lemma 4.1 follows, with $\hat{N}_* := \check{N} - \hat{N}$ and $\hat{f}_G := \mu(e^{-\bar{K}s_0/6}\check{P} + \mathcal{O}(\epsilon_1^{2m+1}))$. \blacksquare

We then apply Lemma 4.1 to the system (73)–(74) with \bar{K} as in (66), with ϵ, α replaced by ϵ_1, α_* and

$$\bar{\gamma} = \gamma_*\sqrt[4]{\mu\bar{K}^{\tau+1}}. \quad (80)$$

where γ_* is as in (78). By the thesis of Lemma 4.1, we conjugate $\hat{\mathcal{H}}_G$ in (73)–(74) to $\bar{\mathcal{H}}_G$ in (76)–(77), with \bar{f}_G satisfying (79), via a symplectic transformation which, by the choice of $\bar{\gamma}$ in (80), is $\mu^{1/12}$ -close to the identity.

³³Sometimes distinction between “Normal Form” and “Averaging” Theory is made, depending on the strength of the remainder. For an exponentially small remainder, as in [32], [38], [7], “Normal Form” Theory is often used (after [38]); for a quadratically-small remainder, “Averaging” Theory is used, after [5]. Normal form Theory is obtained with suitably many steps of averaging.

³⁴In [13, Eq (45)] $\mu\epsilon$ should be replaced by μ . This does not affect the thesis of [13, Theorem 1.4]

³⁵The symbol \check{N} used in [13] is here replaced with \hat{N} , to avoid confusions with (75).

4.4 Step 4: Nehorošev Theory

We apply Nehorošev Theory (i.e., Theorem D.1) to the system $\overline{\mathcal{H}}_G$ in (76)–(77), in the *planar* case, i.e., with $\hat{t}_4 = 0$.

For information on the tools that are used, compare [31], [32], [33] and Appendix D.

In applying Theorem D.1, we shall take

$$\begin{aligned} n_1 = 3, \quad n_2 = 0, \quad V = \bar{D}, \quad B^4 := B_{\epsilon_1/8}^4, \quad \rho := \min\{\bar{\rho}/16, s_0/48, \epsilon_1/8\} \\ H_0(\bar{\Lambda}_1, \bar{\Lambda}_2, \bar{t}_1) := h_{\text{Kep}}(\bar{\Lambda}_1, \bar{\Lambda}_2) + \mu(\hat{N} + \hat{N}_*)(\bar{\Lambda}_1, \bar{\Lambda}_2, \bar{t}_1), \quad P := \mu \bar{f}_G \end{aligned} \quad (81)$$

where $\bar{\rho}$, s_0 and ϵ_1 are as in Lemma 4.1

We have to check³⁶ steepness of $H_0(\bar{\Lambda}_1, \bar{\Lambda}_2, \bar{t}_1)$ and the smallness condition (96) of P . The first check is provided by the following claim.

Claim 4.2 *The function H_0 in (81) is $(g, m, C_1, C_2, \mathbf{a}_1, \mathbf{a}_2, \delta_1, \delta_2)$ -steep, with*

$$g = \hat{g}, \quad m = \hat{m}, \quad \mathbf{a}_i = \hat{\mathbf{a}}_i, \quad \delta_i = \min\{\sqrt{\alpha_*}, \epsilon_1^2\} \hat{\delta}_i, \quad C_i = \mu \alpha_*^2 \hat{C}_i \quad (82)$$

where $(\hat{g}, \hat{m}, \hat{C}_1, \hat{C}_2, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\delta}_1, \hat{\delta}_2)$ suitable numbers independent of α_* , μ , ϵ_1 .

Proof We take, in (75), $\hat{t}_3 = 0$. The system has three degrees of freedom. We firstly prove steepness for a suitable “rescaled” system associated to F. That is, if $\hat{N}_0 := -\frac{\bar{m}_1 \bar{m}_2}{a_2}$ is as in (75) and $\hat{N}_1 := \hat{N} - \hat{N}_0$ we consider the system

$$\begin{aligned} F_{\text{resc}}(\hat{\Lambda}_1, \hat{\Lambda}_2, \hat{t}_2) &:= \bar{m}_1^2 \bar{m}_0 \alpha_* \left(h_{\text{Kep}}^{(1)}(\bar{m}_1 \sqrt{\bar{m}_0 \alpha_*} \hat{\Lambda}_1) + \beta_2 h_{\text{Kep}}^{(2)}(\hat{\Lambda}_2) + \mu \hat{N}_0(\Lambda_2) \right. \\ &\quad \left. + \mu \beta_3 (\hat{N}_1 + \hat{N}_*)(\bar{m}_1 \sqrt{\bar{m}_0 \alpha_*} \hat{\Lambda}_1, \hat{\Lambda}_2, \epsilon_1^2 \hat{t}_2) \right) \end{aligned} \quad (83)$$

with α_* , ϵ_1 as in Lemma 4.1

$$\beta_2 := \alpha_*^{-3/2}, \quad \beta_3 := \mu^{-1} \alpha_*^{-3} \epsilon_1^{-2}. \quad (84)$$

We check that F_{resc} is steep by verifying the three-jet condition: See Appendix D.1. The three-jet condition (100) for the system (83) is

$$\begin{cases} \eta_1 + \beta_2 \alpha_*^{3/2} \left(\frac{\hat{a}_1}{\hat{a}_2} \right)^{3/2} \eta_2 + \beta_3 \alpha_*^3 \epsilon_1^2 \mu \frac{3}{4} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2} \right)^3 \eta_3 = 0 \\ \eta_1^2 + \frac{\bar{m}_1}{\bar{m}_2} \beta_2 \alpha_*^2 \left(\frac{\hat{a}_1}{\hat{a}_2} \right)^2 \eta_2^2 - \beta_3 \alpha_*^3 \mu \epsilon_1^4 \frac{1}{4} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2} \right)^3 \eta_3^2 = 0 \\ \eta_1^3 + \left(\frac{\bar{m}_1}{\bar{m}_2} \right)^2 \beta_2 \alpha_*^{5/2} \left(\frac{\hat{a}_1}{\hat{a}_2} \right)^{5/2} \eta_2^3 + \beta_3 \alpha_*^7 \mu \epsilon_1^6 \frac{9}{16} \frac{\bar{m}_1}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2} \right)^{7/2} \eta_3^3 = 0 \end{cases} \quad (85)$$

where we have used $m_i = \bar{m}_i + O(\mu)$, $M_i = \bar{m}_0 + O(\mu)$ and neglected higher order terms going to zero with μ , ϵ_1 , α_* . If we eliminate η_1 from the first and the second equation and from the first and the third equation, we obtain a homogeneous system of two equations in (η_2, η_3) that, in view of (84), generically, has only solution $\eta_2 = \eta_3 = 0$, implying that also $\eta_1 = 0$. This implies that the function F_{resc} (83) is $(2\hat{g}, \hat{m}/2, \hat{C}_1, \hat{C}_2, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\delta}_1, \hat{\delta}_2)$ -steep with suitable values of $(\hat{g}, \hat{m}, \hat{C}_1, \hat{C}_2, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\delta}_1, \hat{\delta}_2)$ which are of order 1 in μ , α_* , ϵ_1 . This readily implies that F in (81) is $(g, m, C_1, C_2, \mathbf{a}_1, \mathbf{a}_2, \delta_1, \delta_2)$ -steep, with $(g, m, C_1, C_2, \mathbf{a}_1, \mathbf{a}_2, \delta_1, \delta_2)$ as in (82). \blacksquare

³⁶Recall that, for $n_2 = 0$, as it is in our case, condition (97) is void; see Appendix D.

Remark 4.1 In the case of the *spatial* three–body problem, instead of (85), we would have

$$\left\{ \begin{array}{l} \eta_1 + \beta_2 \alpha_*^{3/2} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^{3/2} \eta_2 + \beta_3 \epsilon_1^2 \alpha_*^3 \mu \frac{3}{4} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^3 \eta_3 + \beta_3 \epsilon_1^2 \alpha_*^3 \mu \frac{3}{2} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^3 \eta_4 = 0 \\ \eta_1^2 + \frac{\bar{m}_1}{\bar{m}_2} \beta_2 \alpha_*^2 \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^2 \eta_2^2 - \beta_3 \epsilon_1^4 \alpha_*^3 \mu \frac{1}{4} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^3 \eta_3^2 + \beta_3 \epsilon_1^4 \alpha_*^3 \mu \frac{3}{2} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^3 \eta_3 \eta_4 \\ \quad + 2\beta_3 \epsilon_1^4 \alpha_*^3 \mu \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^3 \eta_4^2 = 0 \\ \eta_1^3 + \left(\frac{\bar{m}_1}{\bar{m}_2}\right)^2 \beta_2 \alpha_*^{5/2} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^{5/2} \eta_2^3 + \beta_3 \epsilon_1^6 \alpha_*^{7/2} \mu \epsilon_1^6 \frac{9}{16} \frac{\bar{m}_1}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^{7/2} \eta_3^3 + \beta_3 \epsilon_1^6 \alpha_*^3 \mu \frac{70}{64} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^3 \eta_3^2 \eta_4 \\ \quad + \beta_3 \epsilon_1^6 \alpha_*^3 \mu \frac{105}{32} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^3 \eta_3 \eta_4^2 + \beta_3 \epsilon_1^6 \alpha_*^3 \mu \frac{105}{16} \frac{\bar{m}_2}{\bar{m}_0} \left(\frac{\hat{a}_1}{\hat{a}_2}\right)^3 \eta_4^3 = 0 \end{array} \right.$$

It is not clear to the author if this system exhibits non–trivial solutions, so the analysis of this case is deferred to a subsequent paper.

We can now complete the

Proof of Theorem 4.1 It remains only to check condition (96), with P, ρ as in (81). In view of (98), (99), (82) and the choice of $\bar{\gamma}$ in (80), we have $\rho \geq \rho_* \min\{\epsilon_1, \sqrt[4]{\mu}\}$ and hence

$$\begin{aligned} M_* &\geq \frac{\tilde{c}}{\rho} \min\{(\mu \alpha_*^2)^q, \rho^q\} \geq \frac{\tilde{c}}{\rho} \min\{(\mu \alpha_*^2)^q, \epsilon_1^q, \bar{\rho}^q\} \\ &\geq \frac{\tilde{c}}{\rho} \min\left\{(\mu \alpha_*^2)^q, \epsilon_1^q, \left(\frac{\bar{\gamma}_*}{2M}\right)^q \mu^{q/4}\right\} \geq \frac{c_*}{\rho} \min\left\{(\mu \alpha_*^2)^q, \epsilon_1^q\right\} \end{aligned}$$

for some $q > 1 > c_*$ depending only on $n_1, n_2, \mathbf{a}_1, \mathbf{a}_2$. Noticing that (79) and Cauchy inequality imply

$$M := \sup |\partial P| = \mu \sup |\partial \bar{f}_G| \leq \tilde{C} \frac{\mu}{\rho} \max\{e^{-\bar{K} s_0/6}, \epsilon_1^{2m+1}\}$$

one sees that condition (96) is met, provided one previously fixes, in Lemma 4.1, $2m+1 \geq q, \bar{K}$ as in (66), with a suitable \bar{K}_* and takes $\epsilon_1 < \left(\frac{c_*}{\tilde{C}}\right)^{1/(2m+1)} (\mu \alpha_*^2)^{q/(2m+1)}$. The thesis then follows, with α, ϵ replaced by α_*, ϵ_1 and $\phi := \check{\phi} \circ \hat{\phi} \circ \bar{\phi} \circ \hat{\phi}^{-1}$. ■

A The Fundamental Theorem and another result in Arnold’s 1963 paper

Here we recall two theorems in [5]. The former is named “The Fundamental Theorem” in [5] and is as follows.

Recall the definition of \mathcal{P}_{ϵ_0} in (60).

Theorem A.1 (V. I. Arnold, [5, p. 143]) *Consider a Hamiltonian of the form*

$$H(I, \varphi, p, q) = H_0(I) + \mu P(I, \varphi, p, q)$$

which is real–analytic on \mathcal{P}_{ϵ_0} where $V \subset \mathbb{R}^{n_1}$ is open and connected, $B_{\epsilon_0}^{2n_2} \subset \mathbb{R}^{2n_2}$ is a ball of radius ϵ_0 around the origin and $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. Assume that

- (i) $I \in V \rightarrow \partial_I H_0$ is a diffeomorphism;
- (ii) P_{av} is in Birkhoff normal form³⁷ of order 6;

³⁷We refer to [22] for information on Birkhoff Theory.

(iii) the matrix β of the “second order Birkhoff invariants”: is not singular: $|\det \beta| \neq 0$ on V .

Then, there exists $\epsilon_0 > 0$ such that, for

$$0 < \epsilon < \epsilon_0, \quad 0 < \mu < \epsilon^8, \quad (86)$$

one can find a set $\mathcal{K}_{\mu, \epsilon} \subset \mathcal{P}_\epsilon \subset \mathcal{P}_{\epsilon_0}$, with

$$\text{meas } \mathcal{K}_{\mu, \epsilon} \geq (1 - \epsilon^{16(n_1+n_2)}) \text{meas } \mathcal{P}_\epsilon$$

formed by the union of H -invariant $(n_1 + n_2)$ -dimensional tori on which the H -motion is analytically conjugated to linear Diophantine³⁸ quasi-periodic motions.

The latter is less general, but used in [5] to prove Theorem 1.2.

Theorem A.2 (V. I. Arnold, [5, p. 128]) Under the same assumptions as in Theorem A.1, but replacing (ii), (iii) and (86) with

(ii)' P_{av} has the form

$$P_{\text{av}}(I, p, q) = N(I, J) + \tilde{N}(I, p, q) \text{ where } J = \left(\frac{p_1^2+q_1^2}{2}, \dots, \frac{p_{n_2}^2+q_{n_2}^2}{2}\right) \text{ and } \tilde{N} = o(\mu);$$

(iii)' the Hessian $\partial_{(I,J)}^2 N$ is non-singular: $\det \partial_{(I,J)}^2 N \neq 0$ on $V \times B_{\epsilon_0}^{2n_2}$

and condition (86) with condition

$$|\mu| < \mu_*$$

one can find a set $\mathcal{K}_\mu \subset \mathcal{P}_{\epsilon_0}$, with

$$\text{meas } \mathcal{K}_\mu \geq (1 - \mu^a) \text{meas } \mathcal{P}_{\epsilon_0}$$

(where a decreases with $n_1 + n_2$) having the same properties as the set \mathcal{K}_ϵ of Theorem A.1.

B Proof of (16), (17) and (68).

The formulae in (16) and (17) are a consequence of Proposition 2.1 and the formulae developed in [37]–[16] (see, e.g., [16, Appendix A]). Indeed, from such papers there results that, if

$$\begin{aligned} a_i &:= \frac{1}{M_i} \left(\frac{\Lambda_i}{m_i}\right)^2, \quad e_i^2 = \frac{\eta_i^2 + \xi_i^2}{\Lambda_i} - \left(\frac{\eta_i^2 + \xi_i^2}{2\Lambda_i}\right)^2 =: 2iu_i u_i^* \bar{e}_i^2 \\ \zeta_i &: \zeta_i - e_i \sin \zeta_i = \lambda_i + \arg(\eta_i, \xi_i) \\ \bar{c} &:= \frac{2\Lambda_1 + 2\Lambda_2 - 2iu_1 u_1^* - 2iu_2 u_2^* - ivv^*}{4(\Lambda_1 - iu_1 u_1^*)(\Lambda_2 - iu_2 u_2^*)}, \quad \bar{s} := \sqrt{2\bar{c}(1 - ivv^* \bar{c})} \end{aligned}$$

then, the expressions of $C^{(2)} \cdot x^{(1)}$, $|C^{(2)}|$, $r_1 = |x^{(1)}|$ and $r_2 = |x^{(2)}|$ in terms of the RPS variables are

$$\begin{aligned} C^{(2)} \cdot x^{(1)} &= \left((\hat{u}_1 v^* - \hat{u}_1^* v) x^{(1)} + i(\hat{u}_1 v^* + \hat{u}_1^* v) x^{(2)} \right) \bar{s} |C^{(2)}| \\ |C^{(2)}| &= \Lambda_2 - iu_2 u_2^*, \quad r_i = a_i (1 - e_i \cos \zeta_i) \\ x_1^{(1)} &:= \frac{1}{M_1} \left(\frac{\Lambda_1}{m_1}\right)^2 (\cos \zeta_1 - e_1), \quad x_2^{(1)} := \frac{1}{M_1} \left(\frac{\Lambda_1}{m_1}\right)^2 \sqrt{1 - e_1^2} \sin \zeta_1 \end{aligned}$$

³⁸I.e., the flow is conjugated to the Kronecker flow $\theta \in \mathbb{T}^{n_1+n_2} \rightarrow \theta + \omega t \in \mathbb{T}^{n_1+n_2}$, with $\omega \in \mathbb{R}^{n_1+n_2}$ satisfying $|\omega \cdot k| \geq \gamma |k|_1^{-\tau}$ for all $k \neq 0$, for suitable $\gamma, \tau > 0$.

with

$$\hat{u}_i := \frac{u_i}{\sqrt{i u_i u_i^*}} = \frac{\eta_i - i \xi_i}{\sqrt{2} \sqrt{\eta_i^2 + \xi_i^2}}, \quad \hat{u}_i^* := \frac{u_i^*}{\sqrt{i u_i u_i^*}} = \frac{\eta_i + i \xi_i}{\sqrt{2} \sqrt{\eta_i^2 + \xi_i^2}}.$$

Then we have

$$\begin{aligned} (\mathbb{C}^{(2)} \cdot x^{(1)})^2 &= \left(i v v^* ((x_1^{(1)})^2 + (x_2^{(1)})^2) + ((\hat{u}_1^*)^2 v^2 + (v^*)^2 \hat{u}_1^2) ((x_1^{(1)})^2 - (x_2^{(1)})^2) \right. \\ &\quad \left. + 2i((\hat{u}_1^*)^2 v^2 - (v^*)^2 \hat{u}_1^2) x_1^{(1)} x_2^{(1)} \right) \bar{\mathfrak{s}}^2 |\mathbb{C}^{(2)}|^2. \end{aligned}$$

and hence, taking the λ_1 -average (recall the relation $d\lambda_2 = (1 - e_2 \cos \zeta_2) d\zeta_2$)

$$\frac{1}{2\pi} \int_{\mathbb{T}} (\mathbb{C}^{(2)} \cdot x^{(1)})^2 d\lambda_1 = \left(i v v^* a_1^2 (1 + \frac{3}{2} e_1^2) + \frac{5}{2} ((u_1^*)^2 v^2 + (v^*)^2 u_1^2) \frac{a_1^2 e_1^2}{2i u_1 u_1^*} \right) \bar{\mathfrak{s}}^2 |\mathbb{C}^{(2)}|^2 \quad (87)$$

Here, we have used

$$\begin{aligned} \frac{1}{2\pi} \int_{\mathbb{T}} ((x_1^{(1)})^2 + (x_2^{(1)})^2) d\lambda_1 &= \frac{1}{2\pi} \int_{\mathbb{T}} r_1^2 = \frac{1}{2\pi} \int_{\mathbb{T}} a_1^2 (1 - e_1 \cos \zeta_1)^3 d\zeta_1 \\ &= a_1^2 (1 + \frac{3}{2} e_1^2) \\ \frac{1}{2\pi} \int_{\mathbb{T}} ((x_1^{(1)})^2 - (x_2^{(1)})^2) d\lambda_1 &= \frac{1}{2\pi} \int_{\mathbb{T}} d\zeta_1 (a_1^2 (\cos 2\zeta_1 + e_1^2 \\ &\quad + e_1^2 \sin^2 \zeta_1 - 2e_1 \cos \zeta_1) (1 - e_1 \cos \zeta_1)) = \frac{5}{2} a_1^2 e_1^2 \\ \frac{1}{2\pi} \int_{\mathbb{T}} x_1^{(1)} x_2^{(1)} d\lambda_1 &= \frac{1}{2\pi} \int_{\mathbb{T}} a_1^2 \sqrt{1 - e_1^2} (\cos \zeta_1 - e_1) (1 - e_1 \cos \zeta_1) \sin \zeta_1 d\zeta_1 = 0 \end{aligned}$$

Note that (87) implies (17). In turn, (16) for $n = 2$ and (17) give (68), with $f := \frac{1}{2\pi} \int_{\mathbb{T}} \frac{d\zeta}{(1 - e_2 \cos \zeta)}$.
(1 - \frac{\eta_2^2 + \xi_2^2}{2\Lambda_2})^2

C Proof of Theorem 3.1

Theorem 3.1 is an easy consequence³⁹ of the following more technical statement.

Theorem C.1 *Under the same notations and assumptions as in Theorem 3.1, one can find γ_* , C_* such that, for any ϵ_0 , one can find positive numbers $\epsilon_1 < \epsilon_0$, μ_* and α_* such that, for any α , μ , γ_1 , $\bar{\gamma}_2$, $\bar{\gamma}$ verifying*

$$|\alpha| < \alpha_*, \quad |\mu| < \mu_*, \quad \mu \bar{\gamma}_2 \leq \gamma_1$$

and

$$\left\{ \begin{array}{l} \gamma_* \sqrt[4]{\mu} (\log \alpha^{-1})^{\tau+1} \leq \bar{\gamma} \leq \gamma_* \\ \gamma_* \max \left\{ \alpha^2, \frac{\sqrt{\mu} (\log \alpha^{-1})^{\tau+1}}{\bar{\gamma}} \right\} < \gamma_1 < \gamma_* \\ \gamma_* \max \left\{ \alpha^2 (\log (\gamma_1^2 / \alpha^3))^{\tau_*+1}, \right. \\ \left. \sqrt{\mu} (\log \alpha^{-1})^{\tau+1} \bar{\gamma}^{-1} \left(\log \left(\frac{\gamma_1^2}{\mu (\log \alpha^{-1})^{2\tau+1} \bar{\gamma}^{-2}} \right) \right)^{\tau+1} \right\} < \bar{\gamma}_2 < \gamma_* \epsilon_0^2, \end{array} \right. \quad (88)$$

³⁹To obtain Theorem 3.1 from Theorem C.1, it is sufficient to choose

$$\bar{\gamma} = \gamma_* \sqrt[4]{\mu} \log(\alpha^{-1})^{\tau+1}, \quad \gamma_1 = \gamma_2 = \gamma_*^2 \max\{\alpha^2, \sqrt[4]{\mu}\} < \gamma_* \epsilon_0^2, \quad C_* := \frac{C}{\epsilon_0^2}.$$

where $\tau > n := n_1 + n_2$, then, one can find a set $\mathcal{K} \subset \mathcal{P}$ formed by the union of H -invariant n -dimensional tori, on which the H -motion is analytically conjugated to linear Diophantine quasi-periodic motions. The set \mathcal{K} is of positive measure and satisfies

$$\text{meas } \mathcal{K} > \left[1 - C(\bar{\gamma} + \gamma_1 + \frac{\bar{\gamma}_2}{\epsilon_0^2} + \alpha^{n_2}) \right] \text{meas } \mathcal{P}_{\epsilon_1} .$$

Furthermore, the flow on each H -invariant torus in \mathcal{K} is analytically conjugated to a translation $\psi \in \mathbb{T}^n \rightarrow \psi + \omega t \in \mathbb{T}^n$ with Diophantine frequencies.

This result is a slight modification of [13, Theorem 1.4] (which, in turn, had been obtained in [37]). Then here we briefly sketch its proof, describing only the necessary changes with respect to [13, Proof of Theorem 1.4] and referring the reader to that paper for more details.

To proceed, we need to recall

- the definition of “two velocities” Diophantine vector⁴⁰ in [13, Eq. (19)];
- the functional setting and notations described at the beginning of [13, §2];
- the “averaging (iterative) Theorem” [13, Lemma A.1];
- the “two-scale KAM Theorem” [13, Proposition 3].

Sketch of proof of Theorem C.1 Let ρ_0, s_0, ϵ_0 (possibly with a smaller value of ϵ_0) be positive numbers such that H in (61) has analytic extension on the complex set

$$\mathcal{P}_{\rho_0, s_0, \epsilon_0} = V_{\rho_0} \times \mathbb{T}_{s_0}^{n_1} \times B_{\epsilon_0}^{2n_2} .$$

Take three numbers $\bar{\gamma}, \gamma_1, \gamma_2 = \mu\bar{\gamma}_2$ verifying (88) and $\mu\bar{\gamma}_2 < \gamma_1$, where γ_* is some large number, depending only on n_1, n_2 , to be chosen below.

As in [13, Proof of Theorem 1.4, Step 1], start with removing, in H , the dependence on φ up to high orders. But, at difference with [13, Proof of Theorem 1.4, Step 1], apply [13, Lemma A.1] (instead of [13, Proposition 1]), with $\ell_1 = n_1, \ell_2 = 0, m = n_2, h = H_0, g \equiv 0, f = \mu P, B = B' = \{0\}, r_p = r_q = \epsilon_0, s = s_0, \rho_p = \rho_q = \epsilon_0/3, \sigma = s_0/3, \Lambda = \{0\}$,

$$e^{-\bar{K}s_0/3} := \kappa \quad \text{i.e.,} \quad \bar{K} = \frac{3}{s_0} \log \kappa^{-1} , \quad (89)$$

$A = \bar{D}, r = \bar{\rho}, \rho = \bar{\rho}/3$, where $\bar{D}, \bar{\rho}$ are defined as in [13, (37)] By [13, (38)], and the choice of $\bar{\gamma}$, the following standard measure estimate holds

$$\text{meas} \left(V \setminus \bar{D} \right) \leq C\gamma_* \sqrt{\mu} (\log \kappa^{-1})^{\tau+1} \text{meas } V$$

where C depends on the C^1 -norm of H_0 . Proceeding as [13, (39)] and the immediately following formula, one sees that the “non-resonance” condition [13, (64)] on $\bar{D}_{\bar{\rho}}$ and the “smallness” condition [13, (65)] are then verified, provided μ is chosen small enough, because of the choice of $\bar{\gamma}$ and γ_* . By the thesis of [13, Lemma A.1], we find a real-analytic symplectomorphism

$$\bar{\phi} : (\bar{I}, \bar{\varphi}, \bar{p}, \bar{q}) \in W_{(\bar{\rho}, \epsilon_0)/3, s_0/3} \rightarrow (I, \varphi, p, q) \in W_{v_0, s_0}$$

where $W_{v_0, s_0} := \bar{D}_{\rho_0} \times \mathbb{T}_{s_0}^{n_1} \times B_{\epsilon_0}$ ($v_0 = (\rho_0, \epsilon_0)$), and, by the choice of \bar{K} in (89), H is transformed into⁴¹

$$\begin{aligned} \bar{H} := H \circ \bar{\phi} &= h + \mu P_{\text{av}} + \mu \bar{P} \\ &= h + \mu N + \mu \tilde{N} + \mu \bar{P} \end{aligned} \quad (90)$$

⁴⁰This is a suitable generalization of the standard definition of Diophantine numbers, introduced in [5].

⁴¹ $\Pi_0 T_{\bar{K}} P = P_{\text{av}} = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} P d\varphi$.

where $P_{\text{av}} = N + \tilde{N}$ corresponds to g_+ of [13, Lemma A.1], \bar{P} corresponds to f_+ and hence, by the choice of \bar{K} in (89), the assumption on \tilde{N} and the thesis [13, (68)] of [13, Lemma A.1], one has that the new perturbation $\mu\tilde{N} + \mu\bar{P}$ verifies

$$\begin{aligned} \|\mu\tilde{N} + \mu\bar{P}\|_{v_0/3, s_0/3} &\leq C\mu \max\left\{\frac{\bar{K}^{2\tau+1}}{\bar{\gamma}^2}\mu, e^{-\bar{K}s_0/3}, \kappa\right\} \\ &\leq C\mu \max\left\{\frac{\bar{K}^{2\tau+1}}{\bar{\gamma}^2}\mu, \kappa\right\} \end{aligned} \quad (91)$$

In view of [13, (69)], the transformation $\bar{\phi}$ verifies

$$|I - \bar{I}|, |p - \bar{p}|, |q - \bar{q}| \leq C\frac{\mu(\log \kappa^{-1})^\tau}{\bar{\gamma}}, \quad |\varphi - \bar{\varphi}| \leq C\frac{\mu(\log \kappa^{-1})^{2\tau+1}}{\bar{\gamma}^2}.$$

Continue as in [13, Proof of Theorem 1.4, Step 5], but replacing the set in [13, (47)], with the set

$$\mathcal{A} := \left\{ J \in \mathbb{R}^{n_2} : \rho_1 < J_i < \epsilon_0^2/9, \quad 1 \leq i \leq n_2 \right\} \quad (92)$$

where $\rho_1 < \epsilon_0^2/9$ will be fixed in the next step, on so as to maximize the measure of preserved tori. Next define \mathcal{D} as in [13, (48)] (but with \mathcal{A} as in (92)) and

$$\rho := \min\{\rho_1, \bar{\rho}/3\}, \quad s := s_0/3 \quad (93)$$

Introduce the change of variables

$$(J, \psi) = ((J_1, J_2), (\psi_1, \psi_2)) \in \mathcal{D}_\rho \times \mathbb{T}_s^{n_1+n_2} \rightarrow (\bar{I}, \bar{\varphi}, \bar{p}, \bar{q})$$

defined as in [13, (49)], but replacing ‘‘checks’’ with ‘‘bars’’, This lets the Hamiltonian (90) into

$$H(J, \psi) = H_0(J_1) + \mu N(J) + \mu(\bar{P} + \tilde{N}), \quad (J, \psi) \in \mathcal{D}_\rho \times \mathbb{T}_s^{n_1+n_2}$$

Next, analogously to [13, Proof of Theorem 1.4, Step 6], construct the Kolmogorov set and estimate its measure via [13, Proposition 3].

To this end, fix γ_1 and $\gamma_2 = \mu\bar{\gamma}_2$, with $\gamma_1, \bar{\gamma}_2$ satisfying $\mu\bar{\gamma}_2 \leq \gamma_1$ and (88). Let ρ_1 in(92)–(93) be chosen so that

$$\rho_1 = \check{c}_1 \max\left\{\sqrt{\kappa}, \frac{\sqrt{\mu}(\log \kappa^{-1})^{\tau+1/2}}{\bar{\gamma}}\right\}.$$

with \check{c}_1 some large number depending only on n_1, n_2 to be fixed below. Note that the needed condition $\rho_1 < \epsilon_0^2/9$ (compare the previous step; Eq. (92)) is satisfied for $\kappa < (\epsilon_0/(3\sqrt{\check{c}_1}))^4$ and⁴² $\mu < \gamma_\star^4(\epsilon_0/(3\sqrt{\check{c}_1}))^8$. The assumption that the frequency map $\omega := \partial(H_0(J_1) + \mu N(J))$ is a diffeomorphism of \mathcal{D}_ρ is trivially satisfied. Moreover, the numbers $M, \bar{M}, \dots, \bar{M}_2$ involved in [13, Proposition 3] may be chosen as in [13, Proof of Theorem 1.4, Step 6], apart for E , which is chosen as⁴³

$$E = C \max\{\mu\kappa, \bar{K}^{2\tau+1}\mu^2\bar{\gamma}^{-2}\}.$$

Then, we can take L as in [13, Proof of Theorem 1.4, Step 6], while

$$K = C \log(E/(\mu\gamma_1^2))^{-1}$$

⁴²Use the definition of $\bar{\gamma}$ in (88).

⁴³Compare, in particular, (91) for the choice of E and recall Equation (93) and the definition of $\bar{\rho}$ and of \bar{K} in (89).

and

$$\hat{\rho} = c \min \left\{ \frac{\gamma_1}{(\log(E/(\mu\gamma_1^2))^{-1})^{\tau+1}}, \frac{\bar{\gamma}_2}{(\log(E/(\mu\gamma_1^2))^{-1})^{\tau+1}}, \frac{\bar{\gamma}}{(\log \kappa^{-1})^{\tau+1}}, \rho_1, \rho_0 \right\}.$$

To check the ‘‘KAM–smallness condition’’ [13, (32)], we divide the two cases $E = C\mu\kappa$ or $E = C\bar{K}^{2\tau+1}\mu^2\bar{\gamma}^{-2}$. If $E = \mu\kappa$,

$$\hat{c}\hat{E} \leq C \max \left\{ \kappa \left(\log \left(\frac{\gamma_1^2}{\kappa} \right) \right)^{2(\tau+1)} \max \left\{ \frac{1}{\gamma_1^2}, \frac{1}{\bar{\gamma}_2^2} \right\}, \frac{\kappa(\log \kappa^{-1})^{2(\tau+1)}}{\bar{\gamma}^2}, \frac{\kappa}{\rho_1^2}, \frac{\kappa}{\rho_0^2} \right\},$$

with a constant C not involving \check{c}_1 . Then, from (88) and $\rho_1 \geq \check{c}_1\sqrt{\kappa}$ there follows

$$\hat{c}\hat{E} < C \max \left\{ \frac{1}{\gamma_*}, \frac{1}{\check{c}_1^2}, \frac{\kappa}{\rho_0^2} \right\} < 1 \quad (94)$$

provided γ_* , $\check{c}_1^2 > C$ and $\kappa < C^{-1}\rho_0^2$. On the other hand, in the case $E = C\mu^2\bar{K}^{2\tau+1}\bar{\gamma}^{-2}$

$$\begin{aligned} \hat{c}\hat{E} \leq C \max & \left\{ \mu(\log \kappa^{-1})^{2\tau+1}\bar{\gamma}^{-2} \left(\log \left(\frac{\gamma_1^2}{\mu(\log \kappa^{-1})^{2\tau+1}\bar{\gamma}^{-2}} \right) \right)^{2(\tau+1)} \max \left\{ \frac{1}{\gamma_1^2}, \frac{1}{\bar{\gamma}_2^2} \right\}, \right. \\ & \left. \frac{\mu(\log \kappa^{-1})^{4(\tau+1)}}{\bar{\gamma}^4}, \frac{\mu(\log \kappa^{-1})^{2\tau+1}\bar{\gamma}^{-2}}{\rho_1^2}, \frac{\mu(\log \kappa^{-1})^{2\tau+1}\bar{\gamma}^{-2}}{\rho_0^2} \right\}, \end{aligned}$$

Using now that $\rho_1 \geq \check{c}_1 \frac{\sqrt{\mu}(\log \kappa^{-1})^{\tau+1/2}}{\bar{\gamma}}$ and again the definition of $\bar{\gamma}$ in (88), we again find an inequality like in (94), but with $\frac{\kappa}{\rho_0^2}$ replaced by $\frac{\sqrt{\mu}}{\rho_0^2\bar{\gamma}_*}$

Finally, since the KAM condition $\hat{c}\hat{E} < 1$ is met, [13, Proposition 3] holds in this case. Then, we can find a set of invariant tori

$$\mathcal{K}_* \subset \bar{D}_r \times \mathbb{T}^{n_1} \times \{2\rho_1 < p_i^2 + q_i^2 < 2(\epsilon_0/3)^2, \forall i\}_r \subset (\mathcal{P}_{\sqrt{2}\epsilon_0/3})_r$$

(with $r < C\bar{\gamma}_2$) satisfying the measure estimate

$$\begin{aligned} \text{meas}(\mathcal{P}_{\sqrt{2}\epsilon_0/3} \setminus \mathcal{K}_*) & \leq \text{meas}(\mathcal{P}_{\sqrt{2}\epsilon_0/3})_r \setminus \mathcal{K}_* \\ & \leq C(\bar{\gamma} + \gamma_1 + \frac{\bar{\gamma}_2}{\epsilon_0^2} + \kappa^{n_2/4}) \text{meas} \mathcal{P}_{\sqrt{2}\epsilon_0/3}. \end{aligned} \quad (95)$$

We omit to detail how (95) follows from [13, (34)]. For example, the reader may easily modify the end of [13, Proof of Theorem 1.4, Step 6].

The theorem is so proved with $\mathcal{K} := \mathcal{K}_* \cap \mathcal{P}_{\epsilon_0/3}$, $\epsilon_1 = \sqrt{2}\epsilon_0/3$, $\kappa_* := \min \{C^{-1/4} \sqrt{\rho_0}, \epsilon_0/(3\sqrt{\check{c}_1})\}$, $\mu_* := \min \{C^{-2}\rho_0^4\gamma_*^4, \gamma_*^4(\epsilon_0/(3\sqrt{\check{c}_1}))^8\}$. ■

D The Theorem by N. N. Nehorošev

Below is a more technical statement of Theorem 1.3, as it follows from [32] and, especially, [33].

The statement in [32]–[33] is based on the notion of ‘‘steepness’’ for a given smooth function $H_0(I) = H_0(I_1, \dots, I_{n_1})$ of n_1 arguments. We shall adopt the definition given in [32]. This definition involves a number of parameters, denoted, in [32], as $(g, m, C_1, \dots, C_{n_1-1}, \delta_1, \dots, \delta_{n_1-1}, \mathbf{a}_1, \dots, \mathbf{a}_{n_1-1})$. Accordingly, we shall call a given function $(g, m, C_1, \dots, C_{n_1-1}, \delta_1, \dots, \delta_{n_1-1}, \mathbf{a}_1, \dots, \mathbf{a}_{n_1-1})$ –steep, if it is steep with such parameters. See [32, p. 28 and p. 36] for details.

Theorem D.1 ([32], p. 30; [33]) *Let $H = H_0(I) + P(I, \varphi, p, q)$ be real-analytic on $\mathcal{P}_\rho := V_\rho \times \mathbb{T}_\rho^{n_1} \times B_\rho^{2n_2}$ and assume that $I \in V \rightarrow H_0(I)$ is $(g, m, C_1, \dots, C_{n_1-1}, \delta_1, \dots, \delta_{n_1-1}, \mathbf{a}_1, \dots, \mathbf{a}_{n_1-1})$ -steep, with $\rho < 1 < m$. Then, one can find $a, b \in (0, 1)$ and⁴⁴ $0 < M_\star < \rho^{1/b}$ such that, if*

$$M := \sup_{\mathcal{P}_\rho} |\partial P| \in (0, M_\star) \quad (96)$$

any trajectory $t \rightarrow \gamma(t) = (I(t), \varphi(t), p(t), q(t))$ solution of H such that

$$(p(t), q(t)) \in B^{2n_2}, \quad \forall 0 \leq t \leq T := \frac{1}{M} e^{\frac{1}{M^a}} \quad (97)$$

verifies

$$|I(t) - I(0)| \leq r := \frac{1}{2} M^b \quad \forall 0 \leq t \leq T.$$

The number M_\star can be taken to be⁴⁵

$$M_\star = \min\left\{\left(\frac{\rho}{2}\right)^{1/b}, M_0\right\} \quad (98)$$

where M_0 verifies

$$M_0 \geq \frac{c_0}{\rho} \min\left\{\left(\frac{C_{n_1-1}}{g}\right)^p, \left(\frac{\rho}{m}\right)^p, \left(\frac{m}{C_r}\right)^p, \left(\frac{1}{\max_r \delta_r}\right)^p, 1\right\} \quad (99)$$

for some $c_0 < 1 < p$ depending only on n_1, n_2 and $\mathbf{a}_1, \dots, \mathbf{a}_{n_1-1}$.

D.1 Steepness conditions

In [32], a function $H_0 = H_0(I)$ of n_1 variables (I_1, \dots, I_{n_1}) is called “quasi-convex” in I if the system

$$\begin{cases} \sum_{j=1}^{n_1} \partial_{I_j} H_0(I) \eta_j = 0 \\ \sum_{j,k=1}^{n_1} \partial_{I_j I_k}^2 H_0(I) \eta_j \eta_k = 0 \end{cases}$$

has the only trivial solution. Concave or convex functions, having definite in sign Hessian $\partial_{I_j I_k}^2 H_0(I)$, are in particular quasi-convex. Moreover, H_0 is said to satisfy the three-jet conditions if, again, the system

$$\begin{cases} \sum_{j=1}^{n_1} \partial_{I_j} H_0(I) \eta_j = 0 \\ \sum_{j,k=1}^{n_1} \partial_{I_j I_k}^2 H_0(I) \eta_j \eta_k = 0 \\ \sum_{j,k,h=1}^{n_1} \partial_{I_j I_k I_h}^3 H_0(I) \eta_j \eta_k \eta_h = 0 \end{cases} \quad (100)$$

⁴⁴We changed a bit notations of [32]. Let us call $\bar{\mathcal{P}}, \bar{\rho}$ the quantities that in the statement of [32], The main theorem, p. 30] are called F, ρ (clearly, s, n, H_1, G, D of [32] correspond to our n_1, n_2, P, V, B^{2n_2}). In the statement of [32], The main theorem, p. 30], condition (97) is required, with \mathcal{P} replaced by $\bar{\mathcal{P}}_{-2r}$, where $\bar{\mathcal{P}}_{-2r}$ is a real set defined as the biggest subset $\mathcal{A} \subset \bar{\mathcal{P}}$ for which $\mathcal{A}_{2r} \subset \bar{\mathcal{P}}$. Plainly $(\bar{\mathcal{P}}_{-2r})_{2r+\bar{\rho}} = \bar{\mathcal{P}}_{\bar{\rho}}$. Letting $\mathcal{P} := \bar{\mathcal{P}}_{-2r}$ and $\rho := 2r + \bar{\rho}$ we have our statement. Our condition $M_\star < \rho^{1/b}$ corresponds to [32]’s assumption $\bar{\rho} > 0$.

⁴⁵See [33, p. 53]. By the previous note, we have to replace ρ in [33, p. 53] with $\bar{\rho} := \rho - 2r$. Note that condition $M_\star < (\frac{\rho}{2})^{1/b}$ implies $\rho \geq \rho - 2r = \rho - M^b \geq \rho - M_\star^b \geq \frac{\rho}{2}$. With this observation, we are allowed to identify $\bar{\rho}$ of [33, p. 53] with our ρ . Letting then M_0, M_1 and M_2 as in [33, p. 53], one sees, using the formulae in [33, pp. 48–57], that $M_1 = \frac{c_1}{\rho^{\frac{1}{b} m_\star}} \left(\frac{C_{n_1-1}}{g}\right)^p$, while, since $\rho < 1 < m$, $M_2 = c_2 \min\left\{\frac{1}{m\rho^2} \left(\frac{\rho}{m}\right)^p, \frac{1}{m\rho^2} \left(\frac{C_{n_1-1}}{g}\right)^p, \frac{1}{\rho^2 m} \left(\frac{m}{C_r}\right)^p, \frac{1}{m\rho^2} \left(\frac{1}{\max_r \delta_r}\right)^p, \frac{1}{m\rho^2}\right\}$. Therefore, $M_0 := \min\{M_1, M_2\}$ verifies the inequality in (99).

has the only trivial solution.

In [31] it is proved that quasi-convex functions and functions satisfying the three-jet condition are steep.

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