# LECTURE 1: RAMSEY THEORY

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# Basic Approach to Policy Design

- Define a feasible set of allocations $\mathscr{C}$.

- Define social preferences $\mathscr{R} : \mathscr{C} \to \mathbb{R}$ over these allocations.

- Solve: $\max\{\mathscr{R}(\alpha), \alpha \in \mathscr{C}\}$.

- What determines $\mathscr{C}$?

- Previously all resource-feasible allocations.

- But required knowledge of agents' types.

# Origins

- Originated in the work of Frank Ramsey in the 1920's.

- Developed in public finance in 1960's, 1970's.

- Applied to macroeconomics from 1980's onwards.

- Benevolent government.

- Has access to linear taxes on agent activities.

- Linear tax on labor income, capital income, consumption.

- What set of feasible allocations is possible in this case? What is $\mathscr{C}$?

# Issues

- Suppose people are different, but their types are unobservable.

- Could demand same tax payment from everyone.

- But not very "fair".

- Linear taxes may be "fairer".

- More able will work harder, earn more income, consume more and will pay more tax.

- But taxes will distort their choices. Efficiency loss.

- Question: How should we design taxes to minimize the latter?

# Issues

- Most Ramsey theory assumes agents are all the same (type).

- And government must use linear taxes. It cannot use a lump sum tax (i.e. demand same tax payment from everyone).

- Assuming agents are the same and ruling out lump sum taxes is a simplification.

- Underlying motivation: people are different, their types are unobservable, linear taxes fairer than lump sum.

- Linear taxes also a better approximation to actual tax policy than lump sum.

# Results

- Basic result: Tax distortions should be smoothed over time and over states of nature.

- Capital income taxes: High initially, then roughly zero.

- Tax rates on labor and consumption income should be stable over time.

- State contingent taxes on assets/ state contingent debt should be use to provide insurance against shocks

- (Monetary policy: nominal interest rates should be close to zero).

# An Environment

- $n$ different consumption goods, $i = 1, \ldots, n$.

- Large population of identical people. Normalize size of this to 1.

- Each person has preferences over consumption goods
  $\{c_1, \ldots, c_n\} \in \mathbb{R}_+^n$ and $l \in [0, L]$:

$$U(c_1, \ldots, c_n, l)$$

  $U$ is twice continuously differentiable and concave. Increasing in each $c_i$, decreasing in $l$.

- Government consumes $\{g_1, \ldots, g_n\}$ of each good.

# An Environment

- CRS Production process: $F(y_1, \ldots, y_n, l)$

- Example: $y_i = a_i l_i$, where $a_i > 0$ is a constant and $l_i$ is amount of labor allocated to good $i$. Then:

$$F(y_1, \ldots, y_n, l) = l - \sum_{i=1}^{n} \frac{y_i}{a_i}.$$

- Unusual but convenient way to specify production.

# An Environment

- Consider static market economy.

- Treat labor as numeraire (wage =1).

- $p_i$: price of $i$-th good.

- $\tau_i$ tax on consumption of $i$-th good.

- People solve: $\max\{U(c_i,\ldots,c_n,l) \text{ s.t. } \sum_i p_i(1+\tau_i)c_i = l\}$.

- Firm solves: $\max\{\sum_i p_i y_i - l \text{ s. t. } F(y_1,\ldots,y_n,l) = 0\}$.

- Government budget constraint: $\sum_i p_i g_i = \sum_i p_i \tau_i c_i$.

- Market clearing: $c_i + g_i = y_i$.

# Competitive equilibrium

**Definition**: Taxes $\tau$, allocations $(c, l, y)$ and prices $p$ form a competitive equilibrium if:

1. $(c, l)$ maximizes person's problem,
2. $(y, l)$ solve firm's problem,
3. the government budget constraint holds,
4. goods markets clear.

$(c, l)$ is then said to be a competitive allocation.

- $U_i := \frac{\partial U}{\partial c_i}$, $U_l := \frac{\partial U}{\partial l}$.

- $F_i := \frac{\partial F}{\partial y_i}$, $F_l := \frac{\partial F}{\partial l}$.

- An allocation is interior if each $c_i > 0$ and $0 < l < L$.

# Characterize competitive allocations

**Proposition**: *Given $g = \{g_i\}$, an interior allocation $(c, l)$ is an interior competitive allocation if and only if:*

$$F(c_1 + g_1, \ldots, c_n + g_n, l) = 0, \qquad \text{(rc)}$$

*and*

$$\sum_i U_i c_i + U_l l = 0. \qquad \text{(ic)}$$

## Proof

- Suppose an interior competitive allocation $(c, l)$, then there are taxes $\tau$, prices $p$ and outputs $y$ such that $(\tau, c, l, p, y)$ forms a competitive equilibrium.

- Firm's optimize and satisfy $F(y_1, \ldots, y_n, l) = 0$. Market clearing implies each $y_i = c_i + g_i$. Gives first condition.

- $(\tau, c, l, p, y)$ must satisfy budget constraints: $\sum_i p_i(1 + \tau_i)c_i = l$.

- And first order conditions: $U_i = \lambda p_i(1 + \tau_i)$ and $U_l = -\lambda$.

- $\lambda$ : Lagrange multiplier on budget constraint.

- Combine conditions to get (ic).

- (ic) called *implementability constraint*.

## Proof

- Reverse nearly as easy.

- Given interior $(c, l)$ satisfying conditions.

- Define $p_i = -\frac{F_i(c_1+g_1,\ldots,c_n+g_n,l)}{F_l(c_1+g_1,\ldots,c_n+g_n,l)}$, $i = 1, \ldots, n$.

- Define $\tau_i = \frac{U_i(c_1,\ldots,c_n,l)}{U_l(c_1,\ldots,c_n,l)} \frac{F_l(c_1+g_1,\ldots,c_n+g_n,l)}{F_i(c_1+g_1,\ldots,c_n+g_n,l)}$. $i = 1, \ldots, n$.

- Set $y_i = c_i + g_i$, $i = 1, \ldots, n$.

- Then by construction people and firm's satisfy first order conditions. And market clearing holds.

- Substituting defined prices and taxes into (ic) gives personal budget constraints.

- Firms and people satisfy FOC's and constraints. Objectives concave. So optimal solutions.

- Government budget constraint by Walras' Law.

## Primal approach

- We have $\mathscr{C}$:

$$\mathscr{C} = \{(c,l) \in \mathbb{R}^n_{++} \times (0,L) | (rc) \text{ and } (ic) \text{ hold } \}$$

- This leaves out some competitive allocations in which $l = 0$, $l = L$ or $c_i = 0$.

- Policy design problem:

$$\max_{\mathscr{C}} U(c_1, \ldots, c_n, l)$$

- Solve. Then recover supporting taxes.

- So far, only taxes on consumption goods, none on labor income.

- Could have taxes on labor income as well. Then $n + 1$ taxes. Only need $n$ of these.

## Solving the design problem

- There exists a solution to policy design problem.

- Write down FOC's.

- Under regularity conditions, these are necessary (but not sufficient).

- Let $\lambda$ be Lagrange multiplier on (ic), $\gamma$ Lagrange multiplier on (rc).

- First order condition $c_i$:

$$(1 + \lambda)U_i - \lambda U_i H_i = \gamma F_i, \qquad \text{(FOC1)}$$

where $H_i := -(\sum_j U_{ji}c_j - U_{li}l)/U_i$.

- First order condition $l$:

$$(1 + \lambda)U_l - \lambda U_l H_l = \gamma F_l, \qquad \text{(FOC2)}$$

where $H_l := -(\sum_j U_{jl}c_j - U_{ll}l)/U_l$.

## Implication for taxes

- Using personal and firm first order conditions, (FOC1) and (FOC2) gives:

$$\frac{\tau_i}{1 + \tau_i} = \frac{\lambda(H_i - H_j)}{1 + \lambda - \lambda H_l}$$

- Combining formulas for taxes on goods $i$ and $j$ gives:

$$\frac{\tau_i/(1 + \tau_i)}{\tau_j/(1 + \tau_j)} = \frac{H_i - H_l}{H_j - H_l}$$

So if $H_i > H_j > H_l$, then $\tau_i > \tau_j$.

- Need stronger assumptions on $U$ to make progress.

- Suppose $U$ is additively separable:

$$U(c_1, \ldots, c_n, l) = \sum_{i=1}^{n} u_i(c_i) + v(l)$$

Then $U_i = u_i'$, $U_{ii} = u_i''$ and $U_{ij} = 0$, $i \neq j$.

- So $H_i(c, l) = -\frac{u_i''(c_i)c_i}{u_i'(c_i)}$, i.e. the elasticity of $u_i'$.

- And $H_l(c, l) = \frac{v''(l)l}{v'(l)}$, i.e. the elasticity of $v'$.

## Implication for taxes

- Suppose $U$ is quasi-linear (no income effects):

$$U(c_1, \ldots, c_n, l) = \sum_{i=1}^{n} u_i(c_i) - \alpha l.$$

- Then $H_l = 0$ and

$$\frac{\tau_i/(1 + \tau_i)}{\tau_j/(1 + \tau_j)} = \frac{\varepsilon_i}{\varepsilon_j}$$

and $\varepsilon_i$ is elasticity of marginal utility.

- $\varepsilon_i$ is also the reciprocal of the price elasticity of demand!

- Implies: tax goods with low elasticities of demand more heavily.

# Implication for taxes: Uniform commodity taxation

- Suppose $U$ is weakly separable across labor, homothetic in consumption goods:

$$U(c_1, \ldots, c_n, l) = W(G(c_1, \ldots, c_n), l),$$

$G$ is homothetic:

$$\frac{U_i(\alpha c, l)}{U_k(\alpha c, l)} \text{ constant in } \alpha \geq 0.$$

- Then commodity taxation is uniform, for all $i, j$:

$$\frac{U_i}{U_j} = \frac{F_i}{F_j}$$

and $\tau_i = \tau_j$.

## Proof

- Personal first order conditions imply:

$$1 + \tau_i = \frac{U_i}{U_l}\frac{F_l}{F_i}$$

So tax rates equal if $U_i/F_i$ equal across goods.

- Homotheticity implies for all $i$, $k$:

$$\sum_j \frac{c_j U_{ij}}{U_i} = \sum_j \frac{c_j U_{kj}}{U_k} = A$$

and $\tau_i = \tau_j$.

## Proof

- Ramsey first order conditions imply:

$$(1 + \lambda)U_i + \lambda \left[ \sum_j c_j U_{ij} + l U_{il} \right] = \gamma F_i$$

- Use $\sum_j c_j U_{ij} = A U_i$ and $U_i = W_1 G_i$.

- To rewrite FOC as:

$$(1 + \lambda)W_1 G_i + \lambda[A W_1 G_i + l W_{12} G_i] = \lambda F_i.$$

- But this means $G_i/F_i = G_j/F_j$ for all $i$ and $j$.

- But then:

$$\frac{U_i}{F_i} = \frac{W_1 G_i}{F_i} = \frac{W_1 G_j}{F_j} = \frac{U_j}{F_j}$$

# LECTURE 2: DYNAMIC RAMSEY THEORY

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# From Static to Dynamic Ramsey Models

- Previously we considered a static economy with linear taxes.

- We showed how to characterize the set of allocations.

- Derived an optimal tax formula.

- Showed that under quasi-linear preferences it implied taxation of low price elasticity goods.

- Showed that under weakly separability of consumption and labor and homotheticity in consumption, it implied uniform taxation.

- Now extend to dynamic setting.

# General Framework

- $t = 0, 1, 2, \ldots$

- In each period a shock. $s_t$ = shock in period $t$. $s_t \in S$.

- History of shocks: $s^t = (s_0, s_1, \ldots, s_t)$. $s^t \in S^t$.

- Probability of a history: $\mu(s^t)$.

- Initial shock $s_0$ is given. So $\mu(s_0) = 1$.

- Later will assume Markov.

# Goods

- We distinguish goods by history at which they are produced/used.

- $c(s^t)$ is consumption at date $t$ after history $s^t$; $l(s^t)$ is labor at date $t$ after history $s^t$.

- $g(s^t)$ is government consumption after history $s^t$. Later will assume $g(s^t) = g(s_t)$.

- $k(s^{t-1})$ is capital produced at $t-1$ after $s^{t-1}$ and used at date $t$.

## Production

- Output: $F(k(s^{t-1}), l(s^t), s_t)$.

- $F : \mathbb{R}_+ \times [0, L] \times S \to \mathbb{R}_+$ is CRS in $(k, l)$.

- And satisfies Inada conditions.

- The resource constraints are for each $s^t$:

$$c(s^t) + g(s^t) + k(s^t) = F(k(s^{t-1}), l(s^t), s_t) + (1 - \delta)k(s^{t-1})$$

- $\delta \in [0, 1]$ is depreciation.

# Preferences

- Preferences of a representative consumer:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \mu(s^t) U(c(s^t), l(s^t))$$

- $\beta \in (0,1)$ and $U$ is increasing in $c$, decreasing in $l$, strictly concave, twice continuously differentiable and satisfies Inada conditions.

# A market economy

- The government levies taxes on labor income $\tau(s^t)$ and capital income $\theta(s^t)$.

- Government debt: one period maturity, state contingent return.

- $b(s^t)$ is the amount of debt issued at $s^t$; $R_b(s^{t+1})$ is its state contingent return.

- Household budget constraint:

$$c(s^t)+k(s^t)+b(s^t) \leq [1-\tau(s^t)]w(s^t)l(s^t)+R_k(s^t)k(s^{t-1})+R_b(s^t)b(s^{t-1})$$

- where: $R_k(s^t) = 1 + [1 - \theta(s^t)][r(s^t) - \delta]$ is gross return on capital after taxes and depreciation.

- Debt limit: $b(s^t) \geq -M$

## Prices and Allocations

- $r(s^t)$ and $w(s^t)$ are before tax returns on capital and wages..

- In a competitive equilibrium, these equal marginal products:

$$r(s^t) = F_k(k(s^{t-1}), l(s^t), s_t)$$

$$w(s^t) = F_l(k(s^{t-1}), l(s^t), s_t)$$

- Let $x(s^t) = \{c(s^t), l(s^t), k(s^t)\}$ denote allocation at $s^t$.

- And $x = \{x(s^t)\}$ is entire allocation.

- $(w, r, R_b) = \{w(s^t), r(s^t), R_b(s^t)\}$ is entire price system.

# Government

- The government's budget constraint is:

$$b(s^t) = R_b(s^t)b(s^{t-1}) + g(s^t) - \tau(s^t)w(s^t)l(s^t) - \theta(s^t)[r(s^t) - \delta]k(s^{t-1})$$

- A policy is $\pi = \{\tau(s^t), \theta(s^t)\}$.

- $b_{-1}$ and $k_{-1}$ are given.

# A Competitive Equilibrium

**Definition**: A competitive equilibrium is a policy $\pi$, an allocation $x$ a bond process $b$ and a price system $q = (w, r, R_b)$ such that:

- Given $\pi$ and $q$, $(x, b)$ maximizes the agent's utility subject to the budget constraints and debt limits;

- Firm's first order conditions hold at $x$ (prices equal marginal products);

- Government's budget constraint holds.

$x$ is then said to be a competitive allocation.

*Remark:* Agent and government budget constraints holding ensure goods market clear; firm's first order conditions holding at allocation imply capital and labor markets clear.

# Solving the government's problem

- Government picks $\pi$ at $t = 0$.

- We assume government can *commit* to implementing continuation of $\pi$ in later periods.

- Given $\pi$, there are a set of possible competitive equilibria $(\pi, x, b, q)$ that are possible.

- Under our assumptions (strict concavity, representative agent), there is at most one.

- Could solve:

$$\max_{\pi} W(x(\pi))$$

where: $W(x)$ gives payoff to agent from allocation $x$ and $x(\pi)$ is competitive allocation given policy $\pi$.

# Primal approach

- We take the primal approach.

- Find set of competitive allocations over goods and labor.

- Recover policy that "implements" this allocation.

# The initial period portfolio

- Representative agent holds portfolio of capital and bonds in initial period $(k_{-1}, b_{-1})$

- The payout from this portfolio is: $R_k(s_0)k_{-1} + R_b(s_0)b_{-1}$.

- The "utility value" of this: $\Phi(s_0) = U_c(s_0)[R_k(s_0)k_{-1} + R_b(s_0)b_{-1}]$,

- where: $R_k(s_0) = 1 + (1 - \theta(s_0))(F_k(s_0) - \delta)$.

- We will call allocation $x$ interior if each $c(s^t) > 0$, $k(s^t) > 0$ and $l(s^t) \in (0, L)$.

# Characterising competitive allocations

***Proposition****: Given* $g$, $k_{-1}$, $b_{-1}(s_0)$, $R_b(s_0)$ *and* $\theta_0(s_0)$, *an interior allocation* $x = \{c, l, k\}$ *is a competitive allocation if and only if resource constraints, for all* $s^t$,

$$c(s^t) + g(s^t) + k(s^t) = F(k(s^{t-1}), l(s^t), s^t) + (1 - \delta)k(s^{t-1}) \qquad \text{(rc)}$$

*and implementability constraint hold:*

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \mu(s^t)[U_c(s^t)c(s^t) + U_l(s^t)l(s^t)] = U_c(s_0)[R_k(s_0)k_{-1} + R_b(s_0)b_{-1}].$$

$$\text{(ic)}$$

**Proof:** Similar to static case. □

# The initial period

- The right hand side of (ic) shows something special about initial period.

- Agent's portfolio is given (sunk). Taxing $k_{-1}$

- Government would like to implement lump sum taxation.

- Not allowed to; only linear taxes and they distort.

- EXCEPT tax imposed on initial capital $k_{-1}$ since sunk (determined).

- Taxing it cannot distort, so a tax on initial capital income acts like a lump sum tax.

- Government taxes this at maximal amount (or enough to finance entire stream of government spending).

# Time inconsistency

- In later periods, government always tempted to tax capital at high level...

- ..having promised in earlier periods that it will not.

- Source of time consistency problem. Commitment assumption rules this out.

# Ramsey problem

$$\max_{\theta(s_0),x} \sum_{t=0}^{\infty} \beta^t \sum_{S^t} \mu(s^t) U(c(s^t), l(s^t), s_t)$$

s.t. to (rc), (ic) and initial period constraint.

## Recursive formulation

- Any allocation satisfying:

$$\Phi(s_0) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \mu(s^t)[U_c(s^t)c(s^t) + U_l(s^t)l(s^t)]$$

satisfies:

$$\Phi(s_0) = U_c(s_0)c(s_0) + U_l(s_0)l(s_0) + \beta \sum_{s_1} \Phi(s^1)\mu(s_1|s_0)$$

where:

$$\Phi(s_1) = \sum_{t=0}^{\infty} \beta^t \sum_{s^t \in S^t} \mu(s^t)[U_c(s^t)c(s^t) + U_l(s^t)l(s^t)]$$

- Suggests a recursive formulation for the government's problem.

# Recursive formulation

**Step 1** Solve:

$$V(\Phi, k, s) = U(c, l, s) + \beta \sum_{s'} V(\Phi(s'), k', s')\mu(s'|s)$$

s.t.

$$c + g(s) + k' = F(k, l, s) + (1 - \delta)k$$

and

$$\Phi = U_c c + U_l l + \beta \sum_{s'} \Phi(s')\mu(s'|s)$$

The $\Phi(s')$: portfolio value promises.

# Recursive formulation

**Step 2** Solve:

$$\max Q(R_b, b, k, s) = U(c, l, s) + \beta \sum_{s'} V(\Phi(s'), k', s') \mu(s'|s)$$

s.t. (rc), $\Phi = U_c c + U_l l + \beta \sum_{s'} \Phi(s') \mu(s'|s)$, initial constraint.

# Alternative formulation

- Let:

$$W(c(s^t), l(s^t), \lambda) = U(c(s^t), l(s^t)) + \lambda[U_c(s^t)c(s^t) + U_l(s^t)l(s^t)]$$

- Then:

$$\max \sum_{t=0}^{\infty} \beta^{t-1} \sum_{s^t} \mu(s^t)\{W(c(s^t), l(s^t), \lambda)$$
$$-\lambda U_c(s_0)[R_k(s^0)k_{-1} + R_b(s^0)b_{-1}]\}$$

s.t. (rc) and initial constraint

# Take first order conditions

- Intratemporal:

$$-\frac{W_l(s^t)}{W_c(s^t)} = F_l(s^t)$$

- Intertemporal:

$$W_c(s^t) = \sum_{s^{t+1}} \beta \mu(s^{t+1}|s^t) W_c(s^{t+1})[1 - \delta + F_k(s^{t+1})].$$

# Capital tax implications

- To simplify remove uncertainty.

- First order conditions reduce to:
  - Intratemporal:
  $$-\frac{W_{lt}}{W_{ct}} = F_{lt}$$

  - Intertemporal:
  $$W_{ct} = \beta W_{ct+1}[1 - \delta + F_{kt+1}].$$

- So if optimal allocation achieves steady state, $W_{ct} = W_{ct+1} = W_c$ and
$$1 = \beta[1 - \delta + F_k].$$

# Capital tax implications

- Now optimal allocation can be implemented in economy with taxes.

- Must satisfy agent's first order conditions:

$$U_{ct} = \beta U_{ct+1}[1 + (1 - \theta_{t+1})(F_{kt+1} - \delta)].$$

- In steady state:

$$1 = \beta[1 + (1 - \theta)(F_k - \delta)].$$

- But compare this to first order condition from Ramsey problem. $\theta = 0$.

- **No capital income taxes in steady state!**

# Zero capital tax result

- Due to Judd (1985) and Chamley (1986).

- Natural to conjecture due to representative agent assumption.

- But result still holds if two agents, worker (no capital income) and capitalist (no labor income).

- Even if government cares only about workers, should not use capital income taxes in steady state.

- Capital income taxes distort capital accumulation, they lower marginal product of labor.

- Make workers sufficiently badly off they are better off with steady state labor income.

## Outside of steady state

- In some cases we can show that after period 0 capital taxes are not used.

- The use of capital taxes is equivalent to the use of consumption taxes.

- The agent's first order conditions with consumption taxes are:

$$(1 - q_t)U_{ct} = \beta(1 - q_{t+1})U_{ct+1}[1 + (F_{kt+1} - \delta)].$$

- Use uniform commodity taxation arguments to show $1 - q_t = 1 - q_{t+1}$, $t > 0$.

- So:.

$$U_{ct} = \beta U_{ct+1}[1 + (F_{kt+1} - \delta)].$$

# Back in steady state

- Zero capital taxes in steady state is robust result.

- Emerges in (dynamically efficient) OLG economies provided age dependent taxes are available.

- Optimal positive capital taxes re-emerge if there are further restrictions on tax systems.

- Example: Conesa, Kitao, Krueger (AER, 2009).

## Other assets

- The basic logic extends to other assets.

- If labor taxes are the return on a human capital asset, then zero labor taxes in long run!

- No taxes on balanced growth path (Jones, Manuelli, Rossi (JET, 1997).

- Related result in monetary policy. Positive nominal interest rates imply that money is taxed.

- Intuition: Fisher equation: $r = \rho + \pi$. If $r = 0$, then real return on money$= -\pi = \rho =$ real return on bonds.

- Prescription for zero nominal interest rates (Friedman Rule).

# A puzzle

- Capital taxes are used.

- Nominal interest rates are not zero.

- Bad policy? Or is something missing from the basic Ramsey model?

## Dividend taxation

- So far have assumed firm returns all revenues net of labor costs and all undepreciated capital.

- Returns $1 + F_k - \delta$. Taxes levied on $F_k - \delta$.

- Alternatively, suppose firm self finances:

$$K_{t+1} = F(K_t, L_t) + (1 - \delta)K_t - w_t L_t - d_t$$

  where $d_t$ dividends.

- Suppose dividends are taxed. Then tax does not distort capital accumulation. Imitates a lump sum tax on ownership.

- Similar results if investment taxes are expensed (tax deductible). E.g. Abel (2007)

- President Bush's (2003) dividend tax cut?

**Ramsey implications for debt policy**

# LECTURE 3: DYNAMIC RAMSEY THEORY: DEBT MANAGEMENT

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# Dynamic Ramsey Models Policy Prescriptions

- Previously we talked about the Ramsey policy implications for asset and labor income taxes.

  - Ramsey policy summary: smooth distortions over states and time.

  - Tax rates on labor and consumption should be roughly zero.

  - Zero capital income taxation in steady state.

  - Monetary policy such that nominal interest rates close to zero.

- Question: How can the government finance fiscal shocks then?

# Implications for Debt Management

- Explicit state-contingent debt: debt returns vary with fiscal shocks.

- Non-contingent debt with taxes on interest income that vary with shocks.

- Partially default on debt during periods of high expenditure, not default in times of low expenditure.

# Using Debt to Absorb Shocks

- Illustrate role of debt in economy with no capital.

- Linear technology: $F(k, l, z) = zl$ where $z$ is TFP shock.

- Resource constraint before:

$$c(s^t) + g(s^t) + k(s^t) = F(k(s^{t-1}), l(s^t), s_t) + (1 - \delta)k(s^{t-1})$$

- Resource constraint now:

$$c(s^t) + g(s^t) = z(s^t)l(s^t)$$

# Consumers

- FOC for labor supply:

$$-\frac{U_l(s^t)}{U_c(s^t)} = (1 - \tau(s^t))z(s^t).$$

- FOC for debt:

$$U_c(s^t) = \sum_{s^{t+1}} \beta\mu(s^{t+1})U_c(s^{t+1})R_b(s^{t+1})/\mu(s^t)$$

# Implementability

- Combine resource and FOC for consumer:

$$U_c(s^t)c(s^t) + U_l(s^t)l(s^t) = U_c(s^t)[\tau(s^t)z(s^t)l(s^t) - g(s^t)].$$

- Left hand side: $U_c(s^t)c(s^t) + U_l(s^t)l(s^t)$
  value of government surplus at $s^t$ in units of marginal utility.

- Implementability

$$\sum_{t,s^t} \beta^t \mu(s^t)[U_c(s^t)c(s^t) + U_l(s^t)l(s^t)] = U_c(s_0)R_0 b_{-1}.$$

# Contingent taxes on returns

- Necessary condition for allocations to be a part of competitive equilibrium

  - Resource constraint

  - Implementability constraint.

- Non-contingent return on debt: $\bar{R}(s^{t-1})$

- State contingent tax on returns: $(1 - \nu(s^t))$

- Therefore:
$$R_b(s^t) = [1 - \nu(s^t)]R(s^{t-1})$$

# Ramsey problem

- FOC for $t \geq 1$

$$z(s^t)U_c(s^t) + U_l(s^t) + \lambda[z(s^t)H_c(s^t) + H_l(s^t)] = 0$$

where

- $H_c(s^t) = U_{cc}(s^t)c(s^t) + U_c(s^t) + U_{cl}l(s^t)$
- $H_l(s^t) = U_{cl}(s^t)c(s^t) + U_l(s^t) + U_{ll}l(s^t)$

- For $t = 0$,

$$z(s^t)U_c(s^t) + U_l(s^t) + \lambda[z(s^t)H_c(s^t) + H_l(s^t)]$$
$$= \lambda[z(s_0)U_{cc}(s_0)][1 - \nu(s_0)]R_1 b_{-1}$$

# Ramsey problem: A result

**Proposition**    For $t \geq 1$, there exist functions, $\bar{c}$, $\bar{l}$ and $\bar{\tau}$ such that the Ramsey consumption and labor allocations, and labor tax rates can be written as

$$c(s^t) = \bar{c}(g_t, z_t), \ \ l(s^t) = \bar{l}(g_t, z_t), \ \ \tau(s^t) = \bar{\tau}(g_t, z_t).$$

Moreover, if $b_{-1} = 0$, then $c(s_0), l(s_0)$ and $\tau(s_0)$ are given by the same functions.

- Allocations and labor tax rates depend only on the current shocks and not on the history of shocks.

- Labor tax rates inherit stochastic properties of underlying shocks.

- If government consumption is persistent, so are labor tax rates.

- If government consumptions is iid, so are labor tax rates. (constant technology)

- Constrast with literature (Barro (79), Mankiw (87)) that tax rates should follow a random walk.

# Understanding Ramsey outcomes: Example 1

- Deterministic example. $z(s^t) = 1$ for all $s^t$.

- Economy alternating between peacetime (odd $t$) and wartime (even $t$). $g_t = G$ when wartime, equal to 0 when peacetime.

- Initial debt: $R_{-1}b_{-1} = 0$

- Using FOC for Ramsey problem + resource constraint:

$$(1 + \lambda)[U_c(0) + U_l(0)] + \lambda c[U_{cc}(0) + 2U_{cl}(0) + U_{ll}(0)] = 0$$

where partials are evaluated at $g_t = 0$.

Example 1 continued

$$(1 + \lambda)[U_c(0) + U_l(0)] + \lambda c[U_{cc}(0) + 2U_{cl}(0) + U_{ll}(0)] = 0$$

- Strict concavity $\implies [U_{cc}(0) + 2U_{cl}(0) + U_{ll}(0)] < 0$.

- Since $\lambda > 0$, then $[U_c(0) + U_l(0)] > 0$.

- And labor supply FOC $\implies U_c + U_l = \tau U_c$.

- Then $\tau(0) > 0$.

- Proposition and implementability condition imply 2-period balance budget:
$$U_c(G)[\tau(G)l(G) - G] + \beta U_c(0)\tau(0)l(0) =$$

- Surplus in peacetime (second term +) $\implies$ deficit in wartime (first term -)

## Example 1 conclusion

- Sell debt $b(G) = G - \tau(G)l(G)$ in wartime

- Pay debt in peacetime.

- Gross return on debt from wartime to peacetime:

$$R(G) = U_c(G)/\beta U_c(0)$$

- Tax rate on debt always zero.

- Environment same as Example 1, except...

- Recurrent wars with long peacetime in between.

- $g_t = G$ for $t = 0, T, 2T, \cdots$, $g_t = 0$ otherwise.

- Initial debt 0.

# Example 2 continued

- Budget balanced over T-period cycle:

$$U_c(G)[\tau(G)l(G) - G] + \beta U_c(0)\tau(0)l(0) + \cdots + \beta^{T-1}U_c(0)\tau(0)l(0) = 0$$

- Run deficit in wartime, a constant surplus in peacetime.

- War debt slowly retired after $T - 1$ periods of peace.

Example 2 conclusion

Decreasing debt in peacetime:

- Sell $G - \tau(G)l(G)$ in wartime.

- 1st period of peacetime sell:

$$[U_c(G)/\beta U_c(0)][G - \tau(G)l(G)] - \tau(0)l(0)$$

- 2nd period of peacetime sell:

$$[U_c(G)/\beta^2 U_c(0)][G - \tau(G)l(G)] - [\tau(0)l(0)/\beta] - \tau(0)l(0)$$

- so on....

- Debt management with stochastic government spending: Example 3.

- Debt management with incomplete markets:
  - What does debt management look like when neither state contingent debt nor state contingent taxes on returns are available?

- Debt management in practice.

# Debt as a shock absorber: Example 3

- Stochastic government spending.

- Spending follow a 2-state Markov process.
  - States $g_t = G$ and $g_t = 0$
  - $\pi = Prob(g_{t+1} = G | g_t = G) = Prob(g_{t+1} = 0 | g_t = 0) > 1/2$
  - Probability of staying in same state larger than a switch.

- Initial conditions: $g_0 = G$ and $R_{-1}b_{-1} > 0$.

# Lecture 4: Debt Management in Theory and Practice

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July, 2011

- Explicit state-contingent debt: debt returns vary with fiscal shocks.

- With state contingent debt, labor tax rates inherit stochastic properties of underlying shocks.

- Allocations and labor tax rates depend only on the current shocks and not on the history of shocks.

- Partially default on debt during periods of high expenditure, not default in times of low expenditure.

# State Contingency

- Non-contingent return on debt: $\bar{R}(s^{t-1})$

- State contingent tax on returns: $(1 - \nu(s^t))$

- Therefore after tax return is:

$$R_b(s^t) = [1 - \nu(s^t)]R(s^{t-1})$$

- Government and consumers care about after-tax return on debt.

## Determining After Tax Returns

- Multiply consumer BC by $\beta^t \mu(s^t) U_c(s^t)$ and sum over all periods and states after $r$

$$\beta^r U_c(s^r)[1 - \nu(s^r)]R(s^{r-1})b(s^{r-1})$$
$$= \beta^r \mu(s^r)H(s^r) + \sum_{t=r+1}^{t} \sum_{s} \beta^t \mu(s^t)H(s^t)$$

where

$$H(s^t) = U_c(s^t)c(s^t) + U_l l(s^t)$$

- LHS: value of after tax debt obligation.

- RHS: Expected PDV of government surpluses.

- After tax returns are determined by equation above.

- What about gross returns and taxes on returns?

# Determining After Tax Returns

- Gross returns and tax on returns cannot be separately determined.

- Multiple $\nu$ and $R$ that satisfy the same after tax return. Use a normalization to resolve indeterminacy.

$$\bar{R}(s^{t-1}) = [\mu(s^{t-1})U_c(s^{t-1})/\sum \beta\mu(s^t)U_c(s^t)$$

- Normalization implies average tax rates on debt returns is 0:

$$\sum_{s^t} \mu(s^t|s^{t-1})U_c(s^t)\nu(s^t) = 0$$

for all $t$ and $s^{t-1}$.

- Stochastic government spending.

- Spending follow a 2-state Markov process.

  - States $g_t = G$ and $g_t = 0$

  - $\pi = Prob(g_{t+1} = G | g_t = G) = Prob(g_{t+1} = 0 | g_t = 0) > 1/2$

  - Probability of staying in same state larger than a switch.

- Initial conditions: $g_0 = G$ and $R_{-1} b_{-1} > 0$.

# Example 3 continued

Government period BC:

$$b(s^t) = [1 - \nu(s^t)]\bar{R}(s^{t-1})b(s^{t-1}) + g(s^t) - \tau(s^t)l(s^t)$$

- End of period debt $b(s^t)$ and interest rate $R(s^t)$ depend on current shock only. (Result from previous lecture)

- Tax rate $v(s^t)$ depends on current and previous period shock.

- Hence end of period debt, interest rate and tax on debt returns:

$$b(g_t), \ R(g_t), \ \nu(g_{t-1}, g_t)$$

- Remember that the Euler equation depends on where you were last period and where you will be this period.

## Example 3: A result

- Suppose that in the solution to the Ramsey problem:

  1. $H(0) > H(G) > 0$; value of govt surplus higher in peacetime
  2. $b(G), b(0) > 0$: debt is positive
  3. $U_c(G) > U_c(0)$

- Then
$$\nu(0, G) > \nu(G, G) > 0 > \nu(0, 0) > \nu(G, 0)$$

- Tax rates are most extreme in periods of transition.

- Debt is taxed in wartime and subsidized in peacetime.

- INTERSTATE smoothing!

## Intuition

- Ramsey policy smooths labor tax rates across states and time.

- Smoothing $\implies$ smaller surplus in wartime than in peacetime.

- With persistent shocks (Markov process), expected PDV of surpluses starting next period is smaller if currently in wartime compared to peacetime.

- End of period debt $b(g_t) =$ expected PDV of surpluses, so smaller in wartime then in peacetime. $b(G) < b(0)$.

- $R(G)b(G) < R(0)b(0)$: debt obligations smaller if there was war last period, rather than peace.

- Suppose it's wartime today: $g_t = G$.

- If debt inherited is higher, tax debt at higher rate.

- If last period was peace but war now, debt inherited is high, tax high.

- If last period was war and still war now, tax low.

- Symmetrically for other cases.

# Asset Markets and Debt Management

- In an environment with uncertainty, properties of optimal policy depend on asset market structure.

- When markets are complete, optimal tax rates are constant across states and time. Intertemporal and interstate smoothing.

- When markets are incomplete (Aiyagari, Marcet, Sargent, Seppala (2002)
    - Analysis is much more complicated.
    - Results depend on the details of the incompleteness.
    - Interstate smoothing is not possible, only intertemporal smoothing.
    - Tax rates show more persistence, regardless of the $g$ shock process.
    - A debt limit can emerge endogenously and near this limit, policy calls for fiscal consolidation, which can involve a labor tax rate hike.

- Ongoing active research area: Optimal debt management when markets are incomplete.

# Asset Markets and Debt Management

How can we deliver state contingency?

- Issue non-contingent *nominal* debt and use inflation to deliver state-contingency.
    - Works well, delivers complete markets outcomes.
    - Problem: Anticipated inflation vs. unanticipated inflation. If there are further costs associated with latter, it may not be desirable to hedge with state contingent inflation.
- Issue non-contingent *real* debt, but with multiple maturities.
    - Different maturities of debt help "complete" markets. (Angeletos (2002), Lustig, Sleet, Yeltekin (2008))
    - Problems: Optimal debt portfolio may be very extreme both in size and in structure.
- Open area of research: Optimal debt portfolio composition.

Positive questions:

- **Question 1**: What are the fiscal channels that stabilize fiscal balances after spending shocks? How do we quantify them?

- **Question 2**: How much fiscal insurance do bond markets provide?

- **Question 3**: Can the maturity composition of govt debt be altered to increase fiscal insurance and hence financing of fiscal shocks?

# Fiscal Management in Practice

Figure: Debt/GDP and Average Maturity: US
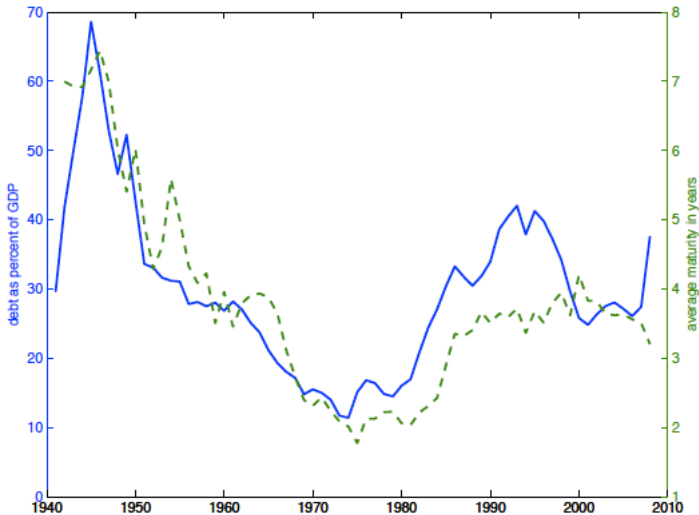
# Fiscal Management in Practice

- Debt/GDP ratio shows periods of build-up and periods of run-down of debt.

- Largest increases: WWII, Cold War.

- Followed by periods of debt obligation decline.

- Evidence of *intertemporal* smoothing.

# Fiscal Shocks

- Fiscal Insurance: Ramsey theory suggest spending/fiscal shocks should be paid by lower returns on debt.

- Theory: The extent of fiscal insurance (interstate smoothing) depends on asset market structure.

- Practice: How much fiscal insurance is there in reality?

- Measuring fiscal insurance requires defining/measuring fiscal shocks.

- One definition of fiscal shocks: Unanticipated war expenses, Ramey (1999, 2011)

- What happens to defense spending, returns on government debt and surpluses after such a shock?

# Fiscal Shocks



Figure: Defense Shocks and Defense Spending
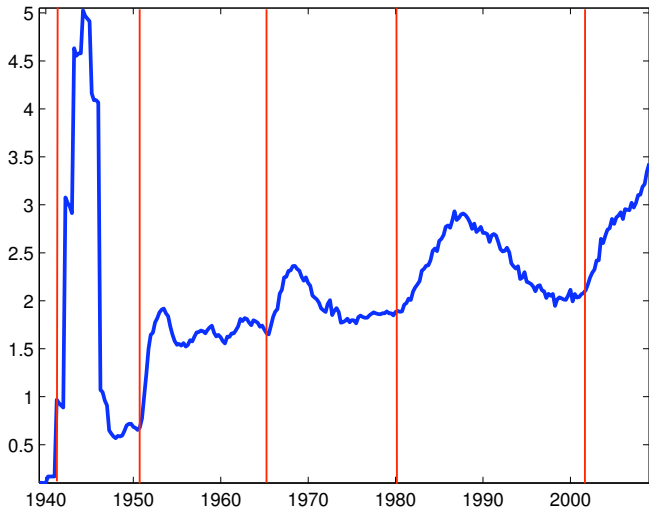
# Empirical approach

Berndt, Lustig, Yeltekin (2011) (Forthcoming, AEJ Macro)

- Step 1: Decompose budget constraint to isolate response to fiscal shocks into 2 channels:

  - $\underbrace{\text{unexpected changes to current and future returns}}_{\textbf{fiscal insurance/debt valuation channel}}$

  - $\underbrace{\text{unexpected changes to current and future fiscal surpluses}}_{\textbf{surplus channel}}$

- Fiscal shocks identified as news to current and future defense spending growth.

# Step 2: Quantifying fiscal adjustment channels

- Use unstructured VARs to estimate news to spending, debt returns and surpluses.

- Measure response of surpluses and debt returns to fiscal shocks.

- Recover fiscal insurance measure:

  - Correlation between
    - Innovations to spending and
    - Innovations to returns

## Government Budget Constraint

- Period by period version of govt dynamic budget constraint

$$B_{t+1} = R_{t+1}^b \left( B_t - S_t \right)$$

  - $B_t$ : real **market** value of outstanding debt at beginning of $t$,
  - $S_t = T_t - G_t$: federal government's real primary surplus,
  - $R_{t+1}^b$: real gross holding return on govt debt between $t$ and $t+1$.

- Growth rate of debt

$$\frac{B_{t+1}}{B_t} = R_{t+1}^b \left( 1 - \frac{S_t}{B_t} \right)$$

- Log-linearize BC to get

$$
\begin{aligned}
(E_{t+1} - E_t)\Delta ns_{t+1} &= (E_{t+1} - E_t)\sum_{j=0}^{\infty} \rho^j r_{t+1+j}^b \\
&- (E_{t+1} - E_t)\sum_{j=1}^{\infty} \rho^j \Delta ns_{t+1+j}
\end{aligned}
$$

- Negative shock to (weighted log) surplus growth today corresponds to
  - negative shock to current and future returns, or
  - positive shock to future surplus growth.

# Fiscal Shocks and Fiscal Adjustment Channels

$$(E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta g_{t+j+1}^{def} =$$

$$- \frac{1}{\mu_g^{def}} \left( (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j r_{t+j+1}^b \right)$$

$$+ \frac{1}{\mu_g^{def}} \left( (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta ns_{t+j+1}^{ndef} \right)$$

**Fiscal Shocks**

$$(E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta g_{t+j+1}^{def} =$$

$$- \frac{1}{\mu_g^{def}} \left( (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j r_{t+j+1}^b \right)$$

$$+ \frac{1}{\mu_g^{def}} \left( (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta n s_{t+j+1}^{ndef} \right)$$

Debt Valuation Channel: Adjustments to debt returns

$$(E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta g_{t+j+1}^{def} =$$

$$-\frac{1}{\mu_g^{def}} \left( (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j r_{t+j+1}^b \right)$$

$$+\frac{1}{\mu_g^{def}} \left( (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta n s_{t+j+1}^{ndef} \right)$$

Surplus Channel: Adjustments to non-defense surplus growth

$$(E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta g_{t+j+1}^{def} =$$

$$- \frac{1}{\mu_g^{def}} \left( (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j r_{t+j+1}^b \right)$$

$$+ \frac{1}{\mu_g^{def}} \left( (E_{t+1} - E_t) \sum_{j=0}^{\infty} \rho^j \Delta n s_{t+j+1}^{ndef} \right)$$

## Our approach in context

Light demands on theory:

- No assumptions about govt. objective.

- No assumptions about private agent behavior.

- No assumptions on market completeness.

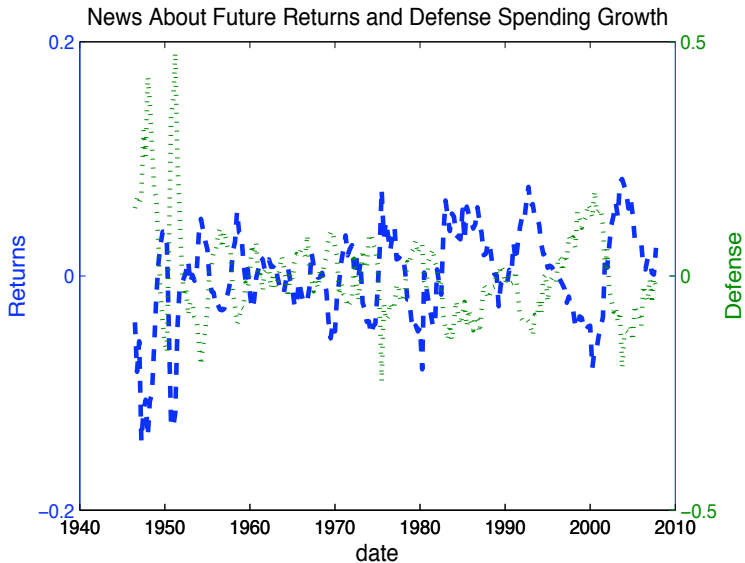- In contrast to empirical fiscal policy literature.

## Summary of findings

- About 10% of fiscal shocks in postwar era financed by debt markets.

- More than 73% by increases in surpluses.

- Future returns:
  - Investors accept low expected returns after fiscal shock
  - Missing from normative models

- Policy in practice looks more like Ramsey with incomplete markets rather than Ramsey with complete markets.

# Fiscal Shocks and Current Returns on Debt



News About Current Returns and Defense Spending Growth
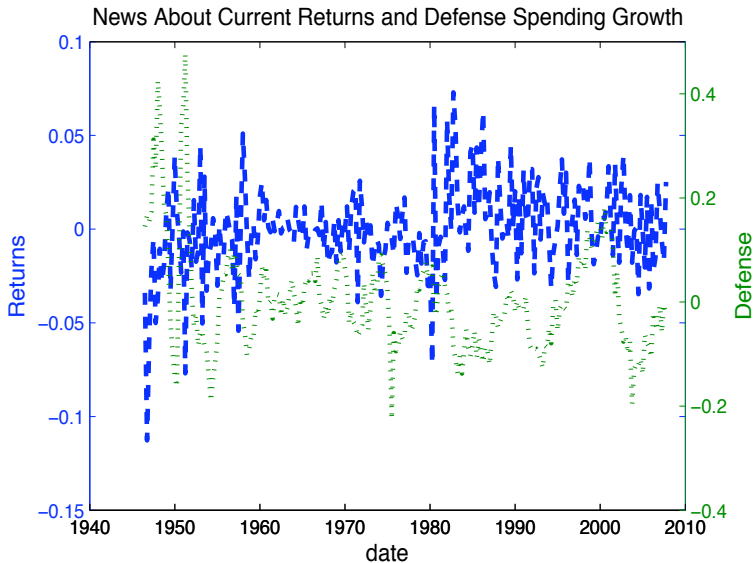
# Fiscal Shocks and Future Returns on Debt



News About Future Returns and Defense Spending Growth

# Fiscal Hedging and Maturity of Debt

- Average return and volatility of LT debt higher than ST debt
  - Campbell (1995): shorten maturity when yield curve is steep to minimize borrowing costs.
  - Barro (1997): shorten maturity when LT debt returns are volatile to tax smooth.

- Both ignore fiscal insurance provided by LT debt.
  - Lustig-Sleet-Yeltekin (2008) : optimal debt management calls for issuing long term debt only

# Real Bond Holding Returns by Maturity

This table reports the average quarterly real holding returns (in percentage terms) on bonds of different maturities (in years). Standard deviations are reported in parentheses. Zero-coupon yield curves are constructed from CRSP data. The sample period is 1946.I-2008.III.

| Maturity | 1 | 5 | 10 | 15 | 20 |
|---|---|---|---|---|---|
| Mean (%) | 0.30 | 0.43 | 0.48 | 0.46 | 0.38 |
| Std dev (%) | (1.52) | (3.58) | (5.74) | (9.78) | (18.05) |

## Maturity Structure and Hedging

- LT debt is more effective at absorbing fiscal risk

- Fraction of expenditure shocks financed:

    - For one year debt: *7.49%*
    - For 20 year debt: *17.22%*

# Maturity Structure of Publicly Held Debt

This plot shows the face value weighted and market value weighted maturity (in years) of publicly held debt between 1939.I-2008.III. The vertical dotted line marks 1946.I, the beginning of the sample period for our empirical analysis.
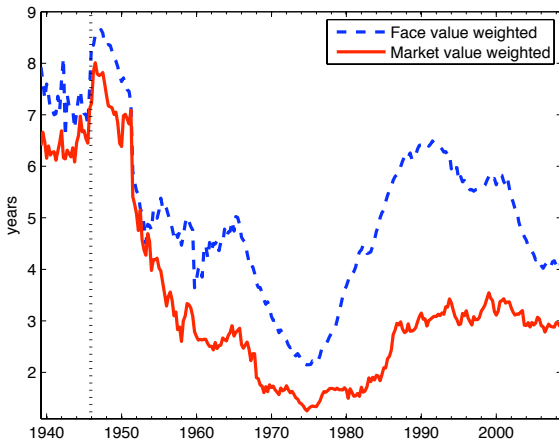
# Conclusion

- Debt management is an emerging, active field in macro policy.

- Theory suggests that interstate smoothing depends on the assumptions regarding asset market structure.

- Empirical work suggests use of bonds markets to hedge fiscal shocks is not large, but it's not negligible either.

- Maturity structure of debt does affect the amount of fiscal insurance.

- More work on completeness of asset markets and on the relationship between maturity and hedging should be done before a solid policy prescription about debt/maturity management emerges.

# LECTURE 5: MIRRLEESIAN THEORY

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June, 2011

# Problems with the Ramsey Models

- In Ramsey models the government is restricted to linear taxes (tax contracts).

- In many Ramsey models, these taxes are the only friction.

- Results driven by government's desire to mitigate this friction.

- By assumption the government cannot use lump sum taxes.

- Many macro-models (esp. older ones) assume all agents are the same.

# Problems with Ramsey framework

- Why should we assume government's use linear taxes? They often do not in practice.

- Better to impose assumptions on "primitives" and derive optimal contract than to impose assumptions directly on contract.

- What are the primitive frictions?

- Key one informational. Agents are different, government cannot observe these differences.

## Mirrlees models

- This approach begins with Mirrlees (1971) (for which one the Nobel prize)

- Active area of research in public economics in 1970's.

- Helped initiate contract theory, key contribution to informational economics..

- Mirrlees model static.

- Dynamic private information models: Green (1987), Phelan-Townsend (1991), Atkeson-Lucas (1992).

- Application: unemployment insurance (Hopenhayn-Nicolini (1997)).

- Application to tax design: Albanesi-Sleet (2006), Kocherlakota (2005).

# Basic framework

- Government faces large population of agents.

- Agents draw private, idiosyncratic skill shocks $\theta$.

- Output given by product of skill and effort: $y = \theta n$.

- Agent's output $y$ is publicly observable.

- In dynamic model skill shocks drawn over lifetime according to some process.

# Static labor tax results

- Optimal labor income tax non-linear.

- Tied to underlying skill distribution.

- Early assumptions on skill distribution implied zero marginal income taxes at the top.

- More recent assumptions (Diamond (1998), Saez (2001)) generate progressive and U-shaped marginal tax schedules.

# Dynamic capital tax results

- It is optimal to distort (deter) savings, even in the long run.

- This should not (necessarily) be done with positive expected capital taxes.

- But with capital taxes that covary negatively with agent consumption.

- Logic: higher wealth deters effort, makes provision of incentives more costly.

- Deterring savings with positive capital tax leads to "double deviations"; save too much, work too little.

# Dynamic estate tax results

- How should estates be taxed?

- If society cares more about children than parents.

- It cares about parents and since parents care about children, it cares about them. It also cares about them in their own right.

- Optimal policy: progressive estate subsidies.

- Subsidies: encourage bequests.

- Progressive: dilute inequality amongst children.

# Dynamic labor taxes

- These should respond to aggregate shocks.

# Credibility/ time consistency results

- If private shocks are persistent, then information is an asset.

- Government tempted to misuse this ex post (knows who is productive).

- Even if shocks are i.i.d., inequality increases over time. Immiseration.

- Can government commit to implementing ever increasing levels of inequality?

- If not progressive capital taxes that damp inequality.

# Primal approach

- Rather than figuring out mapping from taxes to allocations to payoffs.

- We take a "primal approach": find set of feasible allocations.

- Back out taxes.

# Basic mechanism design

- Agent has preferences over actions $a \in \mathscr{A}$ given shock $\theta$: $u(a, \theta)$.

- Shocks are private.

- Principal (government) supplies mechanism: $(\mathscr{M}, f : \mathscr{M} \to \mathscr{A})$

- Agents solve:

$$\max_{\mathscr{M}} u(\theta, f(m))$$

  Delivers agent policy: $m^* : \Theta \to \mathscr{M}$.

- Government payoff:

$$U(\mathscr{M}, f) = E[u(\theta, f(m^*(\theta)))]$$

- Government maximizes over feasible mechanisms. Resource constraints.

# Revelation principle

- All feasible allocations can be implemented with:

  1. direct mechanism $\mathscr{M} = \Theta$

  2. that gives agents incentives to truth-fully report:

$$u(\theta, f(\theta)) \geq u(\theta, f(\theta')).$$

- Revelation principle derived in various places in 1970's.

- Enables us to pin down feasible allocations.

# Static case

- Suppose agent has preferences over consumption and effort:

$$u(c) - v(n)$$

$u$ increasing, concave, differentiable; $v$ increasing, convex, differentiable.

- Output: $y = \theta n$. Probability distribution: $F(\theta)$.

- So:

$$U(c, y, \theta) = u(c) - v\left(\frac{y}{\theta}\right)$$

## Static case

- Preferences:
$$U(c, y, \theta) = u(c) - v\left(\frac{y}{\theta}\right)$$

- These preferences property:
$$\frac{\partial^2 U}{\partial y \partial \theta} = v''(y/\theta)\frac{y}{\theta^2} + v'(y/\theta)\frac{1}{\theta^2} > 0,$$

   marginal cost of producing lower for more productive.

- and single crossing property:
$$\frac{d}{d\theta}\left[\frac{U_y(c, y, \theta)}{U_c(c, y, \theta)}\right] < 0.$$

   indifference curves flatter for more productive types.

# First result

- Given government spending $G$, an allocation $c : \Theta \to \mathbb{R}_+$, $y : \Theta \to \mathbb{R}_+$ is implementable if

  **1** $\forall \theta, \theta',\; u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right) \geq u(c(\theta')) - v\left(\frac{y(\theta')}{\theta}\right);$

  **2** $G + \int_\theta c(\theta) dF(\theta) \leq \int_\theta y(\theta) dF(\theta).$

- Government's mechanism design (MD) problem:

$$\max \int_\theta \left[ u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right) \right] dF(\theta)$$

  over set of implementable allocations.

# A market economy

- Government levies taxes $T$.

- A competitive equilibrium with taxes is a triple $(c^*, y^*, T)$ such that:

  1. $(c^*, y^*)$ solves:

     $$\max u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right)$$

     subject to:

     $$c(\theta) + T(y(\theta)) \leq y(\theta).$$

  2. Government satisfies budget constraint:

     $$G = \int_\theta T(y(\theta)) dF(\theta)$$

# Taxation principle

- Hammond's taxation principle: Any solution to MD problem can be implemented as a competitive allocation in economy with taxes.

- If $(c^*, y^*)$ solves MD, then choose taxes so that:

$$T(y^*(\theta)) = y^*(\theta) - c^*(\theta) \tag{1}$$

- What about $y \neq y^*(\theta)$ for any $\theta$?

# First result

- Implementable mechanisms are monotone: $c$ and $y$ must be increasing in $\theta$.

- Hard to deal with incentive constraints.

- If mechanism is differentiable, agents' first order condition:

$$u'(c(\theta))c'(\theta) - v'(y(\theta)/\theta)y'(\theta)/\theta = 0. \qquad \text{(FOC)}$$

and second order condition:

$$y'(\theta) \geq 0. \qquad \text{(SOC)}$$

- Solve optimal control problem:

$$\max \int_\theta \left[ u(c(\theta)) - v\left(\frac{y(\theta)}{\theta}\right) \right] dF(\theta)$$

subject to: (FOC), (SOC) and resource constraints.

## Quasi-linear case

- In general, Mirrlees problem difficult.

- But quasi-linear case $U(c, y, \theta) = c - v(ytheta)$.

- In this case, tax formula:

$$\frac{T'(y(\theta))}{1 - T'(y(\theta))} = \left(1 + \frac{1}{\varepsilon(\theta)}\right) \frac{1 - F(\theta)}{f(\theta)}$$

Marginal taxes tied to:

1. Labor distortion (labor supply elasticity).

2. "Number" of agents at $\theta$, $f(\theta)$.

3. Number of agents above $\theta$, $1 - F(\theta)$.

- What is $F$? (Saez, RES 2001).

# Dynamic case

- Two period, two shock (to keep life simple!)

- Let $U(c, y, \theta) = u(c) - v(y/\theta)$.

- Preferences:

$$V(c, y) := \sum_{\theta_1} [U(c_1(\theta_1), y_1(\theta_1), \theta_1)$$
$$+ \beta \sum_{\theta_2} [U(c_2(\theta_1, \theta_2), y_2(\theta_1, \theta_2), \theta_2)] p(\theta_2|\theta_1)] p(\theta_1)$$

## Dynamic case: constraints

- Incentive constraints:

$$U(c_1(\theta_1), y_1(\theta_1), \theta_1) + \beta \sum_{\theta_2}[U(c_2(\theta_1, \theta_2), y_2(\theta_1, \theta_2), \theta_2)]p(\theta_2|\theta_1) \geq$$

$$U(c_1(\theta_1'), y_1(\theta_1'), \theta_1) + \beta \sum_{\theta_2}[U(c_2(\theta_1', \theta_2), y_2(\theta_1', \theta_2), \theta_2)]p(\theta_2|\theta_1)$$

(IC1)

and

$$U(c_2(\theta_1, \theta_2), y_2(\theta_1, \theta_2), \theta_2) \geq U(c_2(\theta_1, \theta_2'), y_2(\theta_1, \theta_2'), \theta_2). \quad \text{(IC2)}$$

- Resource constraints:

$$\sum_{\theta_1} c_1(\theta_1)p(\theta_1) + G \leq \sum_{\theta_1} y_1(\theta_1)p(\theta_1) \quad \text{(RC1)}$$

$$\sum_{\theta_1, \theta_2} c_2(\theta_1, \theta_2)p(\theta_2|\theta_1)p(\theta_1) + G \leq \sum_{\theta_1, \theta_2} y_2(\theta_1, \theta_2)p(\theta_2|\theta_1)p(\theta_1).$$

(RC2)

- The problem is:

$$\max V(c, y)$$

s.t. (IC1), (IC2), (RC1), (RC2).

# Inverted Euler equation

- First order conditions yield:

$$\frac{1}{u'(c(\theta_1))} = \frac{1}{\beta R} \sum_{\theta_2} \left[ \frac{1}{u'(c_2(\theta_1, \theta_2))} \right] p(\theta_2|\theta_1) \qquad \text{(IEE)}$$

  where $R$ is the ratio of shadow prices on resource constraints.

- Compare with Euler equation!

- Function $1/x$ is strictly convex for $x > 0$ and so if $z$ is positive random variable: $E[1/z] > 1/E[z]$.

- So:

$$u'(c(\theta_1)) < \beta R \sum_{\theta_2} u'(c_2(\theta_1, \theta_2)) p(\theta_2|\theta_1)$$

## Inverted Euler equation

- Due to Diamond-Mirrlees (1978), Rogerson (1987).

- Generalized by Golosov, Kocherlakota, Tsyvinski (2003).

- who first considered its implications for taxes.

- Suggests we use should use a tax system with a positive capital tax so that:

$$u'(c(\theta_1)) = \beta R(1 - \tau) \sum_{\theta_2} u'(c_2(\theta_1, \theta_2)) p(\theta_2 | \theta_1).$$

- But implications for taxes subtle. Need a tax system such that:

$$u'(c(\theta_1)) = \beta R \sum_{\theta_2} (1 - \tau(y_1(\theta_1), y_2(\theta_1, \theta_2))) u'(c_2(\theta_1, \theta_2)) p(\theta_2 | \theta_1).$$

- Can be chosen so that: $\sum_{\theta_2} \tau(y_1(\theta_1), y_2(\theta_1, \theta_2)) p(\theta_2 | \theta_1) = 0$.