MATHCOUNTS TOOLBOX

Facts, Formulas and Tricks

I. **PRIME NUMBERS** from 1 through 100 (**1 is not prime!**)

2		3		5		7
11		13		17		19
		23		29		
		31		37		
	41		43		47	
		53		59		
		61		67		
	71		73		79	
		83		89		
			97			

II.

FRACTIONS	DECIMALS	PERCENTS
$^1/_2$	.5	50 %
$^1/_3$	$.\overline{3}$	$33.\overline{3}$ %
$^2/_3$	$.\overline{6}$	$66.\overline{6}$ %
$^1/_4$	.25	25 %
$^3/_4$	.75	75 %
$^1/_5$	.2	20 %
$^2/_5$	.4	40 %
$^3/_5$	.6	60 %
$^4/_5$	.8	80 %
$^1/_6$	$.1\overline{6}$	$16.\overline{6}$ %
$^5/_6$	$.8\overline{3}$	$83.\overline{3}$ %
$^1/_8$	.125	12.5 %
$^3/_8$	.375	37.5 %
$^5/_8$	.625	62.5 %
$^7/_8$	.875	87.5 %
$^1/_9$	$.\overline{1}$	$11.\overline{1}$ %
$^1/_{10}$	.1	10 %
$^1/_{11}$	$.\overline{09}$	$9.\overline{09}$ %
$^1/_{12}$	$.08\overline{3}$	$8.\overline{3}$ %
$^1/_{16}$	.0625	6.25 %
$^1/_{20}$	.05	5 %
$^1/_{25}$	.04	4 %
$^1/_{50}$	.02	2 %

III. PERFECT SQUARES AND PERFECT CUBES

$1^2 = 1$	$2^2 = 4$	$3^2 = 9$	$4^2 = 16$	$5^2 = 25$
$6^2 = 36$	$7^2 = 49$	$8^2 = 64$	$9^2 = 81$	$10^2 = 100$
$11^2 = 121$	$12^2 = 144$	$13^2 = 169$	$14^2 = 196$	$15^2 = 225$
$16^2 = 256$	$17^2 = 289$	$18^2 = 324$	$19^2 = 361$	$20^2 = 400$
$21^2 = 441$	$22^2 = 484$	$23^2 = 529$	$24^2 = 576$	$25^2 = 625$

$1^3 = 1$	$2^3 = 8$	$3^3 = 27$	$4^3 = 64$	$5^3 = 125$
$6^3 = 216$	$7^3 = 343$	$8^3 = 512$	$9^3 = 729$	$10^3 = 1000$

IV. SQUARE ROOTS

$\sqrt{1} = 1$	$\sqrt{2} \approx 1.414$	$\sqrt{3} \approx 1.732$	$\sqrt{4} = 2$	$\sqrt{5} \approx 2.236$
$\sqrt{6} \approx 2.449$	$\sqrt{7} \approx 2.646$	$\sqrt{8} \approx 2.828$	$\sqrt{9} = 3$	$\sqrt{10} \approx 3.162$

V. FORMULAS

Perimeter:		Volume:	
Triangle	$p = a + b + c$	Cube	$V = s^3$
Square	$p = 4s$	Rectangular Prism	$V = lwh$
Rectangle	$p = 2l + 2w$	Cylinder	$V = \pi r^2 h$
Circle *(circumference)*	$c = 2\pi r$	Cone	$V = (^1/_3)\pi r^2 h$
	$c = \pi d$	Sphere	$V = (^4/_3)\pi r^3$
		Pyramid	$V = (^1/_3)(\text{area of base})h$

Area:			
Rhombus	$A = (½)d_1d_2$	Circle	$A = \pi r^2$
Square	$A = s^2$	Triangle	$A = (½)bh$
Rectangle	$A = lw = bh$	Right Triangle	$A = (½)l_1l_2$
Parallelogram	$A = bh$	Equilateral Triangle	$A = (¼)\, s^2\sqrt{3}$
Trapezoid	$A = (½)(b_1 + b_2)h$		

Total Surface Area:		Lateral Surface Area:	
Cube	$T = 6s^2$	Rectangular Prism	$L = (2l + 2w)h$
Rectangular Prism	$T = 2lw + 2lh + 2wh$	Cylinder	$L = 2\pi rh$
Cylinder	$T = 2\pi r^2 + 2\pi rh$		
Sphere	$T = 4\pi r^2$		

Distance = Rate × Time

Slope of a Line with Endpoints (x_1, y_1) and (x_2, y_2): slope = $m = \dfrac{(y_2 - y_1)}{(x_2 - x_1)}$

Distance Formula: distance between two points or length of segment with endpoints (x_1, y_1) and (x_2, y_2)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Midpoint Formula: midpoint of a line segment given two endpoints (x_1, y_1) and (x_2, y_2)

$$\left(\frac{x_1 + x_2}{2}\ ,\ \frac{y_1 + y_2}{2}\right)$$

Circles:

Length of an arc = $\left(\dfrac{x}{360}\right)(2\pi r)$, where x is the measure of the central angle of the arc

Area of a sector = $\left(\dfrac{x}{360}\right)(\pi r^2)$, where x is the measure of the central angle of the sector

Combinations (number of groupings when the order of the items in the groups does not matter):

Number of combinations = $\dfrac{N!}{R!(N-R)!}$, where N = # of total items and R = # of items being chosen

Permutations (number of groupings when the order of the items in the groups matters):

Number of permutations = $\dfrac{N!}{(N-R)!}$, where N = # of total items and R = # of items being chosen

Length of a Diagonal of a Square = $s\sqrt{2}$

Length of a Diagonal of a Cube = $s\sqrt{3}$

Length of a Diagonal of a Rectangular Solid = $\sqrt{x^2+y^2+z^2}$, with dimensions x, y and z

Number of Diagonals for a Convex Polygon with N Sides = $\dfrac{N(N-3)}{2}$

Sum of the Measures of the Interior Angles of a Regular Polygon with N Sides = $(N-2)180$

Heron's Formula:

For **any triangle** with side lengths a, b and c, $Area = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = ½(a + b + c)$

Pythagorean Theorem: (Can be used with **all right triangles**)

$a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse

Pythagorean Triples: Integer-length sides for right triangles form Pythagorean Triples – the largest number must be on the hypotenuse. Memorizing the bold triples will also lead to other triples that are multiples of the original.

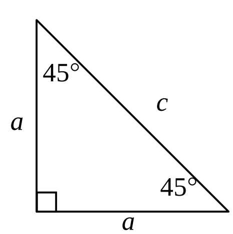
3	**4**	**5**	**5**	**12**	**13**	**7**	**24**	**25**
6	8	10	10	24	26	**8**	**15**	**17**
9	12	15	15	36	39	**9**	**40**	**41**

Special Right Triangles:

$45^o – 45^o – 90^o$

hypotenuse = $\sqrt{2}$ (leg) = $a\sqrt{2}$

leg = $\dfrac{\text{hypotenuse}}{\sqrt{2}} = \dfrac{c}{\sqrt{2}}$

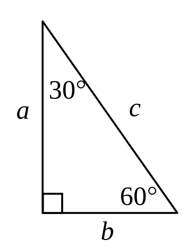


$30^o – 60^o – 90^o$

hypotenuse = 2(shorter leg) = $2b$

longer leg = $\sqrt{3}$ (shorter leg) = $b\sqrt{3}$

shorter leg = $\dfrac{\text{longer leg}}{\sqrt{3}} = \dfrac{\text{hypotenuse}}{2}$



Geometric Mean: $\frac{a}{x} = \frac{x}{b}$ therefore, $x^2 = ab$ and $x = \sqrt{ab}$

Regular Polygon: Measure of a central angle $= \frac{360}{n}$, where n = number of sides of the polygon

Measure of vertex angle $= 180 - \frac{360}{n}$, where n = number of sides of the polygon

Ratio of Two Similar Figures: If the ratio of the measures of corresponding side lengths is A:B, then the ratio of the perimeters is A:B, the ratio of the areas is $A^2 : B^2$ and the ratio of the volumes is $A^3 : B^3$.

Difference of Two Squares: $a^2 - b^2 = (a-b)(a+b)$

Example: $12^2 - 9^2 = (12-9)(12+9) = 3 \cdot 21 = 63$

$144 - 81 = 63$

Determining the Greatest Common Factor (GCF): 5 Methods

1. Prime Factorization (Factor Tree) – Collect all common factors
2. Listing all Factors
3. Multiply the two numbers and divide by the Least Common Multiple (LCM)
   Example: to find the GCF of 15 and 20, multiply $15 \times 20 = 300$,
   then divide by the LCM, 60. The GCF is 5.
4. Divide the smaller number into the larger number. If there is a remainder, divide the remainder into the divisor until there is no remainder left. The last divisor used is the GCF.

   Example: $180\overline{)385}$, -360, remainder 25 $\quad$ $25\overline{)180}$, -175, remainder 5 $\quad$ $5\overline{)25}$, -25, remainder 0

   5 is the GCF of 180 and 385

5. Single Method for finding both the GCF and LCM
   Put both numbers in a lattice. On the left, put ANY divisor of the two numbers and put the quotients below the original numbers. Repeat until the quotients have no common factors except 1 (relatively prime). Draw a "boot" around the left-most column and the bottom row. Multiply the vertical divisors to get the GCF. Multiply the "boot" numbers (vertical divisors and last-row quotients) to get the LCM.

	40	**140**
2	20	70

	40	**140**
2	20	70
10		

	40	**140**
2	20	70
10	2	7

The GCF is $2 \times 10 = 20$
The LCM is
$2 \times 10 \times 2 \times 7 = 280$

VI. DEFINITIONS

Real Numbers: all rational and irrational numbers

Rational Numbers: numbers that can be written as a ratio of two integers

Irrational Numbers: non-repeating, non-terminating decimals; can't be written as a ratio of two integers (i.e. $\sqrt{7}$, π)

Integers: {…, -3, -2, -1, 0, 1, 2, 3, …}

Whole Numbers: {0, 1, 2, 3, …}

Natural Numbers: {1, 2, 3, 4, …}

Common Fraction: a fraction in lowest terms (Refer to "Forms of Answers" in the *MATHCOUNTS School Handbook* for a complete definition.)

Equation of a Line:

Standard form: $Ax + By = C$ with slope = $-\frac{A}{B}$

Slope-intercept form: $y = mx + b$ with slope = m and y-intercept = b

Regular Polygon: a convex polygon with all equal sides and all equal angles

Negative Exponents: $x^{-n} = \frac{1}{x^n}$ *and* $\frac{1}{x^{-n}} = x^n$

Systems of Equations:

$$\begin{array}{r} x + y = 10 \\ \underline{x - y = \ 6} \\ 2x \quad = 16 \\ \mathbf{x \quad = \ 8} \end{array} \quad \rightarrow \quad \begin{array}{r} 8 + y = 10 \\ \mathbf{y = 2} \end{array}$$

(8, 2) is the solution of the system

Mean = Arithmetic Mean = Average

Mode = the number(s) occurring the most often; there may be more than one

Median = the middle number when written from least to greatest
If there is an even number of terms, the median is the average of the two middle terms.

Range = the difference between the greatest and least values

Measurements:

1 mile = 5280 feet
1 square foot = 144 square inches
1 square yard = 9 square feet
1 cubic yard = 27 cubic feet

VII. PATTERNS

Divisibility Rules:

Number is divisible by
- 2: last digit is 0,2,4,6 or 8
- 3: sum of digits is divisible by 3
- 4: two-digit number formed by the last two digits is divisible by 4
- 5: last digit is 0 or 5
- 6: number is divisible by **both** 2 and 3
- 8: three-digit number formed by the last 3 digits is divisible by 8
- 9: sum of digits is divisible by 9
- 10: last digit is 0

Sum of the First N Odd Natural Numbers = N^2
Sum of the First N Even Natural Numbers = $N^2 + N = N(N + 1)$
Sum of an Arithmetic Sequence of Integers: $\frac{N}{2} \times$ (first term + last term), where N = amount of numbers/terms in the sequence

Find the digit in the units place of a particular power of a particular integer
Find the pattern of units digits: 7^1 ends in 7
7^2 ends in 9
7^3 ends in 3
7^4 ends in 1
7^5 ends in 7
(pattern repeats every 4 exponents)
Divide 4 into the given exponent and compare the remainder with the first four exponents.
(a remainder of 0 matches with the exponent of 4)
Example: What is the units digit of 7^{22}?
22 ÷ 4 = 5 r. **2**, so the units digit of 7^{22} is the same as the units digit of 7^2, which is 9.

VIII. FACTORIALS ("*n*!" is read "*n* factorial")

$n! = (n)\times(n-1)\times(n-2)\times\ldots\times(2)\times(1)$ Example: $5! = 5\times4\times3\times2\times1 = 120$
0! = 1
1! = 1
2! = 2
3! = 6
4! = 24
5! = 120
6! = 720
7! = 5040

Notice $\frac{6!}{4!} = \frac{6\times5\times\cancel{4\times3\times2\times1}}{\cancel{4\times3\times2\times1}} = 30$

IX. PASCAL'S TRIANGLE

Pascal's Triangle Used for Probability:
Remember that the first row is row zero (0). Row 4 is 1 4 6 4 1. This can be used to determine the different outcomes when flipping four coins.

1	**4**	**6**	**4**	**1**
way to get 4 heads 0 tails	ways to get 3 heads 1 tail	ways to get 2 heads 2 tails	ways to get 1 head 3 tails	way to get 0 heads 4 tails

For the Expansion of $(a + b)^n$, use numbers in Pascal's Triangle as coefficients.

1	$(a + b)^0 = 1$
1 1	$(a + b)^1 = a + b$
1 2 1	$(a + b)^2 = a^2 + 2ab + b^2$
1 3 3 1	$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
1 4 6 4 1	$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
1 5 10 10 5 1	$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

For 2^n, add all the numbers in the n^{th} row. (Remember the triangle starts with row 0.)

1	$2^0 = 1$
1 1	$2^1 = 1 + 1 = 2$
1 2 1	$2^2 = 1 + 2 + 1 = 4$
1 3 3 1	$2^3 = 1 + 3 + 3 + 1 = 8$
1 4 6 4 1	$2^4 = 1 + 4 + 6 + 4 + 1 = 16$
1 5 10 10 5 1	$2^5 = 1 + 5 + 10 + 10 + 5 + 1 = 32$

X. SQUARING A NUMBER WITH A UNITS DIGIT OF 5

$(n5)^2 = \underline{n \times (n+1)}\ \underline{2}\ \underline{5}$, where n represents the block of digits before the units digit of 5

Examples:

$(35)^2 = \underline{3\times(3+1)}\ \underline{2}\ \underline{5}$
$= \underline{3\times(4)}\ \underline{2}\ \underline{5}$
$= \underline{1\ 2}\ \underline{2}\ \underline{5}$
$= 1{,}225$

$(125)^2 = \underline{12\times(12+1)}\ \underline{2}\ \underline{5}$
$= \underline{12\times(13)}\ \underline{2}\ \underline{5}$
$= \underline{1\ 5\ 6}\ \underline{2}\ \underline{5}$
$= 15{,}625$

XI. BASES

Base 10 = decimal – only uses digits 0 – 9
Base 2 = binary – only uses digits 0 – 1
Base 8 = octal – only uses digits 0 – 7
Base 16 = hexadecimal – only uses digits 0 – 9, A – F (where A=10, B=11, …, F=15)

Changing from Base 10 to Another Base:

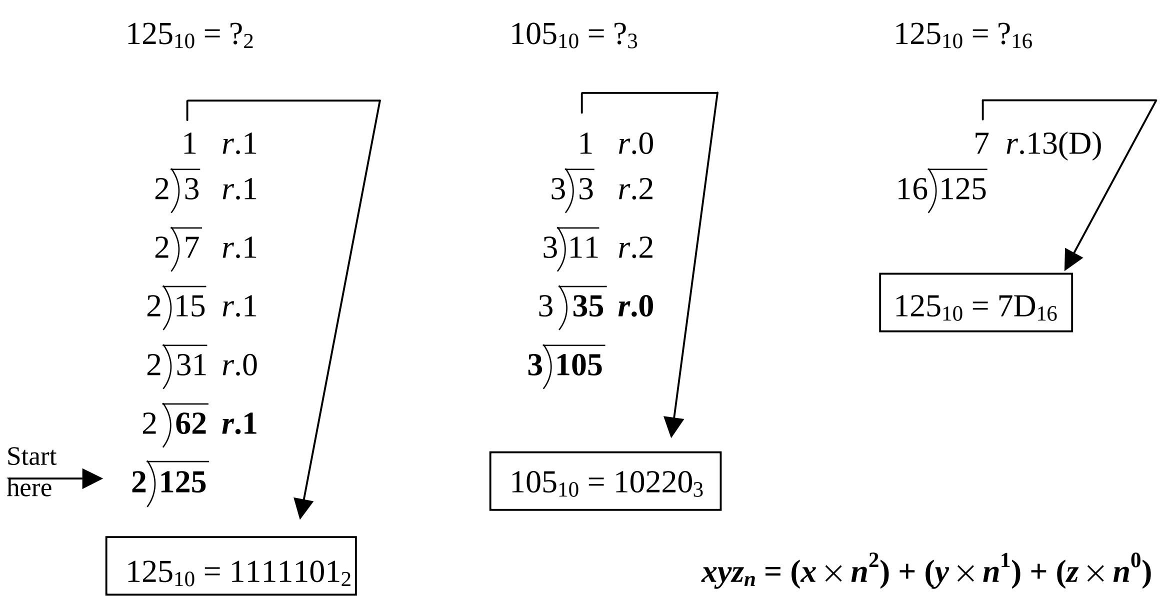
What is the base 2 representation of 125 (or "125 base 10" or "125_{10}")?

We know $125 = 1(10^2) + 2(10^1) + 5(10^0) = 100 + 20 + 5$, but what is it equal to in base 2?
$125_{10} = ?(2^n) + ?(2^{n-1}) + \ldots + ?(2^0)$
The largest power of 2 in 125 is $64 = 2^6$, so we now know our base 2 number will be:
$?(2^6) + ?(2^5) + ?(2^4) + ?(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$ and it will have 7 digits of 1's and/or 0's.

Since there is **one** 64, we have: $\mathbf{1}(2^6) + ?(2^5) + ?(2^4) + ?(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$
We now have 125 – 64 = 61 left over, which is **one** $32 = 2^5$ and 29 left over, so we have:
$\mathbf{1}(2^6) + \mathbf{1}(2^5) + ?(2^4) + ?(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$
In the left-over 29, there is **one** $16 = 2^4$, with 13 left over, so we have:
$\mathbf{1}(2^6) + \mathbf{1}(2^5) + \mathbf{1}(2^4) + ?(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$
In the left-over 13, there is **one** $8 = 2^3$, with 5 left over, so we have:
$\mathbf{1}(2^6) + \mathbf{1}(2^5) + \mathbf{1}(2^4) + \mathbf{1}(2^3) + ?(2^2) + ?(2^1) + ?(2^0)$
In the left-over 5, there is **one** $4 = 2^2$, with 1 left over, so we have:
$\mathbf{1}(2^6) + \mathbf{1}(2^5) + \mathbf{1}(2^4) + \mathbf{1}(2^3) + \mathbf{1}(2^2) + ?(2^1) + ?(2^0)$
In the left-over 1, there is **no** $2 = 2^1$, so we still have 1 left over, and our expression is:
$\mathbf{1}(2^6) + \mathbf{1}(2^5) + \mathbf{1}(2^4) + \mathbf{1}(2^3) + \mathbf{1}(2^2) + \mathbf{0}(2^1) + ?(2^0)$
The left-over 1 is **one** 2^0, so we finally have:
$\mathbf{1}(2^6) + \mathbf{1}(2^5) + \mathbf{1}(2^4) + \mathbf{1}(2^3) + \mathbf{1}(2^2) + \mathbf{0}(2^1) + \mathbf{1}(2^0) = 1111101_2$

Now try ***What is the base 3 representation of 105?***
The largest power of 3 in 105 is 81 = 3^4, so we now know our base 3 number will be:
$?(3^4) + ?(3^3) + ?(3^2) + ?(3^1) + ?(3^0)$ and will have 5 digits of 2's, 1's, and/or 0's.
Since there is **one** 81, we have: $\mathbf{1}(3^4) + ?(3^3) + ?(3^2) + ?(3^1) + ?(3^0)$
In the left-over 105 – 81 = 24, there is **no** 27 = 3^3, so we still have 24 and the expression:
$\mathbf{1}(3^4) + \mathbf{0}(3^3) + ?(3^2) + ?(3^1) + ?(3^0)$
In the left-over 24, there are **two** 9's (or 3^2's), with 6 left over, so we have:
$\mathbf{1}(3^4) + \mathbf{0}(3^3) + \mathbf{2}(3^2) + ?(3^1) + ?(3^0)$
In the left-over 6, there are **two** 3's (or 3^1's), with 0 left over, so we have:
$\mathbf{1}(3^4) + \mathbf{0}(3^3) + \mathbf{2}(3^2) + \mathbf{2}(3^1) + ?(3^0)$
Since there is nothing left over, we have **no** 1's (or 3^0's), so our final expression is:
$\mathbf{1}(3^4) + \mathbf{0}(3^3) + \mathbf{2}(3^2) + \mathbf{2}(3^1) + \mathbf{0}(3^0) = 10220_3$

The following is another fun algorithm for converting base 10 numbers to other bases:



$$\boldsymbol{xyz_n = (x \times n^2) + (y \times n^1) + (z \times n^0)}$$

Notice: Everything in bold shows the first division operation. The first remainder will be the last digit in the base *n* representation, and the quotient is then divided again by the desired base. The process is repeated until a quotient is reached that is less than the desired base. At that time, the final quotient and remainders are read downward.

XII. FACTORS

Determining the Number of Factors of a Number: First find the prime factorization (include the 1 if a factor is to the first power). *Increase* each exponent by 1 and multiply these new numbers together.

Example: How many factors does 300 have?
The prime factorization of 300 is $2^{\mathbf{2}} \times 3^{\mathbf{1}} \times 5^{\mathbf{2}}$. Increase each of the exponents by 1 and multiply these new values: $(2+1) \times (1+1) \times (2+1) = 3 \times 2 \times 3 = 18$. So 300 has 18 factors.

Finding the Sum of the Factors of a Number:

Example: What is the sum of the factors of 10,500?
(From the prime factorization $2^2 \times 3^1 \times 5^3 \times 7^1$, we know 10,500 has $3 \times 2 \times 4 \times 2 = 48$ factors.)
The sum of these 48 factors can be calculated from the prime factorization, too:
$(2^0 + 2^1 + 2^2)(3^0 + 3^1)(5^0 + 5^1 + 5^2 + 5^3)(7^0 + 7^1) = 7 \times 4 \times 156 \times 8 = 34{,}944$.