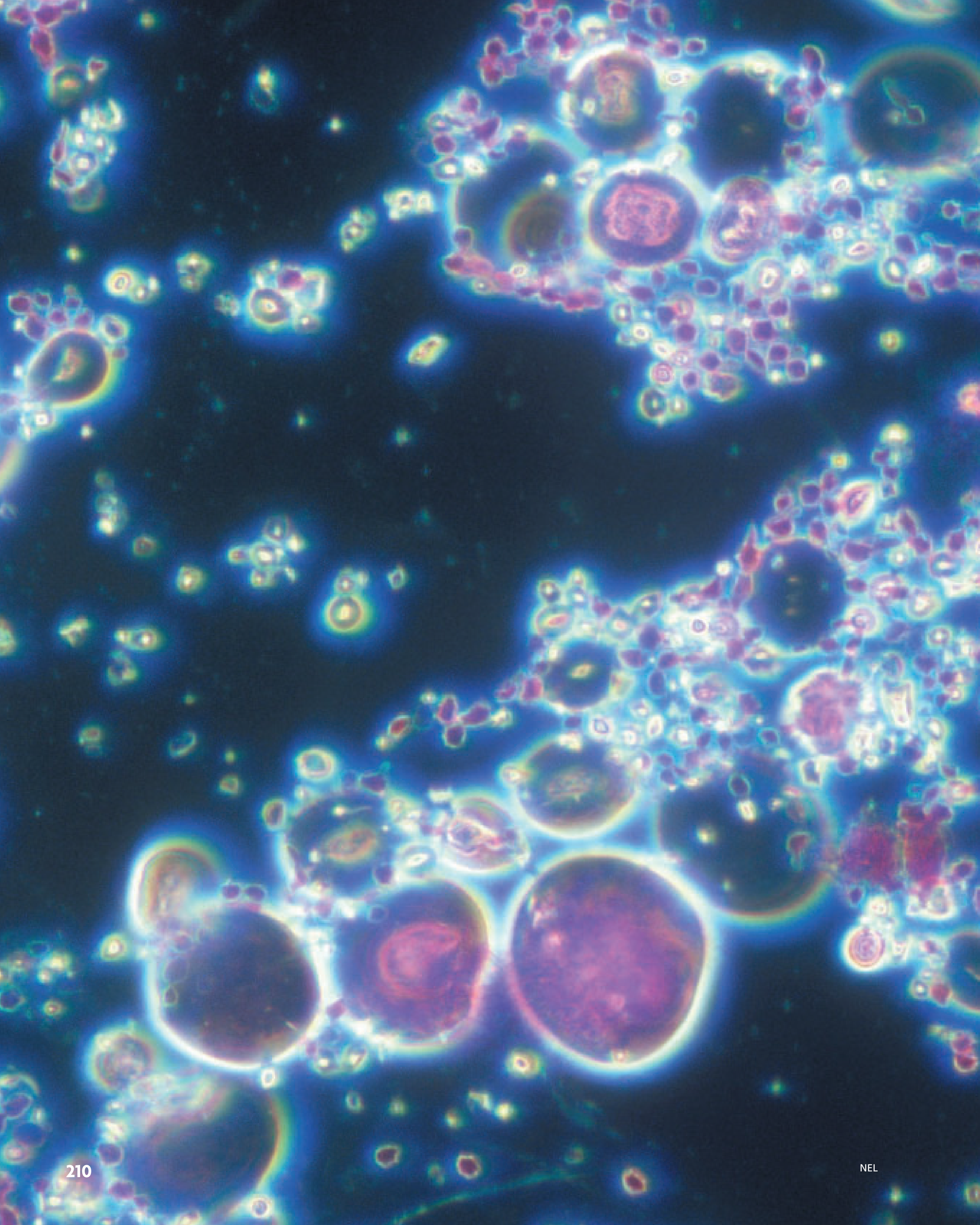
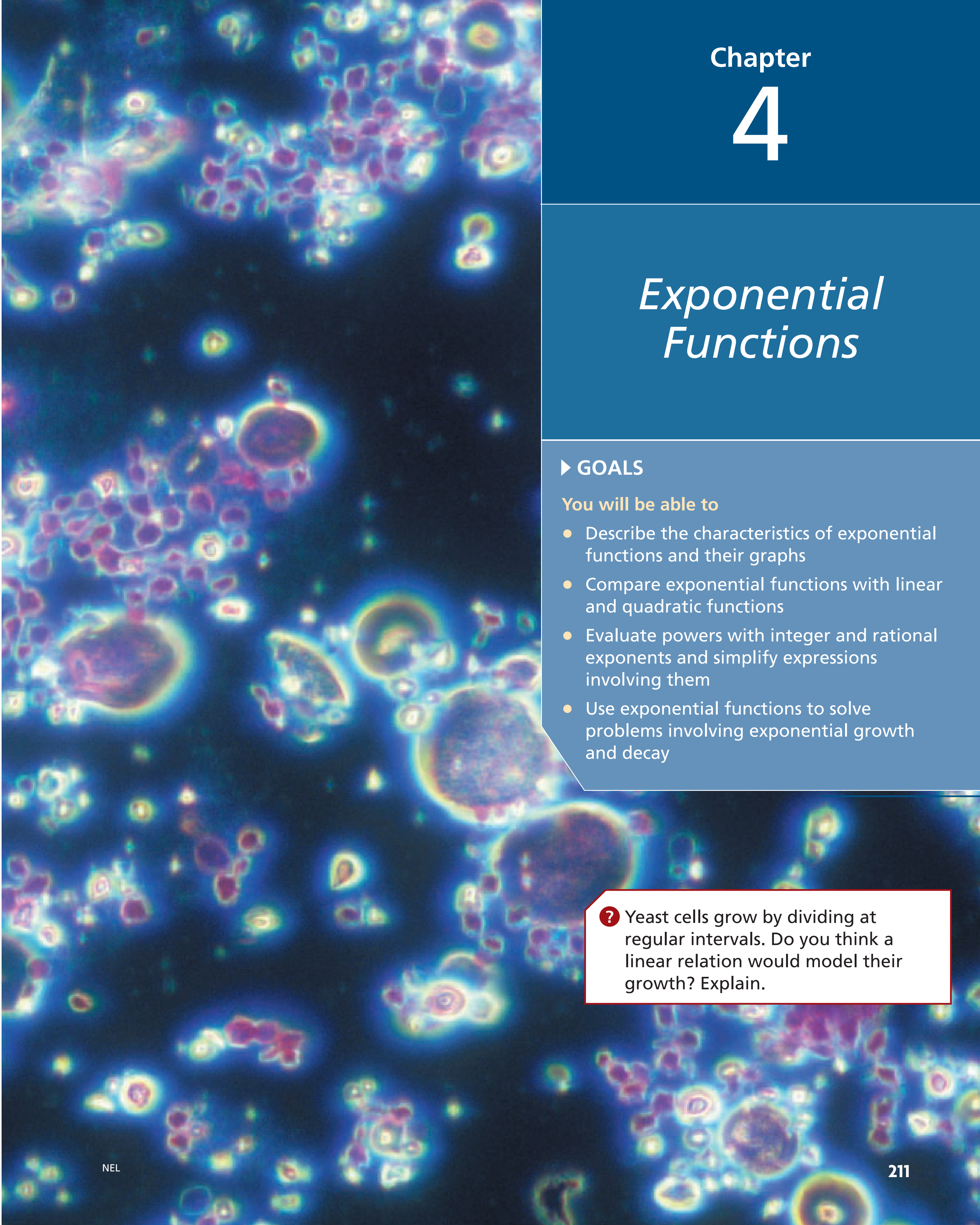




Chapter 4

Exponential Functions

GOALS

You will be able to

- Describe the characteristics of exponential functions and their graphs
- Compare exponential functions with linear and quadratic functions
- Evaluate powers with integer and rational exponents and simplify expressions involving them
- Use exponential functions to solve problems involving exponential growth and decay

? Yeast cells grow by dividing at regular intervals. Do you think a linear relation would model their growth? Explain.

4 Getting Started

SKILLS AND CONCEPTS You Need

Study Aid

- For help, see Essential Skills Appendix.

Question	Appendix
1, 2, 4, and 7	A-3
6	A-2
9	A-7

1. Evaluate.

a) 7^2
b) 2^5
c) 5^{-1}
d) 10^0
e) 100^2
f) 2^{-3}

2. Evaluate.

a) $(-3)^2$
b) $(-3)^3$
c) -4^2
d) $(-4)^2$
e) $(-5)^3$
f) -5^3

3. Predict whether the power $(-5)^{120}$ will result in a positive or negative answer. Explain how you know.

4. Evaluate.

a) $(3^2)^2$
b) $(7^2)^4$
c) $[(-4)^2]^3$
d) $[-(10^2)]^3$
e) $[(2^2)^2]^2$
f) $-[(2^2)^2]^0$

5. Evaluate.

a) $(\sqrt{49})^2$
b) $3\sqrt{64}$
c) $\sqrt{4}\sqrt{16}$
d) $\frac{\sqrt{9}}{\sqrt{81}}$

6. Evaluate.

a) $\frac{5}{8} + \frac{5}{3}$
b) $\frac{5}{8} - \frac{5}{3}$
c) $\frac{7}{8} \div \frac{2}{3}$
d) $\frac{1}{5} - \frac{3}{8}\left(\frac{4}{3}\right)$
e) $-\frac{4}{3} + \left(\frac{9}{10} \div \frac{5}{12}\right)$
f) $-\frac{9}{10}\left(\frac{3}{8} + \frac{7}{3}\right)$

7. Simplify.

a) $a^2(a^5)$
b) $b^{12} \div b^8$
c) $(c^3)^4$
d) $d(d^6)d^3$

8. Determine the exponent that makes each equation true.

a) $9^x = 81$
b) $8^m = 256$
c) $(-5)^a = -125$
d) $-10^r = -100\,000\,000$

9. Evaluate the following formulas for $r = 2.5$ cm and $h = 4.8$ cm.

a) the volume of a cylinder: $V = \pi r^2 h$
b) the volume of a sphere: $V = \frac{4}{3}\pi r^3$

10. For each set of data, calculate the differences. Identify whether or not the data represent a linear or quadratic relationship. Explain.

a)

x	*y*	First Differences	Second Differences
−4	12		
−2	7		
0	2		
2	−3		
4	−8		
6	−13		

b)

x	*y*	First Differences	Second Differences
−3	9		
−2	10		
−1	12		
0	15		
1	19		
2	24		

APPLYING What You Know

Binary Code

The numbers we work with every day are written in base 10.

For example, 235 means $2 \times 10^2 + 3 \times 10^1 + 5 \times 10^0$.

Binary numbers are numbers that are written in base 2.

For example, 1011 means $1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$.
Each number consists of only the digits 0 and 1.

Computers use binary code to do their calculations. Each letter, number, or symbol needs a separate binary code.

? Is there a relationship between the number of digits and the possible number of codes?

A. Copy and complete the table shown.

Number of Digits	Possible Codes	Number of Possible Codes	Possible Codes as a Power of Two
1	0, 1	2	2^1
2	00, 01, 10, 11	4	2^2
3	000, 001, 010, 011, 100, 101, 110, 111	8	
4			
5			
6			
7			

B. Select any two numbers (other than the last two) from the third column, and calculate their product. Write the two numbers and their product as powers of two. Repeat this several times with different pairs of numbers.

C. What is the relationship between the exponents of the powers that you multiplied and the exponent of the resulting product?

D. What rule for multiplying powers with the same base does this suggest?

E. Select any two numbers (other than the first two) from the third column, and divide the greater by the lesser. Write the two numbers and their quotient as powers of two. Repeat this several times with different pairs of numbers.

F. What is the relationship between the exponents of the powers that you divided and the exponent of the resulting quotient?

G. What rule for dividing powers with the same base does this suggest?

H. How can you predict the possible number of codes if you know the number of digits?

4.1 Exploring Growth and Decay

YOU WILL NEED

- two different types of balls that bounce (e.g., basketball, racquetball, soccer ball, golf ball)
- graphing calculator with a motion detector (CBR)

GOAL

Collect data and study the characteristics of rapidly decaying functions.

EXPLORE the Math

When you drop a ball it will bounce several times.

? Is the height of each bounce related to the height of the previous bounce?

A. Set the CBR to "Ball Bounce" mode. Work with a partner. One of you holds the ball, while the other holds the CBR 0.5 m over the ball. When the CBR is triggered, drop the ball.

B. Let the ball bounce at least 5 times while you collect the data. Use the trace key to determine the height of each bounce.

C. Copy the table. In the first row, record the original height of the ball, then record the bounce number and bounce height for next bounces.

Bounce Number	Bounce Height	First Differences
0		
1		
2		

D. Calculate and complete the first-differences column.

E. Repeat parts A to D for two additional starting heights.

F. After recording data for three different starting heights, plot the bounce height versus bounce number on the same graph. Use a different colour for each set of data. Draw a dashed curve through each set of points.

G. Describe the shape of each graph. Does each set of data represent a function? How do you know?

H. State the domain and range of each graph.

I. Repeat the exploration with a different type of ball, and explain how the height of each bounce is related to its previous bounce.

Reflecting

J. Why was a dashed curve used in part F instead of a solid one?

K. Look at the first-differences column. Describe how the bounce height changed from one bounce to the next. Was this pattern the same for each *type* of ball? Explain.

L. Did the type of ball you used influence the graph? Explain. Did the initial height of the drop influence the graph? Explain.

M. What happens to the bounce height as the bounce number increases? If you continue the pattern indefinitely, will the bounce height ever reach zero? Explain.

In Summary

Key Ideas

- Some real-world situations can be modelled by functions whose first differences follow a multiplicative pattern.
- The scatter plots for these situations show increasing or decreasing nonlinear patterns.

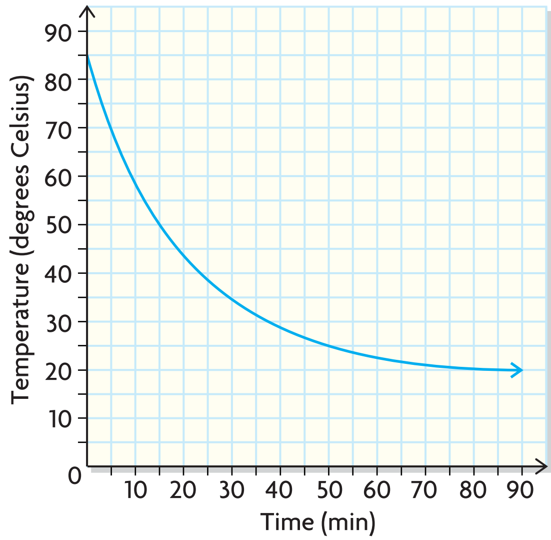
Need to Know

- The domain and range of a function should be considered in terms of the situation it is modelling.

FURTHER Your Understanding

1. A cup of hot cocoa left on a desk in a classroom had its temperature measured once every minute. The graph shows the relationship between the temperature of the cocoa, in degrees Celsius, and time, in minutes.

Cooling Curve for Cocoa

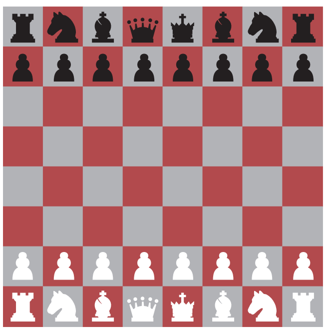


a) What characteristics of this graph are the same as the graph(s) you drew in the ball-bounce experiment?

b) What was the temperature of the cocoa at the start of the experiment?

c) What is the temperature of the classroom?

2. A folktale tells of a man who helps a king solve a problem. In return, the king offers the man anything he desires. The man asks for one grain of rice on a square of a chessboard and then double the number of grains of rice for each subsequent square.



a) Complete the table of values for the first 10 squares.

Number of Squares on the Chessboard	Number of Grains on that Square	First Differences
1	1	
2	2	
3	4	

b) Create a scatter plot of the data in the first two columns.

c) Compare this graph and the first differences of the data with your graphs and data for the ball-bounce experiment. How are they the same and how are they different?

4.2 Working with Integer Exponents

GOAL

Investigate powers that have integer or zero exponents.

LEARN ABOUT the Math

The metric system of measurement is used in most of the world. A key feature of the system is its ease of use. Since all units differ by multiples of 10, it is easy to convert from one unit to another. Consider the chart listing the prefix names and their factors for the unit of measure for length, the metre.

Name	Symbol	Multiple of the Metre	Multiple as a Power of 10
terametre	Tm	1 000 000 000 000	10^{12}
gigametre	Gm	1 000 000 000	10^9
megametre	Mm	1 000 000	10^6
kilometre	km	1 000	10^3
hectometre	hm	100	10^2
decametre	dam	10	10^1
metre	m	1	
decimetre	dm	0.1	
centimetre	cm	0.01	
millimetre	mm	0.001	
micrometre	μm	0.000 1	
nanometre	nm	0.000 01	
picometre	pm	0.000 001	
femtometre	fm	0.000 000 001	
attometre	am	0.000 000 000 001	

? How can powers be used to represent metric units for lengths less than 1 metre?

EXAMPLE 1 Using reasoning to define zero and negative integer exponents

Use the table to determine how multiples of the unit metre that are less than or equal to 1 can be expressed as powers of 10.

Jemila's Solution

Multiples	Powers
1000	10^3
$1000 \div 10 = 100$	$10^3 \div 10 = 10^2$
$100 \div 10 = 10$	$10^2 \div 10 = 10^1$
$10 \div 10 = 1$	$10^1 \div 10 = 10^0$
$1 \div 10 = 0.1 = \frac{1}{10}$	$10^0 \div 10 = 10^{-1}$
$0.1 \div 10 = 0.01 = \frac{1}{100} = \frac{1}{10^2}$	$10^{-1} \div 10 = 10^{-2}$
$0.01 \div 10 = 0.001 = \frac{1}{1000} = \frac{1}{10^3}$	$10^{-2} \div 10 = 10^{-3}$

I think that $x^{-n} = \frac{1}{x^n}$ is the rule for negative exponents.

As I moved down the table, the powers of 10 decreased by 1, while the multiples were divided by 10. To come up with the next row in the table, I divided the multiples and the powers by 10.

If I continue this pattern, I'll get $10^0 = 1$, $10^{-1} = 0.1$, $10^{-2} = 0.01$, etc.

I rewrote each decimal as a fraction and each denominator as a power of 10.

I noticed that $10^0 = 1$ and $10^{-n} = \frac{1}{10^n}$.

I don't think it mattered that the base was 10. The relationship would be true for any base.

EXAMPLE 2 Connecting the concept of an exponent of 0 to the exponent quotient rule

Use the quotient rule to show that $10^0 = 1$.

David's Solution

$$\frac{10^6}{10^6} = 1$$

I can divide any number except 0 by itself to get 1. I used a power of 10.

$$\frac{10^6}{10^6} = 10^{6-6} = 10^0$$

When you divide powers with the same base, you subtract the exponents.

Therefore, $10^0 = 1$.

I applied the rule to show that a power with zero as the exponent must be equal to 1.

Reflecting

A. What type of number results when x^{-n} is evaluated if x is a positive integer and $n > 1$?

B. How is 10^2 related to 10^{-2}? Why do you think this relationship holds for other opposite exponents?

C. Do you think the rules for multiplying and dividing powers change if the powers have negative exponents? Explain.

APPLY the Math

EXAMPLE 3 Representing powers with integer bases in rational form

Evaluate.

a) 5^{-3} **b)** $(-4)^{-2}$ **c)** -3^{-4}

Stergios's Solution

a) $5^{-3} = \frac{1}{5^3}$

$= \frac{1}{125}$

5^{-3} is what you get if you divide 1 by 5^3. I evaluated the power.

b) $(-4)^{-2} = \frac{1}{(-4)^2}$

$= \frac{1}{16}$

$(-4)^{-2}$ is what you get if you divide 1 by $(-4)^2$. Since the negative sign is in the parentheses, the square of the number is positive.

c) $-3^{-4} = -\frac{1}{3^4}$

$= -\frac{1}{81}$

In this case, the negative sign is not inside the parentheses, so the entire power is negative. I knew that $3^{-4} = \frac{1}{3^4}$.

If the base of a power involving a negative exponent is a fraction, it can be evaluated in a similar manner.

Communication *Tip*

Rational numbers can be written in a variety of forms. The term *rational form* means "Write the number as an integer, or as a fraction."

EXAMPLE 4 Representing powers with rational bases as rational numbers

Evaluate $\left(\frac{2}{3}\right)^{-3}$.

Sadira's Solution

$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^{3}}$$

$\left(\frac{2}{3}\right)^{-3}$ is what you get if you divide 1 by $\left(\frac{2}{3}\right)^{3}$.

$$= \frac{1}{\left(\frac{8}{27}\right)}$$

$$= 1 \times \frac{27}{8}$$

Dividing by a fraction is the same as multiplying by its reciprocal, so I used this to evaluate the power.

$$= \frac{27}{8}$$

EXAMPLE 5 Selecting a strategy for expressions involving negative exponents

Evaluate $\frac{3^5 \times 3^{-2}}{(3^{-3})^2}$.

Kayleigh's Solution: Using Exponent Rules

$$\frac{3^5 \times 3^{-2}}{(3^{-3})^2} = \frac{3^{5+(-2)}}{3^{-3\times 2}}$$

I simplified the numerator and denominator separately. Then I divided the numerator by the denominator. I added exponents for the numerator, multiplied exponents for the denominator, and subtracted exponents for the final calculation.

$$= \frac{3^3}{3^{-6}}$$

$$= 3^{3-(-6)}$$

$$= 3^9$$

$$= 19\ 683$$

Derek's Solution: Using a Calculator

I entered the expression into my calculator. I made sure I used parentheses around the entire numerator and denominator so that the calculator would compute those values before dividing.

Tech Support

For help with evaluating powers on a graphing calculator, see Technical Appendix, B-15.

In Summary

Key Ideas

- An integer base raised to a negative exponent is equivalent to the reciprocal of the same base raised to the opposite exponent.

$$b^{-n} = \frac{1}{b^n}, \quad \text{where} \quad b \neq 0$$

- A fractional base raised to a negative exponent is equivalent to the reciprocal of the same base raised to the opposite exponent.

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \left(\frac{b}{a}\right)^n, \quad \text{where} \quad a \neq 0, b \neq 0$$

- A number (or expression), other than 0, raised to the power of zero is equal to 1.

$b^0 = 1$, where $b \neq 0$

Need to Know

- When multiplying powers with the same base, add exponents.

$$b^m \times b^n = b^{m+n}$$

- When dividing powers with the same base, subtract exponents.

$$b^m \div b^n = b^{m-n} \quad \text{if} \quad b \neq 0$$

- To raise a power to a power, multiply exponents.

$$(b^m)^n = b^{mn}$$

- In simplifying numerical expressions involving powers, it is customary to present the answer as an integer, a fraction, or a decimal.
- In simplifying algebraic expressions involving powers, it is customary to present the answer with positive exponents.

CHECK Your Understanding

1. Rewrite each expression as an equivalent expression with a positive exponent.

a) 5^{-4} **c)** $\frac{1}{2^{-4}}$ **e)** $\left(\frac{3}{11}\right)^{-1}$

b) $\left(-\frac{1}{10}\right)^{-3}$ **d)** $-\left(\frac{6}{5}\right)^{-3}$ **f)** $\frac{7^{-2}}{8^{-1}}$

2. Write each expression as a single power with a positive exponent.

a) $(-10)^8(-10)^{-8}$ **c)** $\frac{2^8}{2^{-5}}$ **e)** $(-9^4)^{-1}$

b) $6^{-7} \times 6^5$ **d)** $\frac{11^{-3}}{11^5}$ **f)** $[(7^{-3})^{-2}]^{-2}$

3. Which is the greater power, 2^{-5} or $\left(\frac{1}{2}\right)^{-5}$? Explain.

PRACTISING

4. Simplify, then evaluate each expression. Express answers in rational form.

a) $2^{-3}(2^7)$

b) $(-8)^3(-8)^{-3}$

c) $\frac{5^4}{5^6}$

d) $\frac{3^{-8}}{3^{-6}}$

e) $(4^{-3})^{-1}$

f) $(7^{-1})^2$

5. Simplify, then evaluate each expression. Express answers in rational form.

a) $3^3(3^2)^{-1}$

b) $(9 \times 9^{-1})^{-2}$

c) $\frac{(12^{-1})^3}{12^{-3}}$

d) $\frac{(5^3)^{-2}}{5^{-6}}$

e) $(3^{-2}(3^3))^{-2}$

f) $9^7(9^3)^{-2}$

6. Simplify, then evaluate each expression. Express answers in rational form.

a) $10(10^4(10^{-2}))$

b) $8(8^2)(8^{-4})$

c) $\frac{6^{-5}}{(6^2)^{-2}}$

d) $\frac{4^{-10}}{(4^{-4})^3}$

e) $2^8 \times \left(\frac{2^{-5}}{2^6}\right)$

f) $13^{-5} \times \left(\frac{13^2}{13^8}\right)^{-1}$

7. Evaluate. Express answers in rational form.

a) $16^{-1} - 2^{-2}$

b) $(-3)^{-1} + 4^0 - 6^{-1}$

c) $\left(-\frac{2}{3}\right)^{-1} + \left(\frac{2}{5}\right)^{-1}$

d) $\left(\frac{1}{5}\right)^{-1} + \left(-\frac{1}{2}\right)^{-2}$

e) $5^{-3} + 10^{-3} - 8(1000^{-1})$

f) $3^{-2} - 6^{-2} + \frac{3}{2}(-9)^{-1}$

8. Evaluate. Express answers in rational form.

a) $5^2(-10)^{-4}$

b) $16^{-1}(2^5)$

c) $\frac{12^{-1}}{(-4)^{-1}}$

d) $\frac{(-9)^{-2}}{(3^{-1})^2}$

e) $(8^{-1})\left(\frac{2^{-3}}{4^{-1}}\right)$

f) $\frac{(-5)^3(-25)^{-1}}{(-5)^{-2}}$

9. **K** Evaluate. Express answers in rational form.

a) $(-4)^{-3}$

b) $(-4)^{-2}$

c) $-(5)^{-3}$

d) $-(5)^{-2}$

e) $(-6)^{-3}$

f) $-(6)^{-2}$

10. **T** Without using your calculator, write the given numbers in order from least to greatest. Explain your thinking.

$$(0.1)^{-1}, 4^{-1}, 5^{-2}, 10^{-1}, 3^{-2}, 2^{-3}$$

11. **A** Evaluate each expression for $x = -2$, $y = 3$, and $n = -1$. Express answers in rational form.

a) $(x^n + y^n)^{-2n}$

b) $(x^2)^n(y^{-2n})x^{-n}$

c) $\left(\frac{x^n}{y^n}\right)^n$

d) $\left(\frac{xy^n}{(xy)^{2n}}\right)^{2n}$

12. Kendra, Erik, and Vinh are studying. They wish to evaluate $3^{-2} \times 3$. Kendra notices errors in each of her friends' solutions, shown here.

Erik's solution	Vinh's solution
$3^{-2} \times 3$	$3^{-2} \times 3$
$= 3^{-1}$	$= 3^{-2}$
$= -\frac{1}{3^1}$	$= \frac{1}{3^2}$
$= -\frac{1}{3}$	$= \frac{1}{9}$

a) Explain where each student went wrong.
b) Create a solution that demonstrates the correct steps.

13. Evaluate using the laws of exponents.

a) $2^3 \times 4^{-2} \div 2^2$
b) $(2 \times 3)^{-1}$
c) $\left(\frac{3^{-1}}{2^{-1}}\right)^{-2}$
d) $4^{-1}(4^2 + 4^0)$
e) $\frac{2^5}{3^{-2}} \times \frac{3^{-1}}{2^4}$
f) $(5^0 + 5^2)^{-1}$
g) $\frac{3^{-2} \times 2^{-3}}{3^{-1} \times 2^{-2}}$
h) $\frac{4^{-2} + 3^{-1}}{3^{-2} + 2^{-3}}$
i) $\frac{5^{-1} - 2^{-2}}{5^{-1} + 2^{-2}}$

14. Find the value of each expression for $a = 1$, $b = 3$, and $c = 2$.

a) ac^c
b) $a^c b^c$
c) $(ab)^{-c}$
d) $(b \div c)^{-a}$
e) $(-a \div b)^{-c}$
f) $(a^{-1}b^{-2})^c$
g) $(a^b b^a)^c$
h) $[(b)^{-a}]^{-c}$

15. **a)** Explain the difference between evaluating $(-10)^3$ and evaluating 10^{-3}.
C **b)** Explain the difference between evaluating $(-10)^4$ and evaluating -10^4.

Extending

16. Determine the exponent that makes each equation true.

a) $16^x = \frac{1}{16}$
b) $10^x = 0.01$
c) $2^x = 1$
d) $2^n = 0.25$
e) $25^n = \frac{1}{625}$
f) $12^n = \frac{1}{144}$

17. If $10^{2y} = 25$, determine the value of 10^{-y}, where $y > 0$.

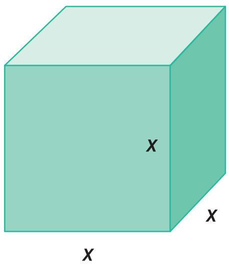
18. Simplify.

a) $(x^2)^{5-r}$
b) $(b^{2m+3m}) \div (b^{m-n})$
c) $(b^{2m+3n}) \div (b^{m-n})$
d) $x^{3(7-r)}x^r$
e) $(a^{10-p})\left(\frac{1}{a}\right)^P$
f) $[(3x^4)^{6-m}]\left(\frac{1}{x}\right)^m$

4.3 Working with Rational Exponents

GOAL

Investigate powers involving rational exponents and evaluate expressions containing them.



The volume of this cube is $V(x) = x^3$ and the area of its base is $A(x) = x^2$. In this cube, x is the side length and can be called

- the square root of A, since if squared, the result is $A(x)$
- the cube root of V, since if cubed, the result is $V(x)$

LEARN ABOUT the Math

? What exponents can be used to represent the side length x as the square root of area and the cube root of volume?

EXAMPLE 1 Representing a side length by rearranging the area formula

Express the side length x as a power of A and V.

Ira's Solution

$A = x^2$

$x = A^n$

$A = (x)(x)$

I used the area formula for the base. Since I didn't know what power to use, I used the variable n to write x as a power of A.

$A = A^n \times A^n$

$A = A^{n+n}$

$A^1 = A^{2n}$

I rewrote the area formula, substituting A^n for x.

Since I was multiplying powers with the same base, I added the exponents.

Therefore,

$1 = 2n$

I set the two exponents equal to each other. I solved this equation.

$\frac{1}{2} = n$

The exponent that represents a square root is $\frac{1}{2}$.

Therefore, $x = A^{\frac{1}{2}} = \sqrt{A}$.

EXAMPLE 2 Representing a side length by rearranging the volume formula

Sienna's Solution

$V = x^3$
$x = V^n$
$V = (x)(x)(x)$

I used the volume formula for a cube. I represented the edge length x as a power of the volume V. I used the variable n.

$V = V^n \times V^n \times V^n$

I rewrote the volume formula, substituting V^n for x.

$V = V^{n+n+n}$

I added the exponents.

$V^1 = V^{3n}$

Therefore,

$1 = 3n$

I set the two exponents equal to each other. I solved this equation.

$\frac{1}{3} = n$

The exponent that represents a cube root is $\frac{1}{3}$.

Therefore, $x = V^{\frac{1}{3}} = \sqrt[3]{V}$.

Reflecting

A. Why could x be expressed as both a square root and a cube root?

B. Make a conjecture about the meaning of $x^{\frac{1}{n}}$. Explain your reasoning.

C. Do the rules for multiplying powers with the same base still apply if the exponents are rational numbers? Create examples to illustrate your answer.

APPLY the Math

EXAMPLE 3 Connecting radical notation and exponents

Express the following in radical notation. Then evaluate.

a) $49^{-\frac{1}{2}}$ **b)** $(-8)^{\frac{1}{3}}$ **c)** $10\,000^{\frac{1}{4}}$

Donato's Solution

a) $49^{-\frac{1}{2}} = \frac{1}{49^{\frac{1}{2}}}$

$= \frac{1}{\sqrt{49}}$

$= \frac{1}{7}$

I wrote the power using the reciprocal of its base and its opposite exponent. An exponent of $\frac{1}{2}$ means square root.

I evaluated the power.

index (plural indices)

the number at the left of the radical sign. It tells which root is indicated: 3 for cube root, 4 for fourth root, etc. If there is no number, the square root is intended.

b) $(-8)^{\frac{1}{3}} = \sqrt[3]{-8}$
$= -2$

An exponent of $\frac{1}{3}$ means cube root. I wrote the root as a radical, using an index of 3. That means the number is multiplied by itself three times to get -8. The number is -2.

c) $10\,000^{\frac{1}{4}} = \sqrt[4]{10\,000}$
$= 10$

An exponent of $\frac{1}{4}$ means the fourth root, since
$10\,000^{\frac{1}{4}} \times 10\,000^{\frac{1}{4}} \times 10\,000^{\frac{1}{4}} \times 10\,000^{\frac{1}{4}} = 10\,000^{1}$.

That number must be 10.

EXAMPLE 4 Selecting an approach to evaluate a power

Evaluate $27^{\frac{2}{3}}$.

Cory's Solutions

$27^{\frac{2}{3}}$

I know that the exponent $\frac{1}{3}$ indicates a cube root. So I used the power-of-a-power rule to separate the exponents:

$\frac{2}{3} = 2 \times \frac{1}{3}$ and $\frac{2}{3} = \frac{1}{3} \times 2$

$= 27^{\frac{1}{3} \times 2}$	$= 27^{2 \times \frac{1}{3}}$
$= (27^{\frac{1}{3}})^2$	$= (27^2)^{\frac{1}{3}}$
$= (\sqrt[3]{27})^2$	$= \sqrt[3]{27^2}$
$= (3)^2$	$= \sqrt[3]{729}$
$= 9$	$= 9$

To see if the order in which I applied the exponents mattered, I calculated the solution in two ways.

In the first way, I evaluated the cube root before squaring the result.

In the other way, I squared the base and then took the cube root of the result.

Both ways resulted in 9.

EXAMPLE 5 Evaluating a power with a rational exponent

Evaluate.

a) $(-27)^{\frac{4}{3}}$ **b)** $(16)^{-0.75}$

Casey's Solutions

a) $(-27)^{\frac{4}{3}} = ((-27)^{\frac{1}{3}})^4$

$= (\sqrt[3]{-27})^4$

$= (-3)^4$

$= 81$

I rewrote the exponent as $4 \times \frac{1}{3}$.
I represented $(-27)^{\frac{1}{3}}$ as $\sqrt[3]{-27}$.
I calculated the cube root of -27.
I evaluated the power.

b) $16^{-0.75} = 16^{-\frac{3}{4}}$

$= \dfrac{1}{16^{\frac{3}{4}}}$

$= \dfrac{1}{(\sqrt[4]{64})^3}$

$= \dfrac{1}{2^3}$

$= \dfrac{1}{8}$

I rewrote the power, changing the exponent from -0.75 to its equivalent fraction.

I expressed $16^{-\frac{3}{4}}$ as a rational number, using 1 as the numerator and $16^{\frac{3}{4}}$ as the denominator.

I determined the fourth root of 64 and cubed the result.

The rules of exponents also apply to powers involving rational exponents.

EXAMPLE 6 Representing an expression involving the same base as a single power

Simplify, and then evaluate $\dfrac{8^{\frac{5}{6}}\sqrt{8}}{8^{\frac{5}{3}}}$.

Lucia's Solution

$\dfrac{8^{\frac{5}{6}}\sqrt{8}}{8^{\frac{5}{3}}} = \dfrac{8^{\frac{5}{6}}8^{\frac{1}{2}}}{8^{\frac{5}{3}}}$

$= \dfrac{8^{\frac{5}{6}+\frac{1}{2}}}{8^{\frac{5}{3}}}$

To simplify, I converted the radical into exponent form.

Since the bases were the same, I wrote the numerator as a single power by adding exponents, then I subtracted exponents to simplify the whole expression.

$$= \frac{8^{\frac{4}{3}}}{8^{\frac{5}{3}}}$$

$$= 8^{\frac{4}{3} - \frac{5}{3}}$$

$$= 8^{-\frac{1}{3}}$$

Once I had simplified to a single power of 8, the number was easier to evaluate.

$$= \frac{1}{8^{\frac{1}{3}}}$$

$$= \frac{1}{2}$$

I checked my work on my calculator.

In Summary

Key Ideas

- A number raised to a rational exponent is equivalent to a radical. The rational exponent $\frac{1}{n}$ indicates the nth root of the base. If $n > 1$ and $n \in N$, then $b^{\frac{1}{n}} = \sqrt[n]{b}$, where $b \neq 0$.
- If the numerator of a rational exponent is not 1, and if m and n are positive integers, then $b^{\frac{m}{n}} = (\sqrt[n]{b})^m = \sqrt[n]{b^m}$, where $b \neq 0$.

Need to Know

- The exponent laws that apply to powers with integer exponents also apply to powers with rational exponents. Included are the product-of-powers rule $a^n \times b^n = (ab)^n$ and the quotient of powers rule $a^n \div b^n = \left(\frac{a}{b}\right)^n$.
- The power button on a scientific calculator can be used to evaluate rational exponents.
- Some roots of negative numbers do not have real solutions. For example, -16 does not have a real-number square root, since whether you square a positive or negative number, the result is positive.
- Odd roots can have negative bases, but even ones cannot.

CHECK Your Understanding

1. Write in radical form. Then evaluate without using a calculator.

a) $49^{\frac{1}{2}}$ c) $(-125)^{\frac{1}{3}}$ e) $81^{\frac{1}{4}}$

b) $100^{\frac{1}{2}}$ d) $16^{0.25}$ f) $-(144)^{0.5}$

2. Write in exponent form, then evaluate. Express answers in rational form.

a) $\sqrt[9]{512}$ c) $\sqrt[3]{27^2}$ e) $\sqrt[5]{\frac{-32}{243}}$

b) $\sqrt[3]{-27}$ d) $(\sqrt[3]{-216})^5$ f) $\sqrt[4]{\left(\frac{16}{81}\right)^{-1}}$

3. Write as a single power.

a) $8^{\frac{2}{3}}(8^{\frac{1}{3}})$ d) $(7^{\frac{5}{6}})^{-\frac{6}{5}}$

b) $8^{\frac{2}{3}} \div 8^{\frac{1}{3}}$ e) $\frac{9^{\frac{-1}{5}}}{9^{\frac{2}{3}}}$

c) $(-11)^2(-11)^{\frac{3}{4}}$ f) $10^{-\frac{4}{5}}(10^{\frac{1}{15}}) \div 10^{\frac{2}{3}}$

PRACTISING

4. Write as a single power, then evaluate. Express answers in rational form.

a) $\sqrt{5}\sqrt{5}$ b) $\frac{\sqrt[3]{-16}}{\sqrt[3]{2}}$ c) $\frac{\sqrt{28}\sqrt{4}}{\sqrt{7}}$ d) $\frac{\sqrt[4]{18}\,(\sqrt[4]{9})}{\sqrt[4]{2}}$

5. Evaluate.

a) $49^{\frac{1}{2}} + 16^{\frac{1}{2}}$ d) $128^{-\frac{5}{7}} - 16^{0.75}$

b) $27^{\frac{2}{3}} - 81^{\frac{3}{4}}$ e) $16^{\frac{3}{2}} + 16^{-0.5} + 8 - 27^{\frac{2}{3}}$

c) $16^{\frac{3}{4}} + 16^{\frac{3}{4}} - 81^{-\frac{1}{4}}$ f) $81^{\frac{1}{2}} + \sqrt[3]{8} - 32^{\frac{4}{5}} + 16^{\frac{3}{4}}$

6. Write as a single power, then evaluate. Express answers in rational form.

a) $4^{\frac{1}{5}}(4^{0.3})$ c) $\frac{64^{\frac{4}{3}}}{64}$ e) $\frac{(16^{-2.5})^{-0.2}}{16^{\frac{3}{4}}}$

b) $100^{0.2}(100^{\frac{-7}{10}})$ d) $\frac{27^{-1}}{27^{\frac{-2}{3}}}$ f) $\frac{(8^{-2})(8^{2.5})}{(8^6)^{-0.25}}$

7. Predict the order of these six expressions in terms of value from lowest to highest. Check your answers with your calculator. Express answers to three decimal places.

a) $\sqrt[4]{623}$ c) $\sqrt[10]{10.24}$ e) $17.5^{\frac{5}{8}}$

b) $125^{\frac{2}{5}}$ d) $80.9^{\frac{1}{4}}$ f) $21.4^{\frac{3}{2}}$

8. The volume of a cube is $0.015\,625\ m^3$. Determine the length of each side.

A

9. Use your calculator to determine the values of $27^{\frac{4}{3}}$ and $27^{1.3333}$. Compare the two answers. What do you notice?

10. Explain why $(-100)^{0.2}$ is possible to evaluate while $(-100)^{0.5}$ is not.

C

11. Write $125^{\frac{-2}{3}}$ in radical form, then evaluate. Explain each of your steps.

K

12. Evaluate.

a) $-256^{0.375}$
b) $15.625^{\frac{4}{3}}$
c) $\sqrt[3]{-0.027^4}$
d) $(-3.375)^{\frac{2}{3}}$
e) $\sqrt[4]{(0.0016)^3}$
f) $(-7776)^{1.6}$

13. The power 4^3 means that 4 is multiplied by itself three times. Explain the meaning of $4^{2.5}$.

14. State whether each expression is true or false.

a) $9^{\frac{1}{2}} + 4^{\frac{1}{2}} = (9 + 4)^{\frac{1}{2}}$
b) $9^{\frac{1}{2}} + 4^{\frac{1}{2}} = (9 \times 4)^{\frac{1}{2}}$
c) $\left(\frac{1}{a} + \frac{1}{b}\right)^{-1} = a + b$
d) $\left(\frac{1}{a} \times \frac{1}{b}\right)^{-1} = ab$
e) $\left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)^6 = x^2 + y^2$
f) $\left[\left(x^{\frac{1}{3}}\right)\left(y^{\frac{1}{3}}\right)\right]^6 = x^2y^2$

15. a) What are some values of m and n that would make $(-2)^{\frac{m}{n}}$ undefined?

T b) What are some values of m and n that would make $(6)^{\frac{m}{n}}$ undefined?

Extending

16. Given that $x^y = y^x$, what could x and y be? Is there a way to find the answer graphically?

17. Mary must solve the equation $1.225 = (1 + i)^{12}$ to determine the value of each dollar she invested for a year at the interest rate i per year. Her friend Bindu suggests that she begin by taking the 12th root of each side of the equation. Will this work? Try it and solve for the variable i. Explain why it does or does not work.

18. Solve.

a) $\left(\frac{1}{16}\right)^{\frac{1}{4}} - \sqrt[3]{\frac{8}{27}} = \sqrt{x^2}$

b) $\sqrt[3]{\frac{1}{8}} - \sqrt[4]{x^4} + 15 = \sqrt[4]{16}$

4.4 Simplifying Algebraic Expressions Involving Exponents

GOAL

Simplify algebraic expressions involving powers and radicals.

LEARN ABOUT the Math

The ratio of the surface area to the volume of microorganisms affects their ability to survive. An organism with a higher surface area-to-volume ratio is more buoyant and uses less of its own energy to remain near the surface of a liquid, where food is more plentiful.

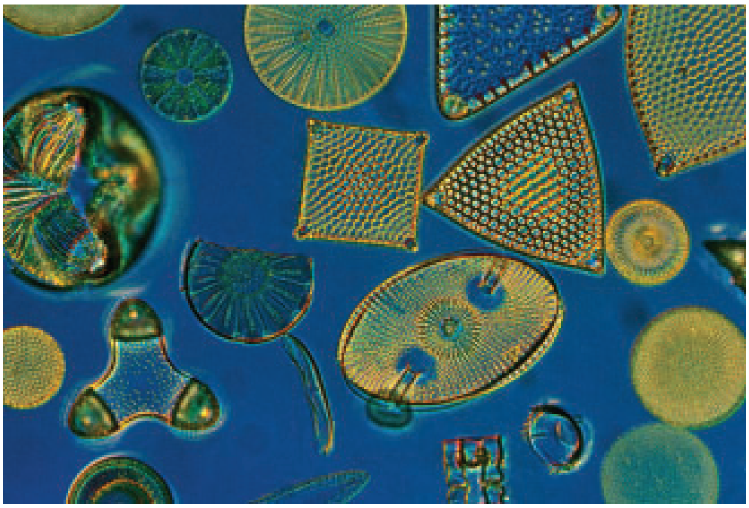
Mike is calculating the surface area-to-volume ratio for different-sized cells. He assumes that the cells are spherical.

For a sphere,

$$SA(r) = 4\pi r^2 \quad \text{and} \quad V(r) = \tfrac{4}{3}\pi r^3.$$

He substitutes the value of the radius into each formula and then divides the two expressions to calculate the ratio.

Radius (μm)	Surface Area/ Volume
1	$\dfrac{4\pi}{\left(\frac{4}{3}\pi\right)}$
1.5	$\dfrac{9\pi}{4.5\pi}$
2	$\dfrac{16\pi}{\left(\frac{32}{3}\pi\right)}$
2.5	$\dfrac{25\pi}{\left(\frac{125}{6}\pi\right)}$
3	$\dfrac{36\pi}{36\pi}$
3.5	$\dfrac{49\pi}{\left(\frac{343}{6\pi}\right)}$



? How can Mike simplify the calculation he uses?

EXAMPLE 1 Representing the surface area-to-volume ratio

Simplify $\frac{SA(r)}{V(r)}$, given that $SA(r) = 4\pi r^2$ and $V(r) = \frac{4}{3}\pi r^3$.

Bram's Solution

$$\frac{SA(r)}{V(r)}$$

I used the formulas for *SA* and *V* and wrote the ratio.

$$= \frac{4\pi r^2}{\frac{4}{3}\pi r^3}$$

The numerator and denominator have a factor of π, so I divided both by π.

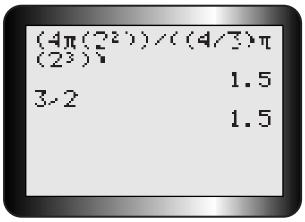
I started to simplify the expression by dividing the coefficients.

$\left(4 \div \frac{4}{3} = 4 \times \frac{3}{4} = 3\right)$

$$= 3r^{-1}$$

The bases of the powers were the same, so I subtracted exponents to simplify the part of the expression involving *r*.

$$= \frac{3}{r}$$



I used a calculator and substituted $r = 2$ in the unsimplified ratio first and my simplified expression next.

Each version gave me the same answer, so I think that they are equivalent, but the second one took far fewer keystrokes!

Reflecting

A. How can you use the simplified ratio to explain why the values in Mike's table kept decreasing?

B. Is it necessary to simplify an algebraic expression before you substitute numbers and perform calculations? Explain.

C. What are the advantages and disadvantages to simplifying an algebraic expression prior to performing calculations?

D. Do the exponent rules used on algebraic expressions work the same way as they do on numerical expressions? Explain by referring to Bram's work.

APPLY the Math

EXAMPLE 2 Connecting the exponent rules to the simplification of algebraic expressions

Simplify $\dfrac{(2x^{-3}y^{2})^{3}}{(x^{3}y^{-4})^{2}}$.

Adnan's Solution

$$\frac{(2x^{-3}y^{2})^{3}}{(x^{3}y^{-4})^{2}} = \frac{(2)^{3}(x^{-3})^{3}(y^{2})^{3}}{(x^{3})^{2}(y^{-4})^{2}}$$

I used the product-of-powers rule to raise each factor in the numerator to the third power and to square each factor in the denominator. Then I multiplied exponents.

$$= \frac{8x^{-9}y^{6}}{x^{6}y^{-8}}$$
$$= 8x^{-9-6}y^{6-(-8)}$$
$$= 8x^{-15}y^{14}$$
$$= \frac{8y^{14}}{x^{15}}$$

I simplified the whole expression by subtracting exponents of terms with the same base.

One of the powers had a negative exponent. To write it with positive exponents, I used its reciprocal.

EXAMPLE 3 Selecting a computational strategy to evaluate an expression

Evaluate the expression $\dfrac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}}$ for $x = -3$ and $n = 2$.

Bonnie's Solution: Substituting, then Simplifying

$$\frac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}} = \frac{(-3)^{2(2)+1}(-3)^{3(2)-1}}{(-3)^{2(2)-5}}$$
$$= \frac{(-3)^{5}(-3)^{5}}{(-3)^{-1}}$$
$$= \frac{(-243)(-243)}{\frac{1}{-3}}$$
$$= -177\ 147$$

I substituted the values for x and n into the expression.

Then I evaluated the numerator and denominator separately, before dividing one by the other.

Alana's Solution: Simplifying, then Substituting

$$\frac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}}$$

Each power had the same base, so I simplified by using exponent rules before I substituted.

$$= \frac{x^{(2n+1)+(3n-1)}}{x^{2n-5}}$$

I added the exponents in the numerator to express it as a single power.

$$= \frac{x^{5n}}{x^{2n-5}}$$

$$= x^{(5n)-(2n-5)}$$

Then I subtracted the exponents in the denominator to divide the powers.

$$= x^{3n+5}$$

$$= (-3)^{3(2)+5}$$

$$= (-3)^{11}$$

$$= -177\ 147$$

Once I had a single power, I substituted -3 for x and 2 for n and evaluated.

EXAMPLE 4 Simplifying an expression involving powers with rational exponents

Simplify $\dfrac{(27a^{-3}b^{12})^{\frac{1}{3}}}{(16a^{-8}b^{12})^{\frac{1}{2}}}$.

Jane's Solution

$$\frac{(27a^{-3}b^{12})^{\frac{1}{3}}}{(16a^{-8}b^{12})^{\frac{1}{2}}} = \frac{27^{\frac{1}{3}}a^{\frac{-3}{3}}b^{\frac{12}{3}}}{16^{\frac{1}{2}}a^{\frac{-8}{2}}b^{\frac{12}{2}}}$$

$$= \frac{3a^{-1}b^{4}}{4a^{-4}b^{6}}$$

In the numerator, I applied the exponent $\frac{1}{3}$ to each number or variable inside the parentheses, using the power-of-a-power rule. I did the same in the denominator, applying the exponent $\frac{1}{2}$ to the numbers and variables.

$$= \frac{3}{4}a^{-1+4}b^{4-6}$$

I simplified by subtracting the exponents.

$$= \frac{3}{4}a^{3}b^{-2}$$

$$= \frac{3a^{3}}{4b^{2}}$$

I expressed the answer with positive exponents.

Sometimes it is necessary to express an expression involving radicals using exponents in order to simplify it.

EXAMPLE 5 Representing an expression involving radicals as a single power

Simplify $\left(\frac{\sqrt[5]{x^8}}{\sqrt{x^3}}\right)^3$.

Albino's Solution

$$\left(\frac{\sqrt[5]{x^8}}{\sqrt{x^3}}\right)^3 = \left(\frac{x^{\frac{8}{5}}}{x^{\frac{3}{2}}}\right)^3$$

Since this is a fifth root divided by a square root, I couldn't write it as a single radical.

I changed the radical expressions to exponential form and used exponent rules to simplify.

$$= (x^{\frac{8}{5}-\frac{3}{2}})^3$$

$$= (x^{\frac{16}{10}-\frac{15}{10}})^3$$

$$= (x^{\frac{1}{10}})^3$$

$$= x^{\frac{3}{10}}$$

$$= \sqrt[10]{x^3}$$

When I got a single power, I converted it to radical form.

In Summary

Key Idea

- Algebraic expressions involving powers containing integer and rational exponents can be simplified with the use of the exponent rules in the same way numerical expressions can be simplified.

Need to Know

- When evaluating an algebraic expression by substitution, simplify prior to substituting. The answer will be the same if substitution is done prior to simplifying, but the number of calculations will be reduced.
- Algebraic expressions involving radicals can often be simplified by changing the expression into exponential form and applying the rules for exponents.

CHECK Your Understanding

1. Simplify. Express each answer with positive exponents.

a) $x^4(x^3)$

b) $(p^{-3})(p)^5$

c) $\frac{m^5}{m^{-3}}$

d) $\frac{a^{-4}}{a^{-2}}$

e) $(y^3)^2$

f) $(k^6)^{-2}$

2. Simplify. Express each answer with positive exponents.

a) $y^{10}(y^4)^{-3}$

b) $(x^{-3})^{-3}(x^{-1})^5$

c) $\dfrac{(n^{-4})^3}{(n^{-3})^{-4}}$

d) $\dfrac{w^4(w^{-3})}{(w^{-2})^{-1}}$

e) $\dfrac{(x^{-1})^4x}{x^{-3}}$

f) $\dfrac{(b^{-7})^2}{b(b^{-5})b^9}$

3. Consider the expression $\dfrac{x^7(y^2)^3}{x^5y^4}$.

a) Substitute $x = -2$ and $y = 3$ into the expression, and evaluate it.

b) Simplify the expression. Then substitute the values for x and y to evaluate it.

c) Which method seems more efficient?

PRACTISING

4. Simplify. Express answers with positive exponents.

a) $(pq^2)^{-1}(p^3q^3)$

b) $\left(\dfrac{x^3}{y}\right)^{-2}$

c) $\dfrac{(ab)^{-2}}{b^5}$

d) $\dfrac{m^2n^2}{(m^3n^{-2})^2}$

e) $\dfrac{(w^2x)^2}{(x^{-1})^2w^3}$

f) $\left(\dfrac{(ab)^{-1}}{a^2b^{-3}}\right)^{-2}$

5. Simplify. Express answers with positive exponents.

a) $(3xy^4)^2(2x^2y)^3$

b) $\dfrac{(2a^3)^2}{4ab^2}$

c) $\dfrac{(10x)^{-1}y^3}{15x^3y^{-3}}$

d) $\dfrac{(3m^4n^2)^2}{12m^{-2}n^6}$

e) $\dfrac{p^{-5}(r^3)^2}{(p^2r)^2(p^{-1})^2}$

f) $\left(\dfrac{(x^3y)^{-1}(x^4y^3)}{(x^2y^{-3})^{-2}}\right)^{-1}$

6. Simplify. Express answers with positive exponents.

K

a) $(x^4)^{\frac{1}{2}}(x^6)^{-\frac{1}{3}}$

b) $\dfrac{9(c^8)^{0.5}}{(16c^{12})^{0.25}}$

c) $\dfrac{\sqrt{25m^{-12}}}{\sqrt{36m^{10}}}$

d) $\sqrt[3]{\dfrac{(10x^3)^2}{(10x^6)^{-1}}}$

e) $\left(\dfrac{(32x^5)^{-2}}{(x^{-1})^{10}}\right)^{0.2}$

f) $\dfrac{\sqrt[10]{1024x^{20}}}{\sqrt[9]{512x^{27}}}$

7. Evaluate each expression. Express answers in rational form with positive exponents.

a) $(16x^6y^4)^{\frac{1}{2}}$ for $x = 2, y = 1$

b) $\dfrac{(9p^{-2})^{\frac{1}{2}}}{6p^2}$ for $p = 3$

c) $\dfrac{(81x^4y^6)^{\frac{1}{2}}}{8(x^9y^3)^{\frac{1}{3}}}$ for $x = 10, y = 5$

d) $\left(\dfrac{(25a^4)^{-1}}{(7a^{-2}b)^2}\right)^{\frac{1}{2}}$ for $a = 11, b = 10$

8. Evaluate. Express answers in rational form with positive exponents.

a) $(\sqrt{10\,000x})^{\frac{3}{2}}$ for $x = 16$

b) $\left(\frac{(4x^3)^4}{(x^3)^6}\right)^{-0.5}$ for $x = 5$

c) $(-2a^2b)^{-3}\sqrt{25a^4b^6}$ for $a = 1$, $b = 2$

d) $\sqrt{\frac{(18m^{-5}n^2)(32m^2n)}{4mn^{-3}}}$ for $m = 10$, $n = 1$

9. Simplify. Express answers in rational form with positive exponents.

a) $(36m^4n^6)^{0.5}(81m^{12}n^8)^{0.25}$

b) $\left(\frac{(6x^3)^2(6y^3)}{(9xy)^6}\right)^{-\frac{1}{3}}$

c) $\left(\frac{\sqrt{64a^{12}}}{(a^{1.5})^{-6}}\right)^{\frac{2}{3}}$

d) $\left(\frac{(x^{18})^{\frac{-1}{6}}}{\sqrt[5]{243x^{10}}}\right)^{0.5}$

10. T If $M = \frac{(16x^8y^{-4})^{\frac{1}{4}}}{32x^{-2}y^8}$, determine values for x and y so that

a) $M = 1$ **b)** $M > 1$ **c)** $0 < M < 1$ **d)** $M < 0$

11. A The volume and surface area of a cylinder are given, respectively, by the formulas

$$V = \pi r^2 h \quad \text{and} \quad SA = 2\pi rh + 2\pi r^2.$$

a) Determine an expression, in simplified form, that represents the surface area-to-volume ratio for a cylinder.

b) Calculate the ratio for a radius of 0.8 cm and a height of 12 cm.

12. If $x = -2$ and $y = 3$, write the three expressions in order from least to greatest.

$$\frac{y^{-4}(x^2)^{-3}y^{-3}}{x^{-5}(y^{-4})^2}, \frac{x^{-3}(y^{-1})^{-2}}{(x^{-5})(y^4)}, (y^{-5})(x^5)^{-2}(y^2)(x^{-3})^{-4}$$

13. C How is simplifying algebraic expressions like simplifying numerical ones? How is it different?

Extending

14. a) The formula for the volume of a sphere of radius r is $V(r) = \frac{4}{3}\pi r^3$. Solve this equation for r. Write two versions, one in radical form and one in exponential form.

b) Determine the radius of a sphere with a volume of $\frac{256\pi}{3}$ m^3.

15. Simplify $\frac{\sqrt{x(x^{2n+1})}}{\sqrt[3]{x^{3n}}}$, $x > 0$.

4 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 4.4, Examples 2, 3, and 4.
- Try Mid-Chapter Review Questions 9, 10, and 11.

Q: How do you evaluate an expression involving a negative exponent?

A: To evaluate a number raised to a negative exponent, you can take the reciprocal of the number, change the sign of the exponent, and then evaluate the equivalent expression.

$$b^{-x} = \frac{1}{b^x}, \quad b \neq 0$$

EXAMPLE

$$5^{-3} = \frac{1}{5^3} = \frac{1}{125} \quad \text{and} \quad \left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$$

Study Aid

- See Lesson 4.2, Examples 1, 3, and 4.
- Try Mid-Chapter Review Questions 1, 2, and 3.

Q: How do you simplify algebraic expressions involving rational exponents?

A: You can use the same exponent rules you use to simplify and evaluate numerical expressions.

Study Aid

- See Lesson 4.3, Examples 3, 4, and 5.
- Try Mid-Chapter Review Questions 5 to 8.

Q: How do you evaluate an expression involving a rational exponent?

A: The denominator of a rational exponent indicates the index of the root of the base. The numerator has the same meaning as an integer exponent. These can be evaluated in two different ways:

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m \quad \text{or} \quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

EXAMPLE

$$\begin{aligned} 27^{\frac{2}{3}} &= (27^{\frac{1}{3}})^2 \\ &= (\sqrt[3]{27})^2 \\ &= 3^2 \\ &= 9 \end{aligned} \quad \text{or} \quad \begin{aligned} 27^{\frac{2}{3}} &= (27^2)^{\frac{1}{3}} \\ &= \sqrt[3]{27^2} \\ &= \sqrt[3]{243} \\ &= 9 \end{aligned}$$

PRACTICE Questions

Lesson 4.2

1. Write as a single power. Express your answers with positive exponents.
   a) $5(5^4)$
   b) $\frac{(-8)^4}{(-8)^5}$
   c) $(9^3)^6$
   d) $\frac{3(3)^6}{3^5}$
   e) $\left(\frac{1}{10}\right)^6\left(\frac{1}{10}\right)^{-4}$
   f) $\left(\frac{(7)^2}{(7)^4}\right)^{-1}$

2. Evaluate. Express answers in rational form.
   a) $4^{-2} - 8^{-1}$
   b) $(4 + 8)^0 - 5^{-2}$
   c) $25^{-1} + 3(5^{-1})^2$
   d) $\left(-\frac{1}{2}\right)^3 + 4^{-3}$

3. Evaluate. Express answers in rational form.
   a) $\left(\frac{4}{7}\right)^2$
   b) $\left(-\frac{2}{5}\right)^3$
   c) $\left(\frac{-2}{3}\right)^{-3}$
   d) $\frac{(-3)^{-2}}{(-3)^{-5}}$

Lesson 4.3

4. What restrictions are there on the value of x in $x^{-\frac{1}{2}}$? Are these restrictions different for $x^{\frac{1}{2}}$? Explain.

5. Evaluate. Express answers in rational form.
   a) $\left(\frac{49}{81}\right)^{\frac{1}{2}}$
   b) $\sqrt{\frac{100}{121}}$
   c) $\left(\frac{16}{9}\right)^{-0.5}$
   d) $\left((-125)^{\frac{1}{3}}\right)^{-3}$
   e) $\sqrt[4]{(-9)^{-2}}$
   f) $\frac{-\sqrt[3]{512}}{\sqrt[5]{-1024}}$

6. Copy and complete the table. Express values in the last column in rational form.

	Exponential Form	Radical Form	Evaluation of Expression
a)	$100^{\frac{1}{2}}$		
b)	$16^{0.25}$		
c)		$\sqrt{121}$	
d)	$(-27)^{\frac{5}{3}}$		
e)	$49^{2.5}$		
f)		$\sqrt[10]{1024}$	

7. Evaluate. Express answers to three decimals.
   a) $-456^{\frac{4}{7}}$
   b) $98^{0.75}$
   c) $\left(\frac{5}{8}\right)^{\frac{5}{8}}$
   d) $(\sqrt[5]{-1000})^3$

8. Evaluate $-8^{\frac{4}{3}}$ and $(-8)^{\frac{4}{3}}$. Explain the difference between the two.

Lesson 4.4

9. Simplify. Express answers with positive exponents.
   a) $\frac{(x^{-3})x^5}{x^7}$
   b) $\frac{(n^{-4})n^{-6}}{(n^{-2})^7}$
   c) $\left(\frac{(y^2)^6}{y^9}\right)^{-2}$
   d) $\frac{(-2x^5)^3}{8x^{10}}$
   e) $(3a^2)^{-3}(9a^{-1})^2$
   f) $\frac{(4r^{-6})(-2r^2)^5}{(-2r)^4}$

10. Simplify. Express answers with positive exponents.
   a) $\frac{x^{0.5}y^{1.8}}{x^{0.3}y^{2.5}}$
   b) $\frac{(mn^3)^{-\frac{1}{2}}}{m^{\frac{1}{2}}n^{-\frac{5}{2}}}$
   c) $\frac{\sqrt{x^2y^4}}{(x^{-2}y^3)^{-1}}$
   d) $\left(\frac{2abc^3}{(2a^2b^3c)^2}\right)^{-2}$
   e) $\frac{\sqrt[4]{81p^8}}{\sqrt{9p^4}}$
   f) $\frac{\sqrt[6]{(8x^6)^2}}{\sqrt[4]{625x^8}}$

11. Evaluate each expression for $a = 2$ and $b = 3$. Express values in rational form.
   a) $\left(\frac{b^3}{a^{\frac{5}{2}}}\right)^2\left(\frac{2a^4}{b^5}\right)$
   b) $\sqrt{\frac{9b^3(ab)^2}{(a^2b^3)^3}}$

12. Simplify.
   a) $(a^{10+2p})(a^{-p-8})$
   b) $(2x^2)^{3-2m}\left(\frac{1}{x}\right)^{2m}$
   c) $[(c)^{2n-3m}](c^3)^m \div (c^2)^n$
   d) $(x^{4n-m})\left(\frac{1}{x^3}\right)^{m+n}$

4.5 Exploring the Properties of Exponential Functions

YOU WILL NEED

- graphing calculator
- graph paper

GOAL

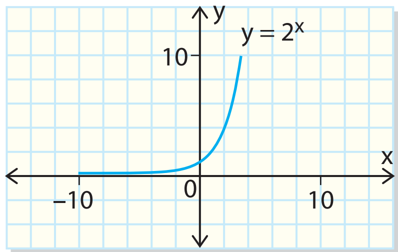
Determine the characteristics of the graphs and equations of exponential functions.

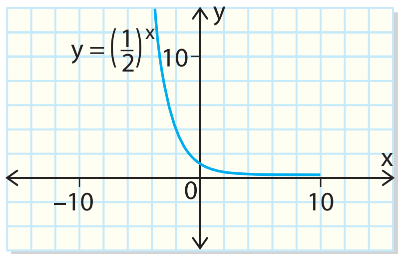
EXPLORE the Math

Functions such as $f(x) = 2^x$ and $g(x) = (\frac{1}{2})^x$ are examples of **exponential functions**. These types of functions can model many different phenomena, including population growth and the cooling of a liquid.

exponential function

a function of the form $y = a(b^x)$





? What are the characteristics of the graph of the exponential function $f(x) = b^x$, and how does it compare with the graphs of quadratic and linear functions?

A. Create a tables of values for each of the following functions. $g(x) = x$, $h(x) = x^2$, and $k(x) = 2^x$, where $-3 \le x \le 5$

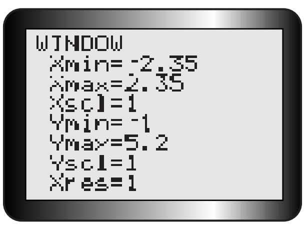
B. In each of your tables, calculate the first and second differences. Describe the difference patterns for each type of function.

C. Graph each function on graph paper and draw a smooth curve through each set of points. Label each curve with the appropriate equation.

D. State the domain and range of each function.

E. For each function, describe how values of the dependent variable, y, change as the values of the independent variable, x, increase and decrease.

F. Use a graphing calculator to graph the functions $y = 2^x$, $y = 5^x$, and $y = 10^x$. Graph all three functions on the same graph. Use the WINDOW settings shown.



G. For each function, state
- the domain and range
- the intercepts
- the equations of any **asymptotes**, if present

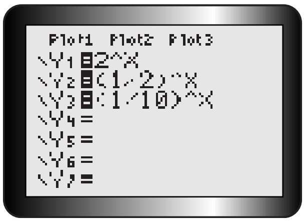
H. Examine the y-values as x increases and decreases. Which curve increases faster as you trace to the right? Which one decreases faster as you trace to the left?

Tech Support
For help tracing functions on the graphing calculator, see Technical Appendix, B-2.

I. Delete the second and third functions $(y = 5^x$ and $y = 10^x)$, and replace them with $y = (\frac{1}{2})^x$ and $y = (\frac{1}{10})^x$ (or $y = 0.5^x$ and $y = 0.1^x)$.

J. For each new function, state
- the domain and range
- the intercepts
- the equations of any asymptotes

K. Describe how each of the graphs of $y = (\frac{1}{2})^x$ and $y = (\frac{1}{10})^x$ differs from $y = 2^x$ as the x-values increase and as they decrease.



L. Investigate what happens when the base of an exponential function is negative. Try $y = (-2)^x$. Discuss your findings.

M. Compare the features of the graphs of $f(x) = b^x$ for each group. Think about the domain, range, intercepts, and asymptotes.
i) different values of b when $b > 1$
ii) different values of b when $0 < b < 1$
iii) values of b when $0 < b < 1$, compared with values of $b > 1$
iii) values of $b < 0$ compared with values of $b > 0$

Reflecting

N. How do the differences for exponential functions differ from those for linear and quadratic functions? How can you tell that a function is exponential from its differences?

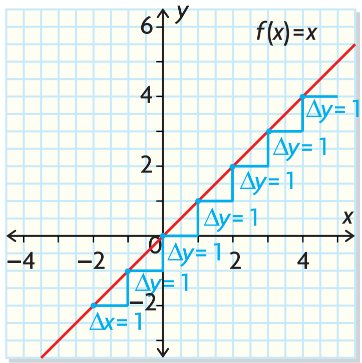
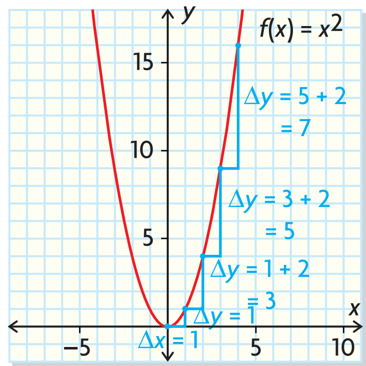
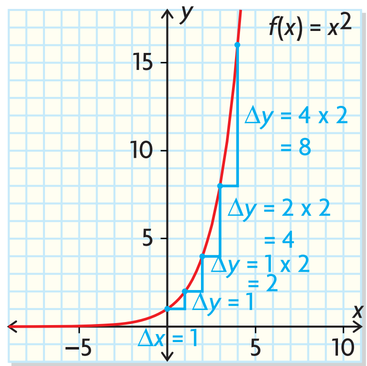
O. The base of an exponential function of the form $f(x) = b^x$ cannot be 1. Explain why this restriction is necessary.

P. Explain how you can distinguish an exponential function from a quadratic function and a linear function by using
- the graphs of each function
- a table of values for each function
- the equation of each function

In Summary

Key Ideas

- Linear, quadratic, and exponential functions have unique first-difference patterns that allow them to be recognized.

Linear	Quadratic	Exponential
Linear functions have constant first differences.	Quadratic functions have first differences that are related by addition. Their second differences are constant.	Exponential functions have first differences that are related by multiplication. Their second finite differences are not constant.
		

- The exponential function $f(x) = b^x$ is
  - an increasing function representing growth when $b > 1$
  - a decreasing function representing decay when $0 < b < 1$

Need to Know

- The exponential function $f(x) = b^x$ has the following characteristics:
  - If $b > 0$, then the function is defined, its domain is $\{x \in \mathbf{R}\}$, and its range is $\{y \in \mathbf{R} \mid y \geq 0\}$.
  - If $b > 1$, then the greater the value of b, the faster the growth.
  - If $0 < b < 1$, then the lesser the value of b, the faster the decay.
  - The function has the x-axis, $y = 0$, as horizontal *asymptote*.
  - The function has a y-intercept of 1.
- Linear, quadratic, and exponential functions can be recognized from their graphs. Linear functions are represented by straight lines, quadratic functions by parabolas, and exponential functions by quickly increasing or decreasing curves with a horizontal asymptote.
- A function in which the variables have exponent 1 (e.g., $f(x) = 2x$) is linear. A function with a single squared term (e.g., $f(x) = 3x^2 - 1$) is quadratic. A function with a positive base (0 and 1 excluded) and variable exponent (e.g., $f(x) = 5^x$) is exponential.

FURTHER Your Understanding

1. Use differences to identify the type of function represented by the table of values.

a)

x	y
−4	5
−3	8
−2	13
−1	20
0	29
1	40

b)

x	y
−5	32
−4	16
−3	8
−2	4
−1	2
0	1

c)

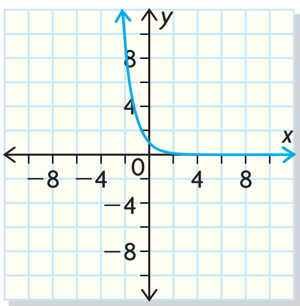
x	y
−2	−2.75
0	−2
2	1
4	13
6	61
8	253

d)

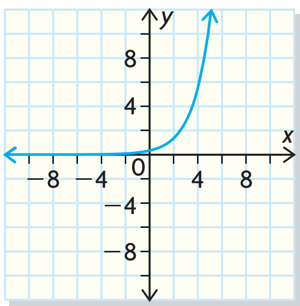
x	y
0.5	0.9
0.75	1.1
1	1.3
1.25	1.5
1.5	1.7
1.75	1.9

2. What type of function is represented in each graph? Explain how you know.

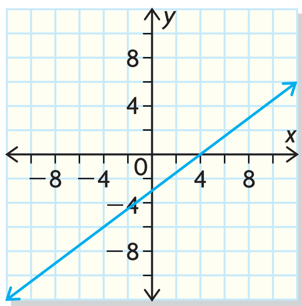
a)



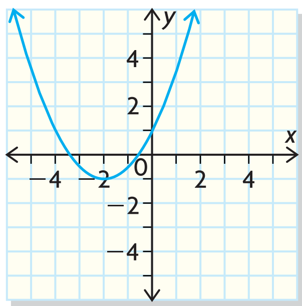
b)



c)



d)



4.6 Transformations of Exponential Functions

YOU WILL NEED

- graphing calculator

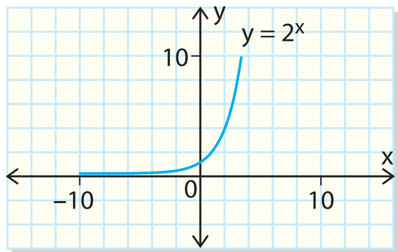
GOAL

Investigate the effects of transformations on the graphs and equations of exponential functions.

INVESTIGATE the Math

Recall the graph of the function $f(x) = 2^x$.

- It is an increasing function.
- It has a y-intercept of 1.
- Its asymptote is the line $y = 0$.



? If $f(x) = 2^x$, how do the parameters a, k, d, and c in the function $g(x) = af(k(x - d)) + c$ affect the size and shape of the graph of $f(x)$?

A. Use your graphing calculator to graph the function $f(x) = 2^x$. Use the window settings shown.

Tech *Support*

You can adjust to these settings by pressing [ZOOM] and [4] [.] [ZDecimal].

B. Predict what will happen to the function $f(x) = 2^x$ if it is changed to

- $g(x) = 2^x + 1$ or $h(x) = 2^x - 1$
- $p(x) = 2^{x+1}$ or $q(x) = 2^{x-1}$

C. Copy and complete the table by graphing the given functions, one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. For each function, sketch the graph on the same grid and describe how its points and features have changed.

<table>
<tr><th>Function</th><th>Sketch</th><th>Table of Values</th><th>Description of Changes of New Graph</th></tr>
<tr><td>$g(x) = 2^x + 1$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^x + 1$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$h(x) = 2^x - 1$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^x - 1$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$p(x) = 2^{x+1}$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^{x+1}$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$q(x) = 2^{x-1}$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^{x-1}$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
</table>

D. Describe the types of transformations you observed in part C. Comment on how the features and points of the original graph were changed by the transformations.

E. Predict what will happen to the function $f(x) = 2^x$ if it is changed to

- $g(x) = 3(2^x)$
- $h(x) = 0.5(2^x)$
- $j(x) = -(2^x)$

F. Create a table like the one in part C using the given functions in part E. Graph each function one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. In your table, sketch the graph on the same grid, complete the table of values, and describe how its points and features have changed.

G. Describe the types of transformations you observed in part F. Comment on how the features and points of the original graph were changed by the transformations.

H. Predict what will happen to the function $f(x) = 2^x$ if it is changed to
- $g(x) = 2^{2x}$
- $h(x) = 2^{0.5x}$
- $j(x) = 2^{-x}$

I. Create a table like the one in part C using the given functions in part H. Graph each function one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. In your table, sketch the graph, complete the table of values, and describe how its points and features have changed.

J. Describe the types of transformations you observed in part I. Comment on how the features and points of the original graph were changed by such transformations.

K. Choose several different bases for the original function. Experiment with different kinds of transformations. Are the changes in the function affected by the value of the base?

L. Summarize your findings by describing the roles that the parameters a, k, d, and c play in the function defined by $f(x) = ab^{k(x-d)} + c$.

Reflecting

M. Which transformations change the shape of the curve? Explain how the equation is changed by these transformations.

N. Which transformations change the location of the asymptote? Explain how the equation is changed by these transformations.

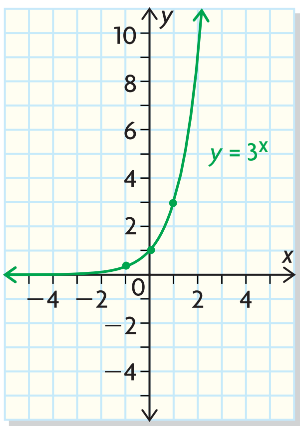
O. Do the transformations affect $f(x) = b^x$ in the same way they affect $f(x) = x$, $f(x) = x^2$, $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, and $f(x) = |x|$? Explain.

APPLY the Math

EXAMPLE 1 Using reasoning to predict the shape of the graph of an exponential function

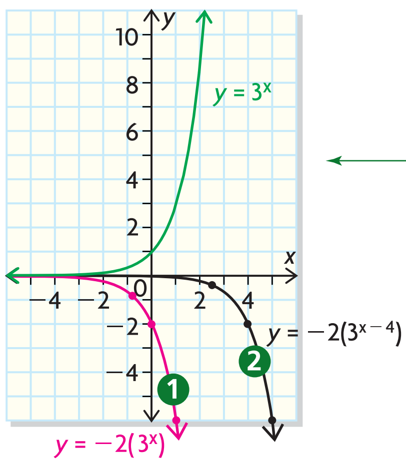
Use transformations to sketch the function $y = -2(3^{x-4})$. State the domain and range.

J.P.'s Solution



I began by sketching the graph of $y = 3^x$.

Three of its key points are (0, 1), (1, 3), and $(-1, \frac{1}{3})$. The asymptote is the *x*-axis, $y = 0$.



The function I really want to graph is $y = -2(3^{x-4})$. The base function, $y = 3^x$, was changed by multiplying all *y*-values by -2, resulting in a vertical stretch of factor 2 and a reflection in the *x*-axis.

Subtracting 4 from *x* results in a translation of 4 units to the right.

I could perform these two transformations in either order, since one affected only the *x*-coordinate and the other affected only the *y*-coordinate. I did the stretch first.

1. With vertical stretches and reflection in the *x*-axis (multiplying by -2, graphed in red), my key points had their *y*-values doubled:

   $(0, 1) \rightarrow (0, -2)$, $(1, 3) \rightarrow (1, -6)$, and $(-1, \frac{1}{3}) \rightarrow (-1, -\frac{2}{3})$

   The asymptote $y = 0$ was not affected.

2. With translations (subtracting 4, graphed in black), the key points changed by adding 4 to the *x*-values:

   $(0, -2) \rightarrow (4, -2)$, $(1, -6) \rightarrow (5, -6)$, and $(-1, -\frac{2}{3}) \rightarrow (3, -\frac{2}{3})$

   This shifted the curve 4 units to the right. The asymptote $y = 0$ was not affected.

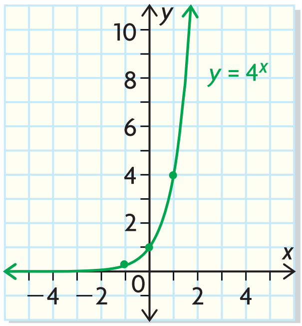
The domain of the original function, $\{x \in \mathbf{R}\}$, was not changed by the transformations.

The range, determined by the equation of the asymptote, was $y > 0$ for the original function. There was no vertical translation, so the asymptote remained the same, but, due to the reflection in the *x*-axis, the range changed to $\{y \in \mathbf{R} \mid y < 0\}$.

EXAMPLE 2 Connecting the graphs of different exponential functions

Use transformations to sketch the graph of $y = 4^{-2x-4} + 3$.

Ilia's Solution



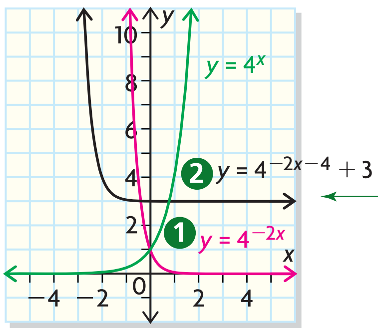
I began by sketching the graph of the base curve, $y = 4^x$. It has the line $y = 0$ as its asymptote, and three of its key points are (0, 1), (1, 4), and $\left(-1, \frac{1}{4}\right)$.

I factored the exponent to see the different transformations clearly:

$y = 4^{-2(x+2)} + 3$

The x-values were multiplied by -2, resulting in a horizontal compression of factor $\frac{1}{2}$, as well as a reflection in the y-axis.

There were two translations: 2 units to the left and 3 units up.



I applied the transformations in the proper order.

The table shows how the key points and the equation of the asymptote change:

Point or Asymptote	Horizontal Stretch and Reflection	Horizontal Translation	Vertical Translation
(0, 1)	(0, 1)	$(-2, 1)$	$(-2, 4)$
(1, 4)	$\left(-\frac{1}{2}, 4\right)$	$\left(-2\frac{1}{2}, 4\right)$	$\left(-2\frac{1}{2}, 7\right)$
$\left(-1, \frac{1}{4}\right)$	$\left(\frac{1}{2}, \frac{1}{4}\right)$	$\left(-1\frac{1}{2}, \frac{1}{4}\right)$	$\left(-1\frac{1}{2}, 3\frac{1}{4}\right)$
$y = 0$	$y = 0$	$y = 0$	$y = 3$

1. There was one stretch and one reflection, each of which applied only to the x-coordinate: a horizontal compression of factor $\frac{1}{2}$ and a reflection in the y-axis (shown in red).
2. There were two translations: 2 units to the left and 3 units up (shown in black).

EXAMPLE 3 Communicating the relationship among different exponential functions

Compare and contrast the functions defined by $f(x) = 9^x$ and $g(x) = 3^{2x}$.

Pinder's Solution: Using Exponent Rules

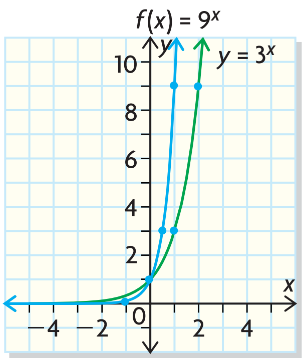
$$\begin{aligned} f(x) &= 9^x \\ &= (3^2)^x \\ &= 3^{2x} \\ &= g(x) \end{aligned}$$

Both functions are the same.

9 is a power of 3, so, to make it easier to compare 9^x with 3^{2x}, I substituted 3^2 for 9 in the first equation.

By the power-of-a-power rule, $f(x)$ has the same equation as $g(x)$.

Kareem's Solution



Both functions are the same.

$f(x) = 9^x$ is an exponential function with a y-intercept of 1 and the line $y = 0$ as its asymptote. Also, $f(x) = 9^x$ passes through the points (1, 9) and $\left(-1, \frac{1}{9}\right)$.

$g(x) = 3^{2x}$ is the base function $y = 3^x$ after a horizontal compression of factor $\frac{1}{2}$. This means that the key points change by multiplying their x-values by $\frac{1}{2}$. The point (1, 3) becomes (0.5, 3) and (2, 9) becomes (1, 9). When I plotted these points, I got points on the curve of $f(x)$.

EXAMPLE 4 Connecting the verbal and algebraic descriptions of transformations of an exponential curve

An exponential function with a base of 2 has been stretched vertically by a factor of 1.5 and reflected in the y-axis. Its asymptote is the line $y = 2$. Its y-intercept is (0, 3.5). Write an equation of the function and discuss its domain and range.

Louise's Solution

$y = a2^{k(x-d)} + c$

I began by writing the general form of the exponential equation with a base of 2.

$y = 1.5(2^{-x}) + c$

Since the function had been stretched vertically by a factor of 1.5, $a = 1.5$. The function has also been reflected in the y-axis, so $k = -1$. There was no horizontal translation, so $d = 0$.

$y = 1.5(2^{-x}) + 2$

Since the horizontal asymptote is $y = 2$ the function has been translated vertically by 2 units, so $c = 2$.

$$\begin{aligned} y &= 1.5(2^{-(0)}) + 2 \\ &= 1.5(1) + 2 \\ &= 3.5 \end{aligned}$$

I substituted $x = 0$ into the equation and calculated the y-intercept. It matched the stated y-intercept, so my equation seemed to represent this function.

The original domain is $\{x \in \mathbf{R}\}$. The transformations didn't change this.

The range changed, since there was a vertical translation. The asymptote moved up 2 units along with the function, so the range is $\{y \in \mathbf{R} \mid y > 2\}$.

In Summary

Key Ideas

- In functions of the form $g(x) = af(k(x - d)) + c$, the constants a, k, d, and c change the location or shape of the graph of $f(x)$. The shape is dependent on the value of the base function $f(x) = b^x$, as well as on the values of a and k.
- Functions of the form $g(x) = af(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the key points of the base function $f(x) = b^x$, one at a time, following the order of operations. The horizontal asymptote changes when vertical translations are applied.

Need to Know

- In exponential functions of the form $g(x) = a\,b^{k(x-d)} + c$:
  - If $|a| > 1$, a vertical stretch by a factor of $|a|$ occurs. If $0 < |a| < 1$, a vertical compression by a factor of $|a|$ occurs. If a is also negative, then the function is reflected in the x-axis.
  - If $|k| > 1$, a horizontal compression by a factor of $|\frac{1}{k}|$ occurs. If $0 < |k| < 1$, a horizontal stretch by a factor of $|\frac{1}{k}|$ occurs. If k is also negative, then the function is reflected in the y-axis.
  - If $d > 0$, a horizontal translation of d units to the right occurs. If $d < 0$, a horizontal translation to the left occurs.
  - If $c > 0$, a vertical translation of c units up occurs. If $c < 0$, a vertical translation of c units down occurs.
  - You might have to factor the exponent to see what the transformations are. For example, if the exponent is $2x + 2$, it is easier to see that there was a horizontal stretch of 2 and a horizontal translation of 1 to the left if you factor to $2(x + 1)$.
  - When transforming functions, consider the order. You might perform stretches and reflections followed by translations, but if the stretch involves a different coordinate than the translation, the order doesn't matter.
  - The domain is always $\{x \in \mathbf{R}\}$. Transformations do not change the domain.
  - The range depends on the location of the horizontal asymptote and whether the function is above or below the asymptote. If it is above the asymptote, its range is $y > c$. If it is below, its range is $y < c$.

CHECK Your Understanding

1. Each of the following are transformations of $f(x) = 3^x$. Describe each transformation.

a) $g(x) = 3^x + 3$

b) $g(x) = 3^{x+3}$

c) $g(x) = \frac{1}{3}(3^x)$

d) $g(x) = 3^{\frac{x}{3}}$

2. For each transformation, state the base function and then describe the transformations in the order they could be applied.

a) $f(x) = -3(4^{x+1})$

b) $g(x) = 2\left(\frac{1}{2}\right)^{2x} + 3$

c) $h(x) = 7(0.5^{x-4}) - 1$

d) $k(x) = 5^{3x-6}$

3. State the y-intercept, the equation of the asymptote, and the domain and range for each of the functions in questions 1 and 2.

PRACTISING

4. Each of the following are transformations of $h(x) = (\frac{1}{2})^x$. Use words to describe the sequence of transformations in each case.

   a) $g(x) = -\left(\frac{1}{2}\right)^{2x}$

   b) $g(x) = 5\left(\frac{1}{2}\right)^{-(x-3)}$

   c) $g(x) = -4\left(\frac{1}{2}\right)^{3x+9} - 6$

5. Let $f(x) = 4^x$. For each function that follows,
   K
   - state the transformations that must be applied to $f(x)$
   - state the y-intercept and the equation of the asymptote
   - sketch the new function
   - state the domain and range

   a) $g(x) = 0.5f(-x) + 2$

   b) $h(x) = -f(0.25x + 1) - 1$

   c) $g(x) = -2f(2x - 6)$

   d) $h(x) = f(-0.5x + 1)$

6. Compare the functions $f(x) = 6^x$ and $g(x) = 3^{2x}$.

7. A cup of hot liquid was left to cool in a room whose temperature was 20 °C.
   C
   The temperature changes with time according to the function $T(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{30}} + 20$. Use your knowledge of transformations to sketch this function. Explain the meaning of the y-intercept and the asymptote in the context of this problem.

8. The doubling time for a certain type of yeast cell is 3 h. The number of cells after t hours is described by $N(t) = N_0 2^{\frac{t}{3}}$, where N_0 is the initial population.

   a) How would the graph and the equation change if the doubling time were 9 h?

   b) What are the domain and range of this function in the context of this problem?

9. Match the equation of the functions from the list to the appropriate graph at the top of the next page.

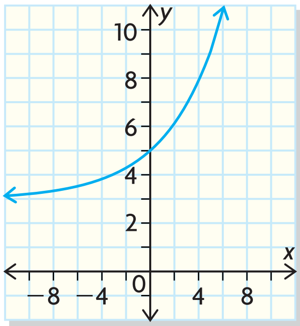
   a) $f(x) = -\left(\frac{1}{4}\right)^{-x} + 3$

   b) $y = \left(\frac{1}{4}\right)^{x} + 3$

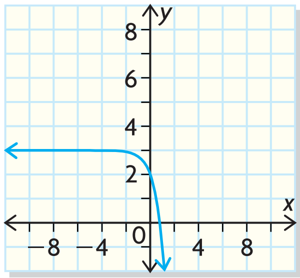
   c) $g(x) = -\left(\frac{5}{4}\right)^{-x} + 3$

   d) $h(x) = 2\left(\frac{5}{4}\right)^{x} + 3$

i)



iii)



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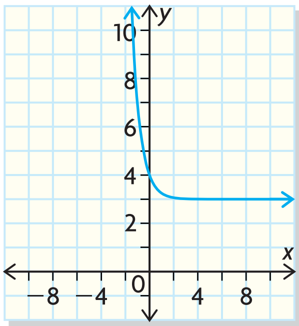

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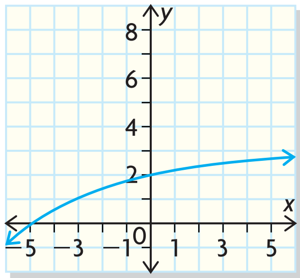
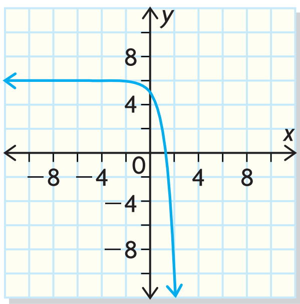
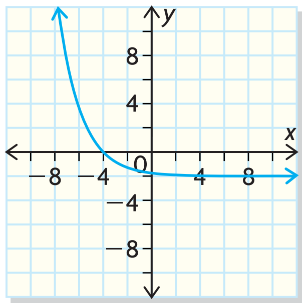


10. Each graph represents a transformation of the function $f(x) = 2^x$. Write an equation for each one.

a)



b)



11. State the transformations necessary (and in the proper order) to transform
T $f(x) = 2^{x+1} + 5$ to $g(x) = \frac{1}{4}(2^x)$.

Extending

12. Use your knowledge of transformations to sketch the function
$f(x) = \frac{-3}{2^{x+2}} - 1$.

13. Use your knowledge of transformations to sketch the function
$g(x) = 4 - 2\left(\frac{1}{3}\right)^{-0.5x+1}$.

14. State the transformations necessary (and in the proper order) to transform
$m(x) = -\left(\frac{3}{2}\right)^{2x-2}$ to $n(x) = -\left(\frac{9}{4}\right)^{-x+1} + 2$.

4.7 Applications Involving Exponential Functions

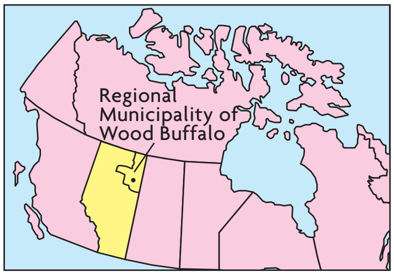
YOU WILL NEED

- graphing calculator

GOAL

Use exponential functions to solve problems involving exponential growth and decay.

The regional municipality of Wood Buffalo, Alberta, has experienced a large population increase in recent years due to the discovery of one of the world's largest oil deposits. Its population, 35 000 in 1996, has grown at an annual rate of approximately 8%.



? How long will it take for the population to double at this growth rate?

LEARN ABOUT the Math

EXAMPLE 1 Selecting a strategy to determine the doubling rate

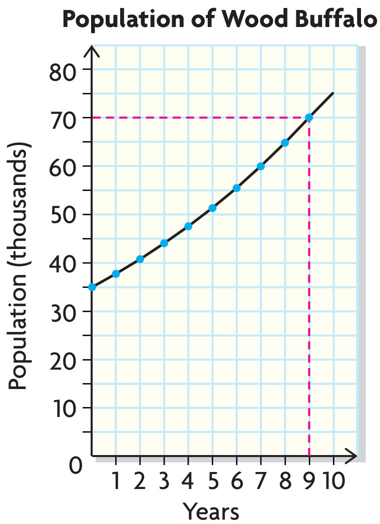
Carter's Solution: Using a Table of Values and a Graph

$$0.08(35) + 35 = 35(0.08 + 1) = 35(1.08)$$

When you add 8% of a number to the number, the new value is 108% of the old one. This is the same as multiplying by 1.08, so I created the table of values by repeatedly multiplying by 1.08.

I did this 10 times, once for each year, and saw that the population doubled to 70 000 after 9 years of growth.

Time (year from 1996)	0	1	2	3	4	5	6	7	8	9	10
Population (thousands)	35.0	37.8	40.8	44.1	47.6	51.4	55.5	60.0	64.8	70.0	75.6



I plotted the points and drew a smooth curve through the data.

I drew a horizontal line across the graph at 70 000 and saw that it touched the curve at 9 years.

Sonja's Solution: Creating an Algebraic Model

$P(1) = 35(1.08) = 37.8$

$P(2) = 37.8(1.08) = 40.8$

Substituting P(1) into P(2):

$P(2) = 35(1.08)(1.08)$

$= 35(1.08)^2$

So, $P(3) = 35(1.08)^2(1.08)$

$= 35(1.08)^3$

Therefore, $P(n) = 35(1.08)^n$

$P(6) = 35(1.08)^6$

$= 35(1.586\ 874\ 323)$

$= 55.540\ 601\ 3$

$P(9) = 35(1.08)^9$

$= 35(1.999\ 004\ 627)$

$= 69.965\ 161\ 95 \doteq 70$

The population would double in approximately 9 years at an 8% rate of growth.

To calculate the population after 1 year, I needed to multiply 35 by 1.08. For each additional year, I repeatedly multiply by 1.08. Repeated multiplication can be represented with exponents. The value of the exponent will correspond to the number for the year.

This led to an algebraic model.

Since population is a function of time, I expressed the relationship in function notation. I used $P(n)$, where the exponent, n, would represent the number of years after 1996 and $P(n)$ would represent the population in thousands.

I guessed that it would take 6 years for the population to double. I substituted $n = 6$ into the expression for the function, but it was too low.

I tried values for n until I got an answer that was close to the target of 70; $n = 9$ was pretty close.

Reflecting

A. Which features of the function indicate that it is exponential?

B. Describe what each part of the equation $P(n) = 35(1.08)^n$ represents in the context of the problem *and* the features of the graph.

C. Compare Carter's and Sonja's solutions. Which approach do you think is better? Why?

APPLY the Math

EXAMPLE 2 Solving an exponential decay problem, given the equation

A 200 g sample of radioactive polonium-210 has a *half-life* of 138 days. This means that every 138 days, the amount of polonium left in a sample is half of the original amount. The mass of polonium, in grams, that remains after t days can be modelled by $M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$.

a) Determine the mass that remains after 5 years.

b) How long does it take for this 200 g sample to decay to 110 g?

Zubin's Solution: Using the Algebraic Model

a) $$5 \text{ years} = 5(365) \text{ days}$$

Since the half-life is measured in days, I converted the number of years to days before substituting into the function.

$$= 1825 \text{ days}$$

$$M(1825) = 200\left(\frac{1}{2}\right)^{\frac{1825}{138}}$$

I used my calculator to determine the answer.

$$\doteq 200(0.000\,104\,5)$$

$$\doteq 0.021$$

There is approximately 0.02 g of polonium-210 left after 5 years.

b) $$M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$$

I began by writing the equation and substituting the amount of the sample remaining.

$$110 = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$$

I needed to isolate t in the equation, so I divided each side by 200.

$$\frac{110}{200} = \left(\frac{1}{2}\right)^{\frac{t}{138}}$$

I didn't know how to isolate t, so I used guess and check to find the answer.

$$0.55 = \left(\frac{1}{2}\right)^{\frac{t}{138}}$$

$$M(100) = 200\left(\frac{1}{2}\right)^{\frac{100}{138}}$$
$$\doteq 121 \text{ g}$$

I knew that if the exponent was 1 ($t = 138$ days), the original amount would be halved, but the amount I needed to find was 110 g, so the exponent needed to be less than 1. I guessed 100 days, which I substituted into the original equation. I calculated the answer.

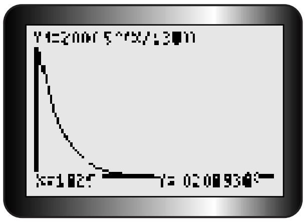
It was too high, which meant that my guess was too low.

$$M(119) = 200\left(\frac{1}{2}\right)^{\frac{119}{138}}$$
$$\doteq 110 \text{ g}$$

I guessed and checked a few more times until I found the answer of approximately 119 days.

Barry's Solution: Using a Graphical Model

a)



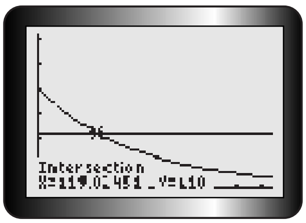
I graphed $M(t)$, then used the value operation. I had to change 5 years into days.

There is about 0.02 g remaining after 5 years.

b)

I graphed $M(t)$, and I graphed a horizontal line to represent 110 g.

I knew that the point where the line met the curve would represent the answer.



I used the "Intersect" operation on the graphing calculator to find the point. The x-value represents the number of days.

It takes approximately 119 days for the sample to decay to 110 g.

Tech Support

For help determining the point(s) of intersection between two functions, see Technical Appendix, B-12.

EXAMPLE 3 Solving a problem by determining an equation for a curve of good fit

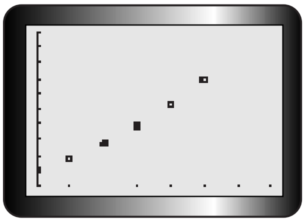
A biologist tracks the population of a new species of frog over several years. From the table of values, determine an equation that models the frog's population growth, and determine the number of years before the population triples.

Year	0	1	2	3	4	5
Population	400	480	576	691	829	995

Tech Support

For help creating scatter plots on a graphing calculator, see Technical Appendix, B-11.

Fred's Solution

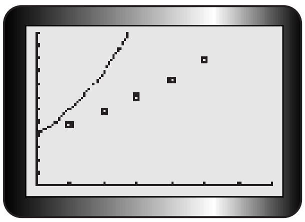


I used my graphing calculator to create a scatter plot.

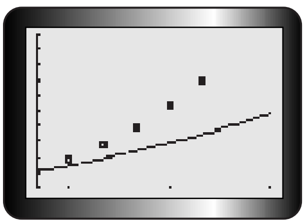
The equation is of the form $P(t) = ab^t$, where

- $P(t)$ represents the population in year t
- a is the initial population
- b is the base of the exponential function

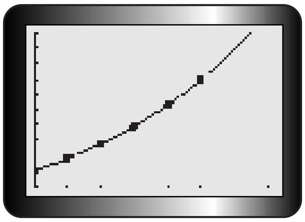
Since the function is increasing, $b > 1$. The initial population occurs when $x = 0$. That means that $a = 400$. If $b = 2$, then the population would have doubled, but it went up by only 80 in the first year, so the value of b must be less than 2.



I tried $b = 1.5$ and entered the equation $P(t) = 400(1.5)^t$ into the equation editor. The graph rose too quickly, so 1.5 is too great for b.



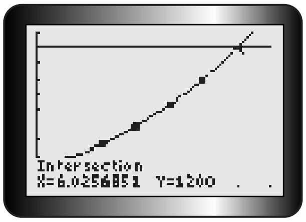
I changed the equation to $P(t) = 400(1.1)^t$. I graphed the equation on the calculator. I checked to see if the curve looked right. It rose too slowly, so b must be between 1.1 and 1.5.



I continued this process until I found a good fit when $b = 1.2$.

The equation that models this population is

$P(t) = 400(1.2)^t$



To determine the year that the population tripled, I graphed the line $P = 1200$ and found the intersection point of the curve and the line.

From this graph, I determined that the frog population tripled in approximately 6 years.

EXAMPLE 4 Representing a real-world problem with an algebraic model

A new car costs $24 000. It loses 18% of its value each year after it is purchased. This is called *depreciation*. Determine the value of the car after 30 months.

Gregg's Solution

$y = ab^x$

The car's value decreases each year. Another way to think about the car *losing* 18% of its value each year is to say that it *keeps* 82% of its value. To determine its value, I multiplied its value in the previous year by 0.82. The repeated multiplication suggested that this relationship is exponential. That makes sense, since this has to be a decreasing function where $0 < b < 1$.

$$V(n) = 24(0.82)^n$$

I used V and n to remind me of what they represented.

The base of the exponential function that models the value of the car is 0.82. The initial value is \$24 000, which is the value of a and the exponent n is measured in years.

$$\begin{aligned} n &= 30 \text{ months} \\ &= 30 \div 12 \text{ years} \\ &= 2.5 \text{ years} \end{aligned}$$

I converted 30 months to years to get my answer.

$$\begin{aligned} V(2.5) &= 24(0.82)^{2.5} \\ &= 24(0.608\ 884\ 097) \\ &\doteq 14.6 \end{aligned}$$

The car is worth about \$14 600 after 30 months.

In Summary

Key Ideas

- The exponential function $f(x) = ab^x$ and its graph can be used as a model to solve problems involving exponential growth and decay. Note that
  - $f(x)$ is the final amount or number
  - a is the initial amount or number
  - for exponential growth, $b = 1 +$ growth rate; for exponential decay, $b = 1 -$ decay rate
  - x is the number of growth or decay periods

Need to Know

- For situations that can be modeled by an exponential function:
  - If the *growth rate* (as a percent) is given, then the base of the power in the equation can be obtained by *adding* the rate, as a decimal, to 1. For example, a growth rate of 8% involves multiplying repeatedly by 1.08.
  - If the *decay rate* (as a percent) is given, then the base of the power in the equation is obtained by *subtracting* the rate, as a decimal, from 1. For example, a decay rate of 8% involves multiplying repeatedly by 0.92.
  - One way to tell the difference between growth and decay is to consider whether the quantity in question (e.g., light intensity, population, dollar value) has increased or decreased.
  - The units for the growth/decay rate and for the number of growth/decay periods must be the same. For example, if light intensity decreases "per metre," then the number of decay periods in the equation is measured in metres, too.

CHECK Your Understanding

1. Solve each exponential equation. Express answers to the nearest hundredth of a unit.

a) $A = 250(1.05)^{10}$

b) $P = 9000\left(\frac{1}{2}\right)^{8}$

c) $500 = N_0(1.25)^{1.25}$

d) $625 = P(0.71)^{9}$

2. Complete the table.

	Function	Exponential Growth or Decay?	Initial Value	Growth or Decay Rate
a)	$V(t) = 20(1.02)^{t}$			
b)	$P(n) = (0.8)^{n}$			
c)	$A(x) = 0.5(3)^{x}$			
d)	$Q(w) = 600\left(\frac{5}{8}\right)^{w}$			

3. The growth in population of a small town since 1996 is given by the function $P(n) = 1250(1.03)^{n}$.

a) What is the initial population? Explain how you know.

b) What is the growth rate? Explain how you know.

c) Determine the population in the year 2007.

d) In which year does the population reach 2000 people?

4. A computer loses its value each month after it is purchased. Its value as a function of time, in months, is modelled by $V(m) = 1500(0.95)^{m}$.

a) What is the initial value of the computer? Explain how you know.

b) What is the rate of depreciation? Explain how you know.

c) Determine the value of the computer after 2 years.

d) In which month after it is purchased does the computer's worth fall below $900?

PRACTISING

5. In 1990, a sum of $1000 is invested at a rate of 6% per year for 15 years.

a) What is the growth rate?

b) What is the initial amount?

c) How many growth periods are there?

d) Write an equation that models the growth of the investment, and use it to determine the value of the investment after 15 years.

6. A species of bacteria has a population of 500 at noon. It doubles every 10 h.
K The function that models the growth of the population, P, at any hour, t, is

$$P(t) = 500\left(2^{\frac{t}{10}}\right).$$

a) Why is the exponent $\frac{t}{10}$?
b) Why is the base 2?
c) Why is the multiplier 500?
d) Determine the population at midnight.
e) Determine the population at noon the next day.
f) Determine the time at which the population first exceeds 2000.

7. Which of these functions describe exponential decay? Explain.
a) $g(x) = -4(3)^x$
b) $h(x) = 0.8(1.2)^x$
c) $j(x) = 3(0.8)^{2x}$
d) $k(x) = \frac{1}{3}(0.9)^{\frac{x}{2}}$

8. A town with a population of 12 000 has been growing at an average rate of 2.5% for the last 10 years. Suppose this growth rate will be maintained in the future. The function that models the town's growth is

$$P(n) = 12(1.025^n)$$

where $P(n)$ represents the population (in thousands) and n is the number of years from now.
a) Determine the population of the town in 10 years.
b) Determine the number of years until the population doubles.
c) Use this equation (or another method) to determine the number of years ago that the population was 8000. Answer to the nearest year.
d) What are the domain and range of the function?

9. A student records the internal temperature of a hot sandwich that has been
A left to cool on a kitchen counter. The room temperature is 19 °C. An equation that models this situation is

$$T(t) = 63(0.5)^{\frac{t}{10}} + 19$$

where T is the temperature in degrees Celsius and t is the time in minutes.
a) What was the temperature of the sandwich when she began to record its temperature?
b) Determine the temperature, to the nearest degree, of the sandwich after 20 min.
c) How much time did it take for the sandwich to reach an internal temperature of 30 °C?

10. In each case, write an equation that models the situation described. Explain what each part of each equation represents.
    a) the percent of colour left if blue jeans lose 1% of their colour every time they are washed
    b) the population if a town had 2500 residents in 1990 and grew at a rate of 0.5% each year after that for t years
    c) the population of a colony if a single bacterium takes 1 day to divide into two; the population is P after t days

11. A population of yeast cells can double in as little as 1 h. Assume an initial population of 80 cells.
    a) What is the growth rate, in percent per hour, of this colony of yeast cells?
    b) Write an equation that can be used to determine the population of cells at t hours.
    c) Use your equation to determine the population after 6 h.
    d) Use your equation to determine the population after 90 min.
    e) Approximately how many hours would it take for the population to reach 1 million cells?
    f) What are the domain and range for this situation?

12. A collector's hockey card is purchased in 1990 for \$5. The value increases by 6% every year.
    a) Write an equation that models the value of the card, given the number of years since 1990.
    b) Determine the increase in value of the card in the 4th year after it was purchased (from year 3 to year 4).
    c) Determine the increase in value of the card in the 20th year after it was purchased.

13. Light intensity in a lake falls by 9% per metre of depth relative to the surface.
    a) Write an equation that models the intensity of light per metre of depth. Assume that the intensity is 100% at the surface.
    b) Determine the intensity of light at a depth of 7.5 m.

14. A disinfectant is advertised as being able to kill 99% of all germs with each application.
    a) Write an equation that represents the percent of germs left with n applications.
    b) Suppose a kitchen countertop has 10 billion (10^{10}) germs. How many applications are required to eliminate all of the germs?

15. **T** A town has a population of 8400 in 1990. Fifteen years later, its population grew to 12 500. Determine the average annual growth rate of this town's population.

16. **C** A group of yeast cells grows by 75% every 3 h. At 9 a.m., there are 200 yeast cells.
    a) Write an equation that models the number of cells, given the number of hours after 9 a.m.
    b) Explain how each part of your equation is related to the given information.

Extending

17. In the year 2002, a single baby girl born in Alberta was given the name Nevaeh.
- Two years later, there were 18 girls (including the first one) with that name.
- By 2005, there were 70 girls with the name (*National Post,* Wed., May 24, 2006, p. A2).

a) Investigate whether or not this is an example of exponential growth.
b) Determine what the growth rate might be, and create a possible equation to model the growth in the popularity of this name.
c) Discuss any limitations of your model.

18. Psychologist H. Ebbinghaus performed experiments in which he had people memorize lists of words and then tested their memory of the list. He found that the percent of words they remembered can be modelled by

$$R(T) = \frac{100}{1 + 1.08T^{0.21}}$$

where $R(T)$ is the percent of words remembered after T hours. This equation is now known as the "forgetting curve," even though it actually models the percent of words remembered!

a) Graph this function with technology. Describe its features and decide whether or not it is an example of exponential decay.
b) Predict the percent of words remembered after 24 h.

Curious | *Math*

Zeno's Paradox

Zeno of Elea (c. 490–425 BCE), a Greek philosopher and mathematician, is famous for his paradoxes that deal with motion. (A paradox is a statement that runs counter to common sense, but may actually be true.) Zeno suggested that it is impossible to get to point *B* from point *A*.

He illustrated his point of view with a story.

Achilles (point *A*) and Tortoise agreed to have a race. Tortoise was given a head start (point *B*). After the race started, Achilles travelled half the distance between himself and Tortoise (point *C*). And again, after a period of time, he travelled half the remaining distance between himself and Tortoise (point *D*). Each time he arrived at the halfway point, there was a new, and smaller, halfway point. So if you look at it this way, there is an infinite number of halfway points and Achilles would never catch up to Tortoise.

1. What is the function that models this problem, if Tortoise was given a head start of 1000 m? Does this function support Zeno's paradox? Explain.

2. Who will win the race between Achilles and Tortoise? Explain.

4 Chapter Review

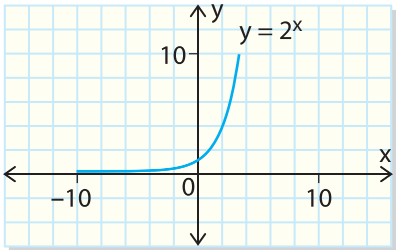
FREQUENTLY ASKED Questions

Q: How can you identify an exponential function from

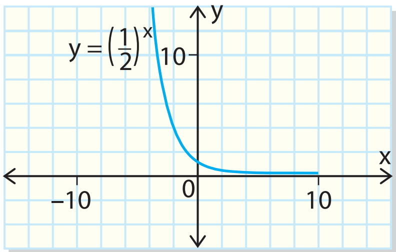
- its equation?
- its graph?
- a table of values?

A: The exponential function has the form $f(x) = b^x$, where the variable is an exponent.

The shape of its graph depends upon the parameter b.



If $b > 1$, then the curve increases as x increases.



If $0 < b < 1$, then the curve decreases as x increases.

In each case, the function has the x-axis (the line $y = 0$) as its horizontal asymptote.

A differences table for an exponential function shows that the differences are never constant, as they are for linear and quadratic functions. They are related by a multiplication pattern.

x	$y = 3^x$	First Differences	Second Differences
0	1		
		2	
1	3		4
		6	
2	9		12
		18	
3	27		36
		54	
4	81		108
		162	
5	243		324
		486	
6	729		

Study Aid

- See Lesson 4.5.
- Try Chapter Review Questions 9 and 10.

Study | ***Aid***

- See Lesson 4.6, Examples 1, 2, and 3.
- Try Chapter Review Questions 11 and 12.

Q: How can transformations help in drawing the graphs of exponential functions?

A: Functions of the form $g(x) = af(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the key points and asymptotes of the parent function $f(x) = b^x$, following an appropriate order—often, stretches and compressions, then reflections, and finally translations.

In functions of the form $g(x) = ab^{k(x-d)} + c$, the constants a, k, d, and c change the location or shape of the graph of $f(x)$. The shape of the graph of $g(x)$ depends on the value of the base of the function, $f(x) = b^x$.

- a represents the vertical stretch or compression factor. If $a < 0$, then the function has also been reflected in the x-axis.
- k represents the horizontal stretch or compression factor. If $k < 0$, then the function has also been reflected in the y-axis.
- c represents the number of units of vertical translation up or down.
- d represents the number of units of horizontal translation right or left.

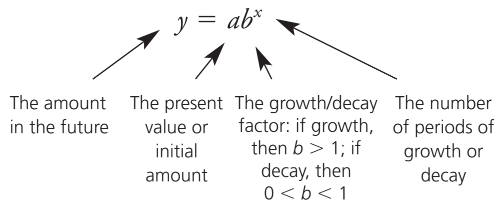
Study | ***Aid***

- See Lesson 4.7, Examples 1, 2, 3 and 4.
- Try Chapter Review Questions 13 to 17.

Q: How can exponential functions model growth and decay? How can you use them to solve problems?

A: Exponential functions can be used to model phenomena exhibiting repeated multiplication of the same factor.

Each formula is modelled after the exponential function



When solving problems, list these four elements of the equation and fill in the data as you read the problem. This will help you organize the information and create the equation you require to solve the problem.

Here are some examples:

Growth	Decay
Cell division (doubling bacteria, yeast cells, etc.): $P(t) = P_0(2)^{\frac{t}{D}}$	Radioactivity or half-life: $N(t) = 100\left(\frac{1}{2}\right)^{\frac{t}{H}}$
Population growth: $P(n) = P_0(1 + r)^n$	Depreciation of assets: $V(n) = V_0(1 - r)^n$
Growth in money: $A(n) = P(1 + i)^n$	Light intensity in water: $V(n) = 100(1 - r)^n$

PRACTICE Questions

Lesson 4.2

1. **a)** If $x > 1$, which is greater, x^{-2} or x^2? Why?
b) Are there values of x that make the statement $x^{-2} > x^2$ true? Explain.

2. Write each as a single power. Then evaluate. Express answers in rational form.

a) $(-7)^3(-7)^{-4}$

b) $\dfrac{(-2)^8}{(-2)^3}$

c) $\dfrac{(5)^{-3}(5)^6}{5^3}$

d) $\dfrac{4^{-10}(4^{-3})^6}{(4^{-4})^8}$

e) $(11)^9\left(\dfrac{1}{11}\right)^7$

f) $\left(\dfrac{(-3)^7(-3)^4}{(-3^4)^3}\right)^{-3}$

Lesson 4.3

3. Express each radical in exponential form and each power in radical form.

a) $\sqrt[3]{x^7}$

b) $y^{\frac{8}{5}}$

c) $(\sqrt{p})^{11}$

d) $m^{1.25}$

4. Evaluate. Express answers in rational form.

a) $\left(\dfrac{2}{5}\right)^{-3}$

b) $\left(\dfrac{16}{225}\right)^{-0.5}$

c) $\dfrac{(81)^{-0.25}}{\sqrt[3]{-125}}$

d) $(\sqrt[3]{-27})^4$

e) $(\sqrt[5]{-32})(\sqrt[6]{64})^5$

f) $\sqrt[6]{((-2)^3)^2}$

5. Simplify. Write with only positive exponents.

a) $a^{\frac{3}{2}}(a^{-\frac{3}{2}})$

b) $\dfrac{b^{0.8}}{b^{-0.2}}$

c) $\dfrac{c\left(c^{\frac{5}{6}}\right)}{c^2}$

d) $\dfrac{d^{-5}d^{\frac{11}{2}}}{(d^{-3})^2}$

e) $((e^{-2})^{\frac{7}{2}})^{-2}$

f) $((f^{-\frac{1}{6}})^{\frac{6}{5}})^{-1}$

6. Explain why $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$, for $a > 0$ and $b > 0$.

Lesson 4.4

7. Evaluate each expression for the given values. Express answers in rational form.

a) $(5x)^2(2x)^3$; $x = -2$

b) $\dfrac{8m^{-5}}{(2m)^{-3}}$; $m = 4$

c) $\dfrac{2w(3w^{-2})}{(2w)^2}$; $w = -3$

d) $\dfrac{(9y)^2}{(3y^{-1})^3}$; $y = -2$

e) $(6(x^{-4})^3)^{-1}$; $x = -2$

f) $\dfrac{(-2x^{-2})^3(6x)^2}{2(-3x^{-1})^3}$; $x = \dfrac{1}{2}$

8. Simplify. Write each expression using only positive exponents. All variables are positive.

a) $\sqrt[3]{27x^3y^9}$

b) $\sqrt{\dfrac{a^6b^5}{a^8b^3}}$

c) $\dfrac{m^{\frac{3}{2}}n^{-2}}{m^{\frac{7}{2}}n^{-\frac{3}{2}}}$

d) $\dfrac{\sqrt[4]{x^{-16}(x^6)^{-6}}}{(x^4)^{-\frac{11}{2}}}$

e) $((-x^{0.5})^3)^{-1.2}$

f) $\dfrac{\sqrt{x^6(y^3)^{-2}}}{(x^3y)^{-2}}$

Lesson 4.5

9. Identify the type of function (linear, quadratic, or exponential) for each table of values.

a)

x	*y*
−5	−38
0	−3
5	42
10	97
15	162
20	237

b)

x	*y*
0	−45
2	−15
4	15
6	45
8	75
10	105

c)

x	y
1	13
2	43
3	163
4	643
5	2 563
6	10 243

d)

x	y
−2	40
−1	20
0	10
1	5
2	2.5
3	1.25

e)

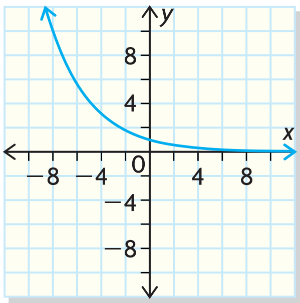
x	y
−2	2000
−1	1000
0	500
1	250
2	125
3	62.5

f)

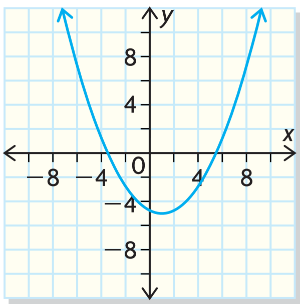
x	y
0.2	−10.8
0.4	−9.6
0.6	−7.2
0.8	−2.4
1	7.2
1.2	26.4

10. Identify each type of function (linear, quadratic, or exponential) from its graph.

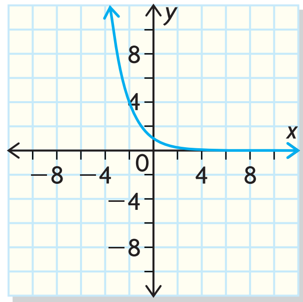
a)



b)



c)



Lesson 4.6

11. For each exponential function, state the base function, $y = b^x$. Then state the transformations that map the base function onto the given function. Use transformations to sketch each graph.

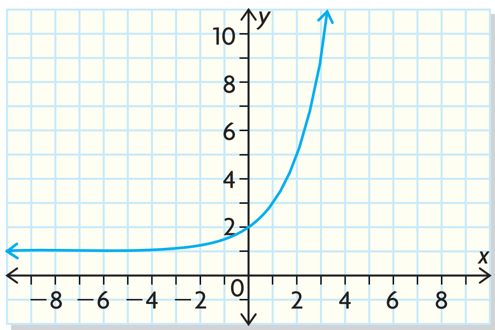
a) $y = \left(\frac{1}{2}\right)^{\frac{x}{2}} - 3$

b) $y = \frac{1}{4}(2)^{-x} + 1$

c) $y = -2(3)^{2x+4}$

d) $y = \frac{-1}{10}(5)^{3x-9} + 10$

12. The exponential function shown has been reflected in the y-axis and translated vertically. State its y-intercept, its asymptote, and a possible equation for it.



Lesson 4.7

13. Complete the table.

	Function	Exponential Growth or Decay?	Initial Value (y-intercept)	Growth or Decay Rate
a)	$V(t) = 100(1.08)^t$			
b)	$P(n) = 32(0.95)^n$			
c)	$A(x) = 5(3)^x$			
d)	$Q(n) = 600\left(\frac{5}{8}\right)^n$			

14. A hot cup of coffee cools according to the equation

$$T(t) = 69\left(\frac{1}{2}\right)^{\frac{t}{30}} + 21$$

where T is the temperature in degrees Celsius and t is the time in minutes.

a) Which part of the equation indicates that this is an example of exponential decay?
b) What was the initial temperature of the coffee?
c) Use your knowledge of transformations to sketch the graph of this function.
d) Determine the temperature of the coffee, to the nearest degree, after 48 min.
e) Explain how the equation would change if the coffee cooled faster.
f) Explain how the graph would change if the coffee cooled faster.

15. The value of a car after it is purchased depreciates according to the formula

$$V(n) = 28\,000(0.875)^n$$

where $V(n)$ is the car's value in the nth year since it was purchased.

a) What is the purchase price of the car?
b) What is the annual rate of depreciation?
c) What is the car's value at the end of 3 years?
d) What is its value at the end of 30 months?
e) How much value does the car lose in its first year?
f) How much value does it lose in its fifth year?

16. Write the equation that models each situation. In each case, describe each part of your equation.
a) the percent of a pond covered by water lilies if they cover one-third of a pond now and each week they increase their coverage by 10%
b) the amount remaining of the radioactive isotope U_{238} if it has a half-life of 4.5×10^9 years
c) the intensity of light if each gel used to change the colour of a spotlight reduces the intensity of the light by 4%

17. The population of a city is growing at an average rate of 3% per year. In 1990, the population was 45 000.
a) Write an equation that models the growth of the city. Explain what each part of the equation represents.
b) Use your equation to determine the population of the city in 2007.
c) Determine the year during which the population will have doubled.
d) Suppose the population took only 10 years to double. What growth rate would be required for this to have happened?

4 Chapter Self-Test

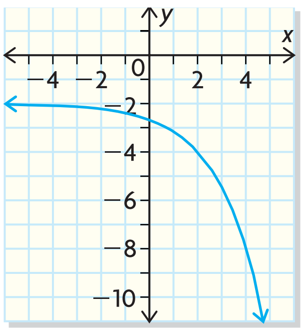
1. The function $f(x) = -\frac{1}{2}(3^{2x+4}) + 5$ is the transformation of the function $g(x) = 3^x$.
   a) Explain how you can tell what type of function $f(x)$ represents just by looking at the equation.
   b) Create a table of values for $f(x)$. Describe how to tell the type of function it is from its table of values.
   c) Describe the transformations necessary (in the proper order) that map $g(x)$ onto $f(x)$. Sketch $f(x)$ and state the equation of its asymptote.

2. Evaluate. Express answers as rational numbers.
   a) $(-5)^{-3}$
   b) $27^{\frac{2}{3}}$

3. Simplify. Use only positive exponents in your final answers.
   a) $(-3x^2y)^3(-3x^{-3}y)^2$
   b) $\frac{(5a^{-1}b^2)^{-2}}{125a^5b^{-3}}$
   c) $\sqrt[5]{\frac{1024(x^{-1})^{10}}{(2x^{-3})^5}}$
   d) $\frac{(8x^6y^{-3})^{\frac{1}{3}}}{(2xy)^3}$

4. A spotlight uses coloured gels to create the different colours of light required for a theatrical production. Each gel reduces the original intensity of the light by 3.6%.
   a) Write an equation that models the intensity of light, *I*, as a function of the number of gels used.
   b) Use your equation to determine the percent of light left if three gels are used.
   c) Explain why this is an example of exponential decay.

5. A small country that had 2 million inhabitants in 1990 has experienced an average growth in population of 4% per year since then.
   a) Write an equation that models the population, *P*, of this country as a function of the number of years, *n*, since 1990.
   b) Use your equation to determine when the population will double (assuming that the growth rate remains stable).



6. Which of these equations correspond to the graph? Explain how you know.
   a) $f(x) = 2(3^{-x}) + 5$
   b) $g(x) = (3^{-2x-4}) - 5$
   c) $h(x) = -0.8(3^{x-3})$
   d) $p(x) = -2\left(3^{\frac{1}{2}x-1}\right) - 2$

7. What are the restrictions on the value of n in $a^{\frac{1}{n}}$ if $a < 0$? Explain.

4 Chapter Task

Modelling Population

Every two years, the United Nations Population Division prepares estimates and projections of world, regional, and national population size and growth.

The estimated population of the world since 1950 is given in the table.

Year	Years since 1950	World Population (billions)
1950	0	2.55
1955	5	2.8
1960	10	3
1965	15	3.3
1970	20	3.7
1975	25	4
1980	30	4.5
1985	35	4.85
1990	40	5.3
1995	45	5.7
2000	50	6.1

? What is the equation of the function you could use to model the world's population?

A. Use the data from the table to create a scatter plot on graph paper.

B. Draw a curve of good fit. What type of function is this? Explain.

C. Using a graphing calculator, create a scatter plot.

D. Use the data in the table to estimate the average growth rate for a 5-year period and the y-intercept of the function.

E. Use the values you found in part D to write an equation for this function. Graph your equation. How well does your graph fit the data?

F. Make any necessary adjustments to your equation. Do so as often as needed, until you are satisfied with the fit.

G. In 2004, the UN predicted that the world's population will peak at 9.2 billion by 2075 and decline slightly to 8.97 billion by 2300. Does the model you found predict these same values?

H. Write a report that summarizes your findings. In your report, include your graph-paper scatter plot and the prediction of the type of function you thought this might be, supported by your reasons. Discuss

- the estimated average 5-year growth rate and the y-intercept of the function
- the original equation you determined
- the changes you made to your original equation and the reasons for those changes
- the calculations you used to check whether your model matches the future population predictions made by the UN
- why the UN predictions may differ from those of your model

Task *Checklist*

- ✔ Did you include all the required elements in your report?
- ✔ Did you properly label the graphs including some values?
- ✔ Did you show your work in your choice of the equation for part E?
- ✔ Did you support your decision in part G?