3.3 Quadratic Equations by Square Roots

Learning objectives

- Solve quadratic equations involving perfect squares.
- Approximate solutions of quadratic equations.
- Solve real-world problems using quadratic functions and square roots.

Introduction

So far you know how to solve quadratic equations by factoring. However, this method works only if a quadratic polynomial can be factored. Unfortunately, in practice, most quadratic polynomials are not factorable. In this section you will continue learning new methods that can be used in solving quadratic equations. In particular, we will examine equations in which we can take the square root of both sides of the equation in order to arrive at the solution(s).

Solve Quadratic Equations Involving Perfect Squares

Let's first examine quadratic equations of the type

$$x^2 - c = 0$$

We can solve this equation by isolating the x^2 term: $x^2 = c$

Once the x^2 term is isolated we can take the square root of both sides of the equation since the x-term has degree 2. Remember that when we take the square root we get two answers: the positive square root and the negative square root:

$$x = \sqrt{c} \qquad \text{and} \qquad x = -\sqrt{c}$$

Often this is written as $x = \pm\sqrt{c}$.

For the equation

$$x^2 + c = 0$$

We can solve this equation by isolating the x^2 term: $x^2 = -c$

Once the x^2 term is isolated we can take the square root of both sides of the equation. Remember that when we take the square root we get two answers: the positive square root and the negative square root:

$$x = \sqrt{-c} = i\sqrt{c} \qquad \text{and} \qquad x = -\sqrt{-c} = i\sqrt{c}$$

Often this is written as $x = \pm i\sqrt{c}$.

Example 1

Solve the following quadratic equations.

a) $x^2 - 4 = 0$

b) $x^2 - 25 = 0$

Solution

a) $x^2 - 4 = 0$

Isolate the x^2 $x^2 = 4$

Take the square root of both sides $x = \sqrt{4}$ and $x = -\sqrt{4}$

Answer $x = 2$ and $x = -2$

b) $x^2 - 25 = 0$

Isolate the x^2	$x^2 = 25$
Take the square root of both sides	$x = \sqrt{25} = 5$ and $x = -\sqrt{25} = -5$

Answer $x = 5$ and $x = -5$

Another type of equation where we can find the solution using the square root is

$$ax^2 - c = 0$$

We can solve this equation by isolating the x^2 term

$$ax^2 = c$$
$$x^2 = \frac{c}{a}$$

Now we can take the square root of both sides of the equation.

$$x = \sqrt{\frac{c}{a}} \qquad \text{and} \qquad x = -\sqrt{\frac{c}{a}}$$

Often this is written as $x = \pm\sqrt{\frac{c}{a}}$.

Example 2

Solve the following quadratic equations.

a) $9x^2 - 16 = 0$

b) $81x^2 - 1 = 0$

Solution

a) $9x^2 - 16 = 0$

$$\text{Isolate the } x^2. \qquad 9x^2 = 16$$
$$x^2 = \frac{16}{9}$$

Take the square root of both sides. $x = \sqrt{\frac{16}{9}} = \frac{4}{3}$ and $x = -\sqrt{\frac{16}{9}} = -\frac{4}{3}$

Answer: $x = \frac{4}{3}$ and $x = -\frac{4}{3}$

b) $81x^2 - 1 = 0$

$$\text{Isolate the } x^2 \qquad 81x^2 = 1$$
$$x^2 = \frac{1}{81}$$

Take the square root of both sides $x = \sqrt{\frac{1}{81}} = \frac{1}{9}$ and $x = -\sqrt{\frac{1}{81}} = -\frac{1}{9}$

Answer $x = \frac{1}{9}$ and $x = -\frac{1}{9}$

As you have seen previously, some quadratic equations have no real solutions.

Example 3

Solve the following quadratic equations.

a) $x^2 + 1 = 0$

b) $4x^2 + 9 = 0$

Solution

a) $x^2 + 1 = 0$

$$\text{Isolate the } x^2 \qquad x^2 = -1$$

Take the square root of both sides: $x = \sqrt{-1} = i$ and $x = -\sqrt{-1} = -i$ or $x = \pm i$

Answer Square roots of negative numbers do not give real number results, so there are **no real solutions** to this equation. The solutions are **complex numbers**.

b) $4x^2 + 9 = 0$

$$\text{Isolate the } x^2 \qquad 4x^2 = -9$$
$$x^2 = -\frac{9}{4}$$

Take the square root of both sides $x = \sqrt{-\frac{9}{4}} = \frac{3}{2}i$ and $x = -\sqrt{-\frac{9}{4}} = -\frac{3}{2}i$

Answer There are **no real solutions.** The solutions are **complex numbers.**

We can also use the square root function in some quadratic equations where one side of the equation is a perfect square. This is true if an equation is of the form

$$(x-2)^2 = 9$$

Both sides of the equation are perfect squares. We take the square root of both sides.

$x-2=3$ and $x-2=-3$

Solve both equations

Answer $x=5$ and $x=-1$

Notice if we graph set the eqaution equal to 0 by subtracting 9 on both sides we get:

$$(x-2)^2 - 9 = 0$$

Now if we graph:

$$y = (x-2)^2 - 9$$

the points where y = 0 are the x-values that solve our equation. The points on a function where y = 0, are the x-intercepts. Notice the graph of $y=(x-2)^2-9$ has x-intercepts of (5, 0) and (-1,0), which are also the solutions to the equation. You can use this technique to check the solutions to quadratic equations as long as the quadratic equation has real solutions. If a quadratic equation has complex solutions with an imaginary part, the graph will not cross the x-axis.

Example 4

Solve the following quadratic equations.

a) $(x-1)^2 = 4$

b) $(x+3)^2 = -16$

Solution

a) $(x-1)^2 = 4$

Take the square root of both sides.	$x-1=2$ and $x-1=-2$
Solve each equation.	$x=3$ and $x=-1$

Answer $x=3$ and $x=-1$

b) $(x+3)^2 = -16$

Take the square root of both sides.	$x+3=4i$ and $x+3=-4i$
Solve each equation.	$x=-3+4i$ and $x=-3-4i$

It might be necessary to factor the left side of the equation as a perfect square before applying the method outlined above.

Example 5

Solve the following quadratic equations.

a) $x^2+8x+16=25$

b) $4x^2-40x+25=-9$

Solution

a) $x^2+8x+16=25$

Factor the right hand side.	$x^2+8x+16=(x+4)^2$	so $(x+4)^2=25$
Take the square root of both sides.	$x+4=5$ and $x+4=-5$	
Solve each equation.	$x=1$ and $x=-9$	

Answer $x=1$ and $x=-9$

b) $4x^2-20x+25=-9$

Factor the right hand side.	$4x^2-20x+25=(2x-5)^2$	so $(2x-5)^2=-9$
Take the square root of both sides.	$2x-5=3i$ and $2x-5=-3i$	
Solve each equation.	$2x=5+3i$ and $2x=5-3i$	
	$x=\frac{5+3i}{2}$ and $x=\frac{5-3i}{2}$	
	$x=\frac{5}{2}+\frac{3i}{2}$ and $x=\frac{5}{2}-\frac{3i}{2}$	

Answer $x=\frac{5}{2}+\frac{3i}{2}$ **and** $x=\frac{5}{2}-\frac{3i}{2}$

Approximate Solutions of Quadratic Equations

We use the methods we learned so far in this section to find approximate solutions to quadratic equations. We can get approximate solutions when taking the square root does not give an exact answer.

Example 6

Solve the following quadratic equations.

a) $x^2-3=0$

b) $2x^2-9=0$

Solution

a)

Isolate the x^2.	$x^2=3$
Take the square root of both sides.	$x=\sqrt{3}$ and $x=-\sqrt{3}$

Answer $x\approx 1.73$ and $x\approx -1.73$

b)

Isolate the x^2.	$2x^2=9$ so $x^2=\frac{9}{2}$
Take the square root of both sides.	$x=\sqrt{\frac{9}{2}}=\frac{3}{\sqrt{2}}$ and $x=-\sqrt{\frac{9}{2}}=-\frac{3}{\sqrt{2}}$

Answer $x \approx 2.12$ and $x \approx -2.12$

Example 7

Solve the following quadratic equations.

a) $(2x+5)^2 = 10$

b) $x^2 - 2x + 1 = 5$

Solution.

a)

$$\begin{array}{rr} \text{Take the square root of both sides.} & 2x+5 = \sqrt{10} \text{ and } 2x+5 = -\sqrt{10} \\ \text{Solve both equations.} & x = \dfrac{-5+\sqrt{10}}{2} \text{ and } x = \dfrac{-5-\sqrt{10}}{2} \end{array}$$

Answer $x \approx -0.92$ and $x \approx -4.08$

b)

$$\begin{array}{rr} \text{Factor the right hand side.} & (x-1)^2 = 5 \\ \text{Take the square root of both sides.} & x-1 = \sqrt{5} \text{ and } x-1 = -\sqrt{5} \\ \text{Solve each equation.} & x = 1+\sqrt{5} \text{ and } x = 1-\sqrt{5} \end{array}$$

Answer $x \approx 3.24$ and $x \approx -1.24$

Solve Real-World Problems Using Quadratic Functions and Square Roots

There are many real-world problems that require the use of quadratic equations in order to arrive at the solution. In this section, we will examine problems about objects falling under the influence of gravity. When objects are **dropped** from a height, they have no initial velocity and the force that makes them move towards the ground is due to gravity. The acceleration of gravity on earth is given by

$$g = -9.8\ m/s^2 \qquad \text{or} \qquad g = -32\ ft/s^2$$

The negative sign indicates a downward direction. We can assume that gravity is constant for the problems we will be examining, because we will be staying close to the surface of the earth. The acceleration of gravity decreases as an object moves very far from the earth. It is also different on other celestial bodies such as the Moon.

The equation that shows the height of an object in free fall is given by

$$y = \frac{1}{2}gt^2 + y_0$$

The term y_0 represents the initial height of the object t is time, and g is the force of gravity. There are two choices for the equation you can use.

$$\begin{array}{ll} y = -4.9t^2 + y_0 & \text{If you wish to have the height in meters.} \\ y = -16t^2 + y_0 & \text{If you wish to have the height in feet.} \end{array}$$

Example 8 Free fall

How long does it take a ball to fall from a roof to the ground 25 feet below?

Solution

$$\begin{aligned}
\text{Since we are given the height in feet, use equation} \qquad & y = -16t^2 + y_0 \\
\text{The initial height is } y_0 = 25 \ feet, \text{ so} \qquad & y = -16t^2 + 25 \\
\text{The height when the ball hits the ground is } y = 0, \text{ so} \qquad & 0 = -16t^2 + 25 \\
\text{Solve for } t \qquad & 16t^2 = 25 \\
& t^2 = \frac{25}{16} \\
& t = \frac{5}{4} \text{ or } t = -\frac{5}{4}
\end{aligned}$$

We can discard the solution $t = -\frac{5}{4}$ since only positive values for time makes sense in this case,

Answer It takes the ball 1.25 seconds to fall to the ground.

Example 9 Free fall

A rock is dropped from the top of a cliff and strikes the ground 7.2 seconds later. How high is the cliff in meters?

Solution

$$\begin{aligned}
\text{Since we want the height in meters, use equation} \qquad & y = -4.9t^2 + y_0 \\
\text{The time of flight is } t = 7.2 \ seconds \qquad & y = -4.9(7.2)^2 + y_0 \\
\text{The height when the ball hits the ground is } y = 0, \text{ so} \qquad & 0 = -4.9(7.2)^2 + y_0 \\
\text{Simplify} \qquad & 0 = -254 + y_0 \text{ so } y_0 = 254
\end{aligned}$$

Answer The cliff is 254 meters high.

Example 10

Victor drops an apple out of a window on the 10^{th} *floor which is 120 feet above ground. One second later Juan drops an orange out of a* 6^{th} *floor window which is 72 feet above the ground. Which fruit reaches the ground first? What is the time difference between the fruits' arrival to the ground?*

Solution Let's find the time of flight for each piece of fruit.

For the Apple we have the following.

$$\begin{aligned}
\text{Since we have the height in feet, use equation} \qquad & y = -16t^2 + y_0 \\
\text{The initial height } y_0 = 120 \ feet. \qquad & y = -16t^2 + 120 \\
\text{The height when the ball hits the ground is } y = 0, \text{ so} \qquad & 0 = -16t^2 + 120 \\
\text{Solve for } t \qquad & 16t^2 = 120 \\
& t^2 = \frac{120}{16} = 7.5 \\
& t = 2.74 \text{ or } t = -2.74 \ seconds.
\end{aligned}$$

For the orange we have the following.

$$\begin{aligned} \text{The initial height } y_0 = 72 \text{ } feet. \qquad & 0 = -16t^2 + 72 \\ \text{Solve for } t. \qquad & 16t^2 = 72 \\ & t^2 = \frac{72}{16} = 4.5 \\ & t = 2.12 \text{ or } t = -2.12 \text{ } seconds \end{aligned}$$

But, don't forget that the orange was thrown out one second later, so add one second to the time of the orange. It hit the ground 3.12 seconds after Victor dropped the apple.

Answer The apple hits the ground first. It hits the ground 0.38 seconds before the orange. (Hopefully nobody was on the ground at the time of this experiment.)

Review Questions

Solve the following quadratic equations.

1. $x^2 - 1 = 0$
2. $x^2 - 100 = 0$
3. $x^2 + 16 = 0$
4. $9x^2 - 1 = 0$
5. $4x^2 - 49 = 0$
6. $64x^2 - 9 = 0$
7. $x^2 - 81 = 0$
8. $25x^2 - 36 = 0$
9. $x^2 + 9 = 0$
10. $x^2 - 16 = 0$
11. $x^2 - 36 = 0$
12. $16x^2 - 49 = 0$
13. $(x-2)^2 = 1$
14. $(x+5)^2 = 16$
15. $(2x-1)^2 - 4 = 0$
16. $(3x+4)^2 = 9$
17. $(x-3)^2 + 25 = 0$
18. $x^2 - 6 = 0$
19. $x^2 - 20 = 0$
20. $3x^2 + 14 = 0$
21. $(x-6)^2 = 5$
22. $(4x+1)^2 - 8 = 0$
23. $x^2 - 10x + 25 = 9$
24. $x^2 + 18x + 81 = 1$
25. $4x^2 - 12x + 9 = 16$
26. $(x+10)^2 = 2$
27. $x^2 + 14x + 49 = 3$
28. $2(x+3)^2 = 8$
29. Susan drops her camera in the river from a bridge that is 400 feet high. How long is it before she hears the splash?
30. It takes a rock 5.3 seconds to splash in the water when it is dropped from the top of a cliff. How high is the cliff in meters?

Review Answers

1. $x = 1, x = -1$
2. $x = 10, x = -10$
3. No real solution.
4. $x = \frac{1}{3}, x = -\frac{1}{3}$
5. $x = \frac{7}{2}, x = -\frac{7}{2}$
6. $x = \frac{3}{8}, x = -\frac{3}{8}$
7. $x = 9, x = -9$
8. $x = \frac{6}{5}, x = -\frac{6}{5}$
9. No real solution.
10. $x = 4, x = -4$
11. $x = 6, x = -6$
12. $x = \frac{7}{4}, x = -\frac{7}{4}$
13. $x = 3, x = 1$
14. $x = -1, x = -9$
15. $x = \frac{3}{2}, x = -\frac{1}{2}$
16. $x = -\frac{1}{3}, x = -\frac{7}{3}$
17. No real solution.
18. $x \approx 2.45, x \approx -2.45$
19. $x \approx 4.47, x \approx -4.47$
20. No real solution.
21. $x \approx 8.24, x \approx 3.76$
22. $x \approx 0.46, x \approx -0.96$
23. $x = 8, x = 2$
24. $x = -8, x = -10$
25. $x = \frac{7}{2}, x = -\frac{1}{2}$
26. $x \approx -8.59, x \approx -11.41$
27. $x \approx -5.27, x \approx -8.73$
28. $x = -1, x = -5$
29. $t = 5\ seconds$
30. $y_0 = 137.6\ meters$