Solved Examples for Chapter 13

Example for Section 13.5

Consider the following unconstrained optimization problem:

$$\text{Maximize} \quad f(\mathbf{x}) = 2x_1x_2 - 2x_1^2 - x_2^2.$$

Thus, $\dfrac{\partial f}{\partial x_1} = -4x_1 + 2x_2$, $\dfrac{\partial f}{\partial x_2} = 2x_1 - 2x_2$.

(a) Starting from the initial trial solution $(x_1, x_2) = (1, 1)$, interactively apply the gradient search procedure with $\varepsilon = 0.5$ to obtain an approximate solution.

Since $\varepsilon = 0.5$, we will stop the gradient search procedure if $\left| \dfrac{\partial f}{\partial x_j} \right| \leq 0.5$ for both $j = 1$ and

2. Starting from $(x_1, x_2) = (1, 1)$, the first iteration proceeds as follows.

$$\nabla f(\mathbf{x}) = (-4x_1 + 2x_2, 2x_1 - 2x_2) = (-2, 0) \quad \text{at } \mathbf{x} = (1, 1).$$
$$\mathbf{x} + t\nabla f(\mathbf{x}) = (1 + t(-2), 1 + t(0)) = (1 - 2t, 1).$$
$$f(\mathbf{x} + t\nabla f(\mathbf{x})) = 2(1 - 2t)(1) - 2(1 - 2t)^2 - 1^2 = -8t^2 + 4t - 1.$$

To find the value of t that maximizes $f(\mathbf{x} + t\nabla f(\mathbf{x}))$ for $t \geq 0$, we set

$$\frac{df(\mathbf{x} + t\nabla f(\mathbf{x}))}{dt} = -16t + 4 = 0,$$

so

$$t^* = 0.25.$$

Therefore,

$$\mathbf{x} + t^*\nabla f(\mathbf{x}) = (1, 1) + 0.25(-2, 0) = (0.5, 1)$$

is the new trial solution to use for iteration 2.

Letting $\mathbf{x}'$ always denote the trial solution being used for the current iteration, the following table shows all the iterations of the gradient search procedure, which terminates at the beginning of iteration 4 (because of the stopping rule) with $(x_1, x_2) = (0.25, 0.5)$ as the desired approximation of the optimal solution.

Iteration	x'	$\nabla f(\mathbf{x}')$	$\mathbf{x}' + t\nabla f(\mathbf{x}')$	$f(\mathbf{x}' + t\nabla f(\mathbf{x}'))$	t^*	$\mathbf{x}' + t^*\nabla f(\mathbf{x}')$
1	(1, 1)	(-2, 0)	(1-2t, 1)	$-8t^2 + 4t -1$	0.25	(0.5, 1)
2	(0.5, 1)	(0, -1)	(0.5, 1-t)	$-t^2 + t - 0.5$	0.5	(0.5, 0.5)
3	(0.5, 0.5)	(-1, 0)	(0.5-t, 0.5)	$-2t^2 + t - 0.25$	0.25	(0.25, 0.5)
4	(0.25, 0.5)	(0, -0.5)				

(b) Solve the system of linear equations obtained by setting $\nabla f(x) = 0$ to obtain the exact solution.

Solving the system,

$$\frac{\partial f}{\partial x_1} = -4x_1 + 2x_2 = 0,$$

$$\frac{\partial f}{\partial x_2} = 2x_1 - 2x_2 = 0,$$

we obtain the exact optimal solution, $(x_1^*, x_2^*) = (0, 0)$.

(c) Referring to Fig. 13.14 as a sample for a similar problem, draw the path of trial solutions you obtained in part (a). Then show the apparent continuation of this path with your best guess for the next three trial solutions [based on the pattern in part (a) and in Fig. 13.14]. Also show the exact optimal solution from part (b) toward which this sequence of trial solutions is converging.

The path of trial solutions from part (a), the guess for the next three trial solutions, and the exact optimal solution all are shown in the following figure.
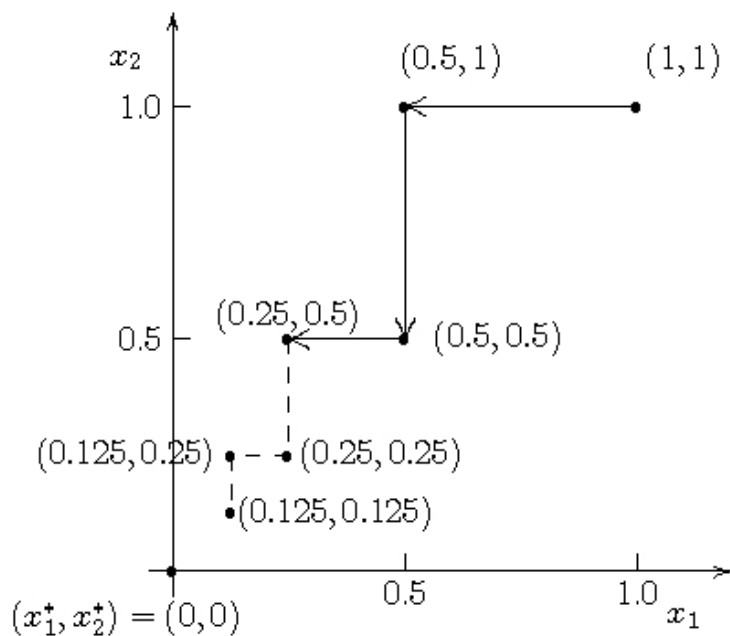
(d) Apply the automatic routine for the gradient search procedure (with ε = 0.01) in your IOR Tutorial to this problem.

As shown next, the IOR Tutorial finds the solution, $(x_1, x_2) = (0.004, 0.008)$, which is a close approximation of the exact optimal solution at $(x_1, x_2) = (0, 0)$.

Number of Terms (<=10) : 3

(To make a new number above take effect, you must press the ENTER key.)

For each term, enter the coefficient on the left and the exponents on the right.

Max f(X) = 2   x1 1   x2 1   +   -2   x1 2   x2 0   +   -1   x1 0   x2 2

Initial Trial Solution : (x1 , x2) = (1   , 1   )

Error Tolerance :

abs(df/dxj) <= 0.01

Initial Trial Solution : (x1 , x2) = (1 , 1)

The Solution is : (x1 , x2) = (0.004 , 0.008)

grad f(x1, x2) =(0 , -0.008)

Error Tolerance is : 0.01

OK

A: Nonlinear Programming P: Automatic Gradient Search Procedure

Example for Section 13.6

Consider the following linearly constrained optimization problem:

$$\text{Maximize} \quad f(\mathbf{x}) = \ln(1 + x_1 + x_2),$$
subject to
$$x_1 + 2 x_2 \leq 5$$
and
$$x_1 \geq 0, \ x_2 \geq 0,$$

where ln denotes the natural logarithm.

(a) Verify that this problem is a convex programming problem.

$$\nabla f(x_1, x_2) = (1/(1+x_1+x_2),\ 1/(1+x_1+x_2))$$

Using the concavity test for a function of two variables given in Appendix 2 (see Table A2.1), since

$$\frac{\partial^2 f}{\partial x_1^2} = -\frac{1}{(1+x_1+x_2)^2} \le 0 \quad \text{for all } x_1 \ge 0, x_2 \ge 0,$$

$$\frac{\partial^2 f}{\partial x_2^2} = -\frac{1}{(1+x_1+x_2)^2} \le 0 \quad \text{for all } x_1 \ge 0, x_2 \ge 0,$$

and

$$\frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]^2 = 0 \quad \text{for all } x_1 \ge 0, x_2 \ge 0,$$

the objective function is concave. Also, it is easy to see that the feasible region is a convex set. Hence, this is a convex programming program.

(b) Use the KKT conditions to derive an optimal solution.

The KKT conditions are:

1(j = 1). $\qquad \dfrac{1}{1+x_1+x_2} - u \ \le 0$

2(j = 1). $\qquad x_1(\ \dfrac{1}{1+x_1+x_2} - u\ ) = 0$

1(j = 2). $\qquad \dfrac{1}{1+x_1+x_2} - 2u \ \le 0$

2(j = 2). $\qquad x_2(\ \dfrac{1}{1+x_1+x_2} - 2u\ ) = 0$

3 . $\qquad\qquad x_1 + 2 x_2 - 5 \ \le \ 0$
4. $\qquad\qquad u\ (x_1 + 2 x_2 - 5) = 0$
5. $\qquad\qquad x_1 \ge 0,\ x_2 \ge 0,$
6. $\qquad\qquad u \ge 0.$

Suppose $u > 0$. This implies that $x_1 + 2 x_2 \ = \ 5$, from condition 4. Suppose $x_2 = 0$, so $x_1 = 5$. Then $u = 1/6$ from condition 2 (j=1). It is easy to verify that $(x_1, x_2) \ = (5, 0)$ and $u = 1/6$ satisfy the KKT conditions listed above. Hence, $(x_1, x_2) \ = (5, 0)$ is an optimal solution.

(c) Use intuitive reasoning to demonstrate that the solution obtained in part (b) is indeed optimal. [Hint: Note that ln(1+ x₁+x₂) is a monotonic strictly increasing function of (1+ x₁+x₂).]

From the hint, the function $\ln(1+ x_1+x_2)$ is a monotonically increasing function of $(1+ x_1+x_2)$. Hence, the original problem is equivalent to the following linear programming problem:

> Maximize $\quad Z = 1+ x_1+x_2,$
> subject to
> $$x_1 + 2 x_2 \le 5$$
> and
> $$x_1 \ge 0, \ x_2 \ge 0.$$

It then is straightforward (e.g., by using the graphical method) to find that $(x_1, x_2) = (5, 0)$ is the optimal solution for this linear programming problem, and so for the original problem as well.

Example for Section 13.7

Jim Matthews, Vice-President for Marketing of the J.R. Nickel Company, is planning advertising campaigns for two unrelated products. These two campaigns need to use some of the same resources. Therefore, Jim knows that his decisions on the levels of the two campaigns need to be made jointly after considering these resource constraints. In particular, letting x_1 and x_2 denote the levels of campaigns 1 and 2, respectively, these constraints are $4x_1 + x_2 \le 20$ and $x_1 + 4x_2 \le 20$.

In facing these decisions, Jim is well aware that there is a point of diminishing returns when raising the level of an advertising campaign too far. At that point, the cost of additional advertising becomes larger than the increase in net revenue (excluding advertising costs) generated by the advertising. After careful analysis, he and his staff estimate that the net profit from the first product (including advertising costs) when conducting the first campaign at level x_1 would be $3x_1 - (x_1 - 1)^2$ in millions of dollars. The corresponding estimate for the second product is $3x_2 - (x_2 - 2)^2$.

This analysis led to the following quadratic programming model for determining the levels of the two advertising campaigns:

> Maximize $\quad Z = 3x_1 - (x_1 - 1)^2 + 3x_2 - (x_2 - 2)^2,$
> subject to
> $$4x_1 + \ x_2 \le 20$$
> $$x_1 + 4x_2 \le 20$$
> and
> $$x_1 \ge 0, \ x_2 \ge 0.$$

(a) Obtain the KKT conditions for this problem in the form given in Sec. 13.6.

The KKT conditions of the given quadratic program are

1(j=1). $\quad -2(x_1-1) +3 -4 u_1 - u_2 \le 0,$
2(j=1). $\quad x_1(-2(x_1-1) +3 -4 u_1 - u_2) = 0,$

1(j=2). $-2(x_2-2) +3 - u_1 - 4\,u_2 \leq 0,$
2(j=2). $x_2(-2(x_2-2) +3 - u_1 - 4\,u_2) = 0,$
3(j=1). $4\,x_1 + \ x_2 - 20 \leq 0,$
3(j=2). $x_1 + 4\,x_2 - 20 \leq 0,$
4(j=1). $u_1(4\,x_1 + x_2 - 20) = 0,$
4(j=2). $u_2(x_1 + 4\,x_2 - 20) = 0,$
5. $x_1 \geq 0, \ x_2 \geq 0,$
6. $u_1 \geq 0, \ u_2 \geq 0.$

(b) You are given the information that the optimal solution does not lie on the boundary of the feasible region. Use this information to derive the optimal solution from the KKT conditions.

Since the optimal solution does not lie on the boundary of the feasible region, we know that

$$x_1 > 0, \ \ x_2 > 0, \ \ 4\,x_1 + \ x_2 - 20 < 0, \ \ \text{and} \ \ x_1 + \ 4\,x_2 - 20 < 0.$$

From conditions 2(j=1) and 2(j=2), we then have

$$-2(x_1-1) +3 -4\,u_1 - u_2 = 0 \quad \text{and} \quad -2(x_2-2) +3 - u_1 - 4\,u_2 = 0.$$

From conditions 4(j=1) and 4(j=2), we also have

$$u_1 = u_2 = 0.$$

Hence, we obtain

$$-2\,x_1 + 5 = 0,$$
$$-2\,x_2 + 7 = 0.$$

Solving for x_1 and x_2, the optimal solution is $x_1 = 5/2, x_2 = 7/2$.

(c) Now suppose that this problem is to be solved by the modified simplex method. Formulate the linear programming problem that is to be addressed explicitly, and then identify the additional complementarity constraint that is enforced automatically by the algorithm.

After introducing nonnegative slack variables (denoted by y_1, y_2, v_1, and v_2, respectively), conditions 1 and 3 of the KKT conditions can be reexpressed as

1(j = 1).	$-2x_1$		$- 4u_1$	$- u_2$	$+ y_1$			$= -5$
1(j = 2).		$- 2x_2$	$-u_1$	$-4u_2$		$+ y_2$		$= -7$
3(j = 1).	$4x_1$	$+ x_2$					$+ v_1$	$= 20$
3(j = 2).	x_1	$+ 4x_2$					$+v_2$	$= 20.$

Thus, conditions 2 and 4 can be reexpressed as

2($j = 1$). $x_1 y_1 = 0$
2($j = 2$). $x_2 y_2 = 0$
4($j = 1$). $u_1 v_1 = 0$
4($j = 2$). $u_2 v_2 = 0$

These conditions 2 and 4 are equivalent to

$$x_1 y_1 + x_2 y_2 + u_1 v_1 + u_2 v_2 = 0.$$

To find an optimal solution for the original problem, we need to find a feasible solution for the KKT conditions, including this last equation, the new expressions for conditions 1 and 3, and conditions 5 and 6, plus nonnegativity constraints on y_1, y_2, v_1, and v_2. However, the new expressions for conditions 1 and 3 are in the form of linear programming constraints, so we can use a procedure for finding an initial BF solution for these constraints. Using phase 1 of the two-phase method (presented in Sec. 4.6), we need to multiply through the expressions for condition 1 by (-1) and introduce artificial variables (denoted by z_1 and z_2) in order to obtain positive right-hand sides.

$$
\begin{aligned}
2x_1 && + 4u_1 &+ u_2 & -y_1 && +z_1 && = 5 \\
&& 2x_2 + u_1 &+ 4u_2 && -y_2 && + z_2 &= 7
\end{aligned}
$$

Phase 1 then involves minimizing the sum of the artificial variables, subject to the linear programming constraints.
Therefore, the linear programming problem that is to be addressed explicitly by the modified simplex method is

Minimize $Z = z_1 + z_2,$
subject to:
$$
\begin{aligned}
2x_1 && + 4u_1 + u_2 &- y_1 & + z_1 &&&& = 5 \\
&& 2x_2 + u_1 + 4u_2 && - y_2 & + z_2 &&& = 7 \\
&& 4x_1 + x_2 &&&& + v_1 && = 20 \\
&& x_1 + 4x_2 &&&&& + v_2 & = 20
\end{aligned}
$$
and
$$x_1 \geq 0,\ x_2 \geq 0,\ u_1 \geq 0,\ u_2 \geq 0,\ y_1 \geq 0,\ y_2 \geq 0,\ z_1 \geq 0,\ z_2 \geq 0,\ v_1 \geq 0,\ v_2 \geq 0.$$

The additional complementarity constraint is $x_1 y_1 + x_2 y_2 + u_1 v_1 + u_2 v_2 = 0.$

(d) Apply the modified simplex method to the problem as formulated in part (c).

Applying the modified simplex algorithm to the linear programming problem in part (c), after restoring proper form from Gaussian elimination to the initial system of equations by algebraically eliminating z_1 and z_2 from Eq. (0), we obtain the following simplex tableaux:

Basic Variable	Eq	Z	x_1	x_2	u_1	u_2	y_1	y_2	z_1	z_2	v_1	v_2	Right Side
						Coefficient of:							
Z	(0)	-1	-2	-2	-5	-5	1	1	0	0	0	0	-12
z_1	(1)	0	2	0	4	1	-1	0	1	0	0	0	5
z_2	(2)	0	0	2	1	4	0	-1	0	1	0	0	7
v_1	(3)	0	4	1	0	0	0	0	0	0	1	0	20
v_2	(4)	0	1	4	0	0	0	0	0	0	0	1	20
Z	(0)	-1	0	-2	-1	-4	0	1	1	0	0	0	-7
x_1	(1)	0	1	0	2	0.5	-0.5	0	0.5	0	0	0	2.5
z_2	(2)	0	0	2	1	4	0	-1	0	1	0	0	7
v_1	(3)	0	0	1	-8	-2	2	0	-2	0	1	0	10
v_2	(4)	0	0	4	-2	-0.5	0.5	0	-0.5	0	0	1	17.5
Z	(0)	-1	0	0	0	0	0	0	1	1	0	0	0
x_1	(1)	0	1	0	2	0.5	-0.5	0	0.5	0	0	0	2.5
x_2	(2)	0	0	1	0.5	2	0	-0.5	0	0.5	0	0	3.5
v_1	(3)	0	0	0	-8.5	-4	2	0.5	-2	-0.5	1	0	6.5
v_2	(4)	0	0	0	-4	-8.5	0.5	2	-0.5	-2	0	1	3.5

Note that x_1 (or x_2) is the initial entering basic variable instead of u_1 or u_2 because the latter variables are prohibited from playing this role by the restricted-entry rule since v_1 and v_2 already are basic variables. In the second tableau, x_2 is the entering basic variable instead of u_2 for the same reason.

The final tableau shows that $x_1 = 2.5$, $x_2 = 3.5$ is optimal with $u_1 = u_2 = 0$.

(e) Use the computer to solve the quadratic programming problem directly.

Using IOR Tutorial to solve the quadratic programming problem obtains the optimal solution, $(x_1, x_2) = (2.5, 3.5)$.

Example for Section 13.8

Reconsider the scenario described in the preceding example. This description is repeated below.

Jim Matthews, Vice-President for Marketing of the J.R. Nickel Company, is planning advertising campaigns for two unrelated products. These two campaigns need to use some of the same resources. Therefore, Jim knows that his decisions on the levels of the two campaigns need to be made jointly after considering these resource constraints. In particular, letting x_1 and x_2 denote the levels of campaigns 1 and 2, respectively, these constraints are $4x_1 + x_2 \le 20$ and $x_1 + 4x_2 \le 20$.

In facing these decisions, Jim is well aware that there is a point of diminishing returns when raising the level of an advertising campaign too far. At that point, the cost of additional advertising becomes larger than the increase in net revenue (excluding advertising costs) generated by the advertising. After careful analysis, he and his staff estimate that the net profit from the first product (including advertising costs) when

conducting the first campaign at level x_1 would be $3x_1 - (x_1 - 1)^2$ in millions of dollars. The corresponding estimate for the second product is $3x_2 - (x_2 - 2)^2$.

This analysis led to the following quadratic programming model for determining the levels of the two advertising campaigns:

Maximize $Z = 3x_1 - (x_1 - 1)^2 + 3x_2 - (x_2 - 2)^2$,

subject to
$$4x_1 + x_2 \leq 20$$
$$x_1 + 4x_2 \leq 20$$

and
$$x_1 \geq 0, \ x_2 \geq 0.$$

(a) Use the separable programming formulation presented in Sec. 13.8 to formulate an approximate linear programming model for this problem. Use 0, 2.5, and 5 as the breakpoints for both x_1 and x_2 for the piecewise linear functions.

The objective function can be written as

$$Z = f_1(x_1) + f_2(x_2),$$

where

$$f_1(x_1) = 3x_1 - (x_1 - 1)^2,$$
$$f_2(x_2) = 3x_2 - (x_2 - 2)^2.$$

Then,

$$f_1(0) = -1, \ f_1(2.5) = 5.25, \ f_1(5) = -1,$$
$$f_2(0) = -4, \ f_2(2.5) = 7.25, \ \text{and} \ f_2(5) = 6.$$

Hence, the slopes of the respective segments of the piecewise linear functions are

$$s_{11} = \frac{5.25 - (-1)}{2.5 - 0} = 2.5, \qquad s_{12} = \frac{-1 - 5.25}{5 - 2.5} = -2.5,$$

$$s_{21} = \frac{7.25 - (-4)}{2.5 - 0} = 4.5, \qquad s_{22} = \frac{6 - 7.25}{5} - 2.5 = -0.5.$$

Let $x_{11}, x_{12}, x_{21},$ and x_{22} be the variables that correspond to the respective segments of the piecewise linear functions, so that

$$x_1 = x_{11} + x_{12},$$
$$x_2 = x_{21} + x_{22}.$$

Then the approximate linear programming model is

Maximize $Z = 2.5\, x_{11} - 2.5\, x_{12} + 4.5\, x_{21} - 0.5\, x_{22}$,

subject to

$$4\, x_{11} + 4\, x_{12} + x_{21} + x_{22} \le 20$$
$$x_{11} + x_{12} + 4\, x_{21} + 4\, x_{22} \le 20$$

and

$$0 \le x_{ij} \le 2.5, \quad \text{for } i, j = 1, 2.$$

(b) Use the computer to solve the model formulated in part (a). Then reexpress this solution in terms of the original variables of the problem.

We use Solver to find the optimal solution, $(x_{11}, x_{12}, x_{21}, x_{22}) = (2.5, 0, 2.5, 0)$ with $Z = 17.5$. In terms of the original variables, we have

$$x_1 = x_{11} + x_{12} = 2.5,$$
$$x_2 = x_{21} + x_{22} = 2.5.$$

(For the original model, $Z = 12.5$, where the difference arises from the fact that the approximate model drops constants, $f_1(0) = -1$ and $f_2(0) = -4$, from the objective function.)

(c) To improve the approximation, now use 0, 1, 2, 3, 4, and 5 as the breakpoints for both x₁ and x₂ for the piecewise linear functions and repeat parts (a) and (b).

As before,

$$Z = f_1(x_1) + f_2(x_2).$$

The values of these functions at the breakpoints are

$$f_1(0) = -1,\ f_1(1) = 3,\ f_1(2) = 5,\ f_1(3) = 5,\ f_1(4) = 3,\ f_1(5) = -1,$$

$$f_2(0) = -4,\ f_2(1) = 2,\ f_2(2) = 6,\ f_2(3) = 8,\ f_2(4) = 8,\ \text{and}\ f_2(5) = 6.$$

Hence, the slopes of the respective segments of the piecewise linear functions are

$$\begin{aligned}
&s_{11} = 4, &&s_{12} = 2, &&s_{13} = 0, &&s_{14} = -2, &&s_{15} = -4,\\
&s_{21} = 6, &&s_{22} = 4, &&s_{23} = 2, &&s_{24} = 0, &&s_{25} = -2.
\end{aligned}$$

Introducing a new variable x_{ij} ($i = 1, 2$; $j = 1, 2, 3, 4, 5$) for each of these segments, we have

$$x_1 = x_{11} + x_{12} + x_{13} + x_{14} + x_{15},$$
$$x_2 = x_{21} + x_{22} + x_{23} + x_{24} + x_{25}.$$

Then the approximate linear programming model is

Maximize $Z = 4x_{11} + 2x_{12} + 0x_{13} - 2x_{14} - 4x_{15} + 6x_{21} + 4x_{22} + 2x_{23} + 0x_{24} - 2x_{25},$
subject to

$$4x_{11} + 4x_{12} + 4x_{13} + 4x_{14} + 4x_{15} + x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 20$$
$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + 4x_{21} + 4x_{22} + 4x_{23} + 4x_{24} + 4x_{25} \leq 20$$

and

$$0 \leq x_{ij} \leq 1, \quad \text{for } i = 1, 2 \text{ and } j = 1, 2, 3, 4, 5.$$

We next use Solver to find the optimal solution, $(x_{11}, x_{12}, x_{15}, x_{14}, x_{15}, x_{21}, x_{22}, x_{23}, x_{24},$
$x_{25}) = (1, 1, 0, 0, 0, 1, 1, 1, 0, 0)$ with $Z = 18$. In terms of the original variables, we have

$$x_1 = x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 2,$$
$$x_2 = x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 3.$$

Note that we improve Z from 17.5 to 18 (or from 12.5 to 13 for the original model).

Example 1 for Section 13.9

Consider the following constrained optimization problem:

Maximize $f(\mathbf{x}) = \ln(1 + x_1 + x_2)$
subject to
$$x_1 + 2x_2 \leq 5$$
and
$$x_1 \geq 0, \ x_2 \geq 0.$$

While analyzing this same problem in the Solved Example for Section 13.6, we found
that it is, in fact, a convex programming problem and that the optimal solution is

$$(x_1, x_2) = (5, 0).$$

**Starting from the initial trial solution $(x_1, x_2) = (1, 1)$, use one iteration of the
Frank-Wolfe algorithm to obtain the optimal solution, $(x_1, x_2) = (5, 0)$. Then use a
second iteration to verify that it is an optimal solution (because it is replicated
exactly). Explain why exactly the same results would be obtained on these two
iterations with any other initial trial solution.**

To start each iteration, we will need to evaluate

$$\nabla f(\mathbf{x}) = \left(\frac{1}{1 + x_1 + x_2}, \ \frac{1}{1 + x_1 + x_2} \right)$$

at the current trial solution $\mathbf{x}$, and then set $(c_1, c_2) = \nabla f(\mathbf{x})$, before proceeding as
described in Sec. 13.9.

Iteration 1:

At the initial trial solution $x^{(0)} = (1, 1)$, we have

$$\nabla f(1, 1) = (1/3, 1/3) = (c_1, c_2).$$

We then solve the following linear programming problem:

Maximize $Z = (1/3) x_1 + (1/3) x_2$,
subject to
$$x_1 + 2 x_2 \leq 5$$
and
$$x_1 \geq 0, \ x_2 \geq 0.$$

The optimal solution is $\mathbf{x}^{(1)}_{LP} = (5, 0)$.

For the variable t $(0 \leq t \leq 1)$, we then define the function $h(t)$ as $h(t) = f(\mathbf{x})$ where

$$\mathbf{x} = \mathbf{x}^{(0)} + t (\mathbf{x}_{LP}^{(1)} - \mathbf{x}^{(0)}) = (1, 1) + t (4, -1) = (1+4t, 1-t), \ \text{ for } 0 \leq t \leq 1.$$

Hence,

$$h(t) = \ln(1+1+4t+1-t) = \ln(3+3t), \ \text{ for } 0 \leq t \leq 1.$$

To choose the value of t, we solve the following problem:

Maximize $h(t) = \ln (3 + 3t)$,
subject to
$$0 \leq t \leq 1,$$

which yields $t^* = 1$. Hence,

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} + t^* (\mathbf{x}_{LP}^{(1)} - \mathbf{x}^{(0)}) = (5, 0),$$

which happens to be the optimal solution.

Iteration 2:

At the new trial solution, $\mathbf{x}^{(1)} = (5, 0)$, we have

$$\nabla f(5, 0) = (1/6, 1/6) = (c_1, c_2).$$

We then solve the following linear programming problem:

Maximize $Z = (1/6) x_1 + (1/6) x_2$,
subject to
$$x_1 + 2 x_2 \leq 5$$
and
$$x_1 \geq 0, \ x_2 \geq 0.$$

The optimal solution is $\mathbf{x}^{(2)}{}_{LP} = (5, 0)$. Since $\mathbf{x}^{(2)}{}_{LP} - \mathbf{x}^{(1)} = (0, 0)$, this implies that $\mathbf{x}^{(2)} = (5, 0) = \mathbf{x}^{(k)}$ for all $k = 2, 3, \dots$. Hence, $(x_1, x_2) = (5, 0)$ is indeed the optimal solution.

Results for Any Other Initial Trial Solution:

Now consider any other initial trial solution $\mathbf{x}^{(0)} \neq (1, 1)$. Since $x_1 \geq 0, x_2 \geq 0$,

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \frac{1}{1 + x_1 + x_2} > 0.$$

Therefore, the beginning of iteration 1 immediately leads to

$$c_1 = c_2 > 0,$$

so the resulting linear programming problem,

Maximize $Z = c_1 x_1 + c_2 x_2$,
subject to
$$x_1 + 2x_2 \leq 5$$
and
$$x_1 \geq 0, \ x_2 \geq 0,$$

has the optimal solution $\mathbf{x}_{LP}^{(1)} = (5, 0)$, just as above, so the algorithm thereafter proceeds as outlined above. Consequently, the results obtained at the end of the two iterations are exactly the same as above when the initial trial solution was $(x_1, x_2) = (1, 1)$.

Example 2 for Section 13.9

Consider the following convex programming problem:

Minimize $\quad f(\mathbf{x}) = \dfrac{(x_1 + 1)^3}{3} + x_2,$

subject to
$$x_1 \geq 1 \quad \text{and} \quad x_2 \geq 0.$$

(a) If SUMT were applied directly to this problem, what would be the unconstrained function P(x; r) to be minimized at each iteration?

Since minimizing $f(x)$ is equivalent to maximizing $-f(x)$, the formula for $P(\mathbf{x}; r)$ given in Sec. 13.9 yields

$$P(\mathbf{x};r) = -f(x) - r\left(\frac{1}{x_1 - 1} + \frac{1}{x_2}\right) = -\frac{(x_1 + 1)^3}{3} - x_2 - r\left(\frac{1}{x_1 - 1} + \frac{1}{x_2}\right).$$

(b) Derive the minimizing solution of P(x; r) analytically, and then give this solution for r = 1, 10^{-2}, 10^{-4}, 10^{-6}.

The optimal solution $\mathbf{x}^*$ of $P(\mathbf{x}; r)$ satisfies $\nabla_{\mathbf{x}} P(\mathbf{x}; r) = 0$.
In particular, we are solving these two equations:

$$\frac{\partial P(\mathbf{x};r)}{\partial x_1} = -(x_1 + 1)^2 + r\frac{1}{(x_1 - 1)^2} = 0,$$

$$\frac{\partial P(\mathbf{x};r)}{\partial x_2} = -1 + \frac{r}{x_2^2} = 0.$$

The solution is

$$(x_1^*, x_2^*) = (\sqrt{1 + \sqrt{r}}, \sqrt{r}).$$

The solution for various r is given in the table below.

r	x_1^*	x_2^*
1	1.4142	1
10^{-2}	1.0488	0.1
10^{-4}	1.0050	0.01
10^{-6}	1.0005	0.001

Note that $(x_1, x_2) \rightarrow (1, 0)$ as $r \rightarrow 0$, so $(x_1, x_2) = (1, 0)$ is optimal.

(c) Beginning with the initial trial solution (x$_1$, x$_2$) = (2, 1), use the automatic procedure in your IOR Tutorial to apply SUMT to this problem (in maximization form) with r = 1, 10^{-2}, 10^{-4}, 10^{-6}.

Using IOR Tutorial to solve when starting with $(x_1, x_2) = (2,1)$ results in the following:

r	x_1	x_2	$f(x)$
	2	1	10
1	1.4142	1	5.69
10^{-2}	1.0488	0.1	2.967
10^{-4}	1.0050	0.01	2.697
10^{-6}	1.0005	0.001	2.67