Algebraic Topology

M382C

Michael Starbird
&
Cynthia Verjovsky Marcotte

Fall 2007

Contents

Chapter 1

Introduction

Abstracting and generalizing essential features of familiar objects often lead to the development of important mathematical ideas. One goal of geometrical analysis is to describe the relationships and features that make up the essential qualities of what we perceive as our physical world. The strategy is to find ideas that we view as central and then to generalize those ideas and to explore those more abstract extensions of what we perceive directly.

Much of topology is aimed at exploring abstract versions of geometrical objects in our world. The concept of geometrical abstraction dates back at least to the time of Euclid (*c.* 225 B.C.E.) The most famous and basic spaces are named for him, the Euclidean spaces. All of the objects that we will study in this course will be subsets of the Euclidean spaces.

1.1 Basic Examples

Definition ($\mathbb{R}^n$)**.** *We define* real *or* Euclidean n-space*, denoted by* $\mathbb{R}^n$*, as the set*

$$\mathbb{R}^n := \{(x_1, x_2, \ldots, x_n) | x_i \in \mathbb{R} \textit{ for } i = 1, \ldots, n\}.$$

We begin by looking at some basic subspaces of $\mathbb{R}^n$.

Definition (standard n-disk)**.** *The* n-dimensional disk*, denoted* $\mathbb{D}^n$ *is defined as*

$$\begin{aligned}\mathbb{D}^n &:= \{(x_1, \ldots, x_n) \in \mathbb{R}^n | 0 \le x_i \le 1 \textit{ for } i = 1, \ldots, n\ \} \\ &\cong \overbrace{[0,1] \times [0,1] \times \cdots \times [0,1]}^{n \textit{ times}} \subset \mathbb{R}^n.\end{aligned}$$

For example, $\mathbb{D}^1 = [0, 1]$. $\mathbb{D}^1$ is also called the unit interval, sometimes denoted by I.

Definition (standard n-ball, standard n-cell)**.** *The* n-dimensional ball *or* cell*, denoted* $\mathbb{B}^n$*, is defined as:*

$$\mathbb{B}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1^2 + \ldots + x_n^2 \leq 1\}.$$

Fact 1.1. *The standard n-ball and the standard n-disk are compact and homeomorphic.*

Definition (standard n-sphere)**.** *The* n-dimensional sphere*, denoted* $\mathbb{S}^n$*, is defined as*

$$\mathbb{S}^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} | x_0^2 + \ldots + x_n^2 = 1\}.$$

Note. $\operatorname{Bd} \mathbb{B}^{n+1} = \mathbb{S}^n$

As usual, the term n-sphere will apply to any space homeomorphic to the standard n-sphere.

Question 1.2. *Describe $\mathbb{S}^0$, $\mathbb{S}^1$, and $\mathbb{S}^2$. Are they homeomorphic? If not, are there any properties that would help you distinguish between them?*

1.2 Simplices

One class of spaces in $\mathbb{R}^n$ we will be studying will be *manifolds* or *k-manifolds*, which are made up of pieces that locally look like $\mathbb{R}^k$, put together in a "nice" way. In particular, we will be studying manifolds that use triangles (or their higher-dimensional equivalents) as the basic building blocks.

Since k-dimensional "triangles" in $\mathbb{R}^n$ (called *simplices*) are the basic building blocks we will be using, we begin by giving a vector description of them.

Definition (1-simplex)**.** *Let v_0, v_1 be two points in $\mathbb{R}^n$. If we consider v_0 and v_1 as vectors from the origin, then $\sigma^1 = \{\mu v_1 + (1 - \mu)v_0 \mid 0 \leq \mu \leq 1\}$ is the straight line segment between v_0 and v_1. σ^1 can be denoted by $\{v_0 v_1\}$ or $\{v_1 v_0\}$ (the order the vertices are listed in doesn't matter). The set σ^1 is called a* 1-simplex *or* edge *with vertices (or 0-simplices) v_0 and v_1.*

Definition (2-simplex). *Let v_0, v_1, and v_2 be three non-collinear points in $\mathbb{R}^n$. Then*

$$\sigma^2 = \{\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 \mid \lambda_0 + \lambda_1 + \lambda_2 = 1 \textit{ and } 0 \le \lambda_i \le 1 \forall i = 0, 1, 2\}$$

is a triangle with edges $\{v_0 v_1\}$, $\{v_1 v_2\}$, $\{v_0 v_2\}$ and vertices v_0, v_1, and v_2. The set σ^2 is a 2-simplex *with vertices v_0, v_1, and v_2 and edges $\{v_0 v_1\}$, $\{v_1 v_2\}$, and $\{v_0 v_2\}$. $\{v_0 v_2 v_2\}$ denotes the 2-simplex σ^2 (where the order the vertices are listed in doesn't matter).*

Note that the plural of simplex is *simplices*.

Definition (n-simplex and face of a simplex). *Let $\{v_0, v_2, \ldots, v_n\}$ be a set affine independent points in $\mathbb{R}^N$. Then an n-*simplex *σ^n (of dimension n), denoted $\{v_0 v_1 v_2 \ldots v_n\}$, is defined to be the following subset of $\mathbb{R}^N$:*

$$\sigma^n = \left\{ \lambda_0 v_0 + \lambda_1 v_1 + ... + \lambda_n v_n \,\middle|\, \sum_{i=0}^{n} \lambda_i = 1\ ;\ 0 \le \lambda_i \le 1,\ i = 0, 1, 2, \ldots, n \right\}.$$

An i-simplex whose vertices are any subset of $i+1$ of the vertices of σ^n is an (i-dimensional) face *of σ^n. The face obtained by deleting the v_m vertex from the list of vertices of σ^n is often denoted by $\{v_0 v_1 v_2 \ldots \widehat{v_m} \ldots v_n\}$. (Note that it is an $(n-1)$-simplex.)*

Exercise 1.3. *Show that the faces of a simplex are indeed simplices.*

Fact 1.4. *The standard n-ball, standard n-disk and the standard n-simplex are compact and homeomorphic.*

We will use the terms n-disk, n-cell, n-ball interchangeably to refer to any topological space homeomorphic to the standard n-ball.

1.3 Simplicial Complexes

Simplices can be assembled to create polyhedral subsets of $\mathbb{R}^n$ known as complexes. These simplicial complexes are the principal objects of study for this course.

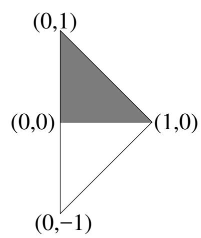
Definition (finite simplicial complex). *Let T be a finite collection of simplices in $\mathbb{R}^n$ such that for every simplex σ_i^j in T, each face of σ_i^j is also a simplex in T and any two simplices in T are either disjoint or their intersection is a face of each. Then the subset K of $\mathbb{R}^n$ defined by $K = \bigcup \sigma_i^j$*

running over all simplices σ_i^j *in* T *is a* finite simplicial complex *with* triangulation T*, denoted* (K,T)*. The set* K *is often called the* underlying space *of the simplicial complex. If* n *is the maximum dimension of all simplices in* T*, then we say* (K,T) *is of dimension* n*.*

Example 1. *Consider* (K,T) *to be the simplicial complex in the plane where*

$$\begin{aligned} T \; = \{ \; & \{(0,0)(0,1)(1,0)\}, \{(0,0)(0,-1)\}, \{(0,-1)(1,0)\}, \\ & \{(0,0)(0,1)\}, \{(0,1)(1,0)\}, \{(1,0)(0,0)\}, \\ & \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(0,-1)\}\} \, . \end{aligned}$$

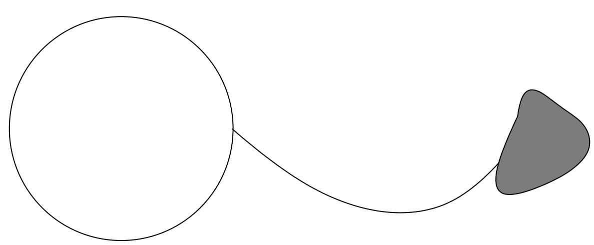
So K *is a filled in triangle and a hollow triangle as pictured.*



Exercise 1.5. *Draw a space made of triangles that is* not *a simplicial complex, and explain why it is not a simplicial complex.*

We have started by making spaces using simplices as building blocks. But what if we have a space, and we want to break it up into simplices? If J is a topological space homeomorphic to K where K is a the underlying space of a simplicial complex (K,T) in $\mathbb{R}^m$, then we say that J is *triangulable.*

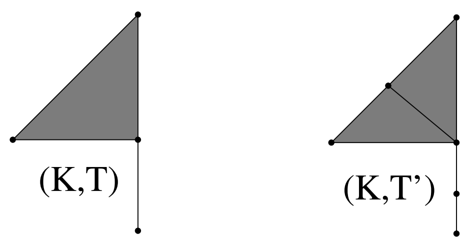
Exercise 1.6. *Show that the following space is triangulable:*



by giving a triangulation of the space.

Definition (subdivision)**.** *Let (K, T) be a finite simplicial complex. Then T' is a subdivision of T if (K, T') is a finite simplicial complex, and each simplex in T' is a subset of a simplex in T.*

Example 2. *The following picture illustrates a finite simplicial complex and a subdivision of it.*



There is a standard subdivision of a triangulation that later will be useful:

Definition (derived subdivision)**.**

1. *Let σ^2 be a 2-simplex with vertices v_0, v_1, and v_2. Then $p = \frac{1}{3}v_0 + \frac{1}{3}v_1 + \frac{1}{3}v_2$ is the* barycenter *of σ^2.*

2. *Let T be a triangulation of a simplicial 2-complex with 2-simplices $\{\sigma_i\}_{i=1}^{k}$. The* first derived subdivision *of T, denoted T', is the union of all vertices of T with the collection of 2-simplices obtained from T by breaking each σ_i in T into six pieces as shown, together with their edges and vertices, and finally the edges and vertices obtained by breaking each edge that is not a face of a 2-simplex into two edges. Notice that the new vertices are the barycenter of each σ_i in T and the center of each edge in T. The* second derived subdivision*, denoted T'', is $(T')'$, the first derived subdivision of T', and so on.(See Figure 1.1)*

Example 3. *Figure 1.2 illustrates a finite simplicial complex and the second derived subdivision of it.*

1.4 2-manifolds

The concept of the real line and the Euclidean spaces produced from the real line are fundamental to a large part of mathematics. So it is natural to

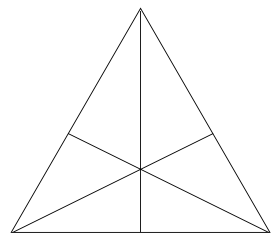


Figure 1.1: Barycentric subdivision of a 2-simplex

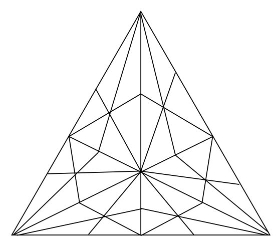


Figure 1.2: Second barycentric subdivision of a 2-simplex

be particularly interested in topological spaces that share features with the Euclidean spaces. Perhaps the most studied spaces considered in topology are those that look locally like the Euclidean spaces. The most familiar such space is the 2-sphere since it is modelled by the surface of Earth, particularly in flat places like Kansas or the middle of the ocean. If you are on a ship in the middle of the Pacific Ocean, the surrounding terrain looks like the surrounding terrain if you were living on a plane, which is Euclidean 2-space or $\mathbb{R}^2$. The concept of a space being locally homeomorphic to $\mathbb{R}^2$ is sufficiently important that it has a name, in fact, two names. A space locally homeomorphic to $\mathbb{R}^2$ is called a *surface* or 2-*manifold*. The 2-sphere is a surface as is the *torus* (which looks like an inner-tube or the surface of a doughnut).

Definition (2-manifold or surface)**.** *A* 2-manifold *or* surface *is a separable, metric space Σ^2 such that for each $p \in \Sigma^2$, there is a neighborhood U of p that is homeomorphic to $\mathbb{R}^2$.*

1.4.1 2-manifolds as simplicial complexes

For now, we will restrict ourselves to 2-manifolds that are subspaces of R^n and that are triangulated.

Definition (triangulated 2-manifold)**.** *A* triangulated compact 2-manifold *is a space homeomorphic to a subset M^2 of $\mathbb{R}^n$ such that M^2 is the underlying space of a simplicial complex (M^2, T).*

Example 4. *The tetrahedral surface below, with triangulation*

$$\begin{aligned} T &= \{\{v_0v_1v_2\}, \{v_0v_1v_3\}, \{v_0v_2v_3\}, \{v_1v_2v_3\}, \\ &\quad \{v_0v_1\}, \{v_0v_2\}, \{v_0v_3\}, \{v_1v_2\}, \{v_1v_3\}, \{v_2v_3\}, \\ &\quad \{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}\} \end{aligned}$$

is a triangulated 2-manifold (homeomorphic to $\mathbb{S}^2$).

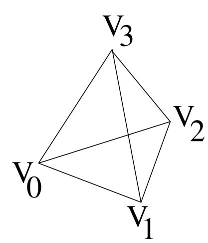


Figure 1.3: Tetrahedral surface

The following theorem asserts that every compact 2-manifold is triangulable, but its proof entails some technicalities that would take us too far afield. So we will analyze triangulated 2-manifolds and simply note here without proof that our results about triangulated 2-manifolds actually hold in the topological category as well.

Theorem 1.7. *A compact, 2-manifold is homeomorphic to a compact, triangulated 2-manifold, in other words, all compact 2-manifolds are triangulable.*

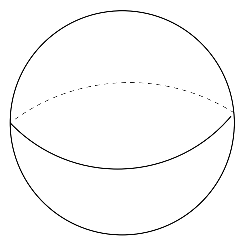
Definitions (1-skeleton and dual 1-skeleton)**.**

1. *The* 1-skeleton *of a triangulation T equals $\bigcup\{\sigma_j \mid \sigma_j$ is a 1-simplex in $T\}$ and is denoted $T^{(1)}$.*

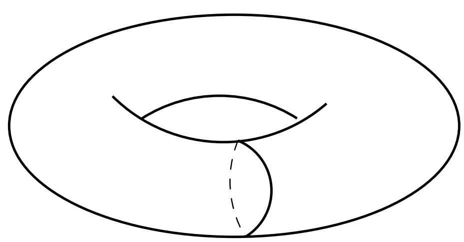
2. *The* dual 1-skeleton *of a triangulation T equals $\bigcup\{\sigma_j \mid \sigma_j$ is an edge of a 2-simplex in T' and neither vertex of σ_j is a vertex of a 2-simplex of $T\}$. An edge in the dual 1-skeleton has each of its ends at the barycenters of 2-simplices of the original triangulation, that is, physically each edge in the dual 1-skeleton is composed of two segments, each running from the barycenter of a 2-simplex to the middle of the edge they share in the original triangulation. So an edge in the dual 1-skeleton is the union of two 1-simplices in T'.*

Examples 5. *The following are triangulable 2-manifolds:*

a. $\mathbb{S}^2$



b. $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$ *or any other space homeomorphic to the boundary of a doughnut, the torus.*



c. Double torus:(See Figure 1.4)

The following example cannot be embedded in $\mathbb{R}^3$; however, it can be embedded in R^4.

d. The Klein bottle, denoted $\mathbb{K}^2$:(See Figure 1.5)

There is another 2-manifold that cannot be embedded in $\mathbb{R}^3$ that we will study, which requires the use of the quotient or identification topology (see Appendix A):

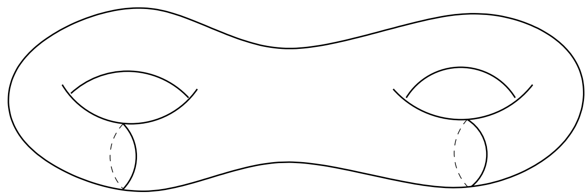


Figure 1.4: The double torus or surface of genus 2

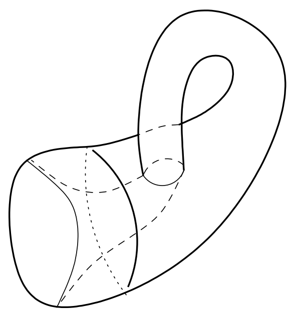


Figure 1.5: The Klein Bottle

e. The projective plane, denoted $\mathbb{RP}^2$, := *space of all lines through* $\mathbf{0}$ *in* $\mathbb{R}^3$ *where the basis for the topology is the collection of open cones with the cone point at the origin.*

Exercise 1.8.

1. *Show* $\mathbb{RP}^2 \cong \mathbb{S}^2/\langle x \sim -x \rangle$*, that is, the* 2*-sphere with diametrically opposite points identified.*

2. *Show that* $\mathbb{RP}^2$ *is also homeomorphic to a disk with two edges on its boundary (called a* bigon*), identified as indicated in Figure 1.6.*

3. *Show that* $\mathbb{RP}^2 \cong$ *Möbius band with a disk attached to its boundary (See Figure 1.7).*

Exercise 1.9. *Show that* $\mathbb{T}^2$ *as defined above is homeomorphic to the surface in* $\mathbb{R}^3$ *parametrized by:*

$$\left\{(\theta, 1 + \frac{1}{2}\cos\phi, \frac{1}{2}\sin\phi \,|0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi\right\}$$

in cylindrical coordinates.

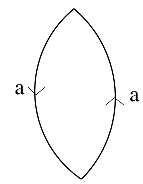


Figure 1.6: $\mathbb{R}\mathrm{P}^2$

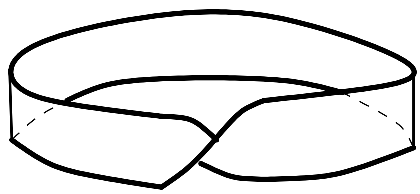


Figure 1.7: The Möbius band

There is a way of obtaining more 2-manifolds by "connecting" two or more together. For instance, the double torus looks like two tori that have been joined together.

Definition (connected sum)**.** *Let M_1^2 and M_2^2 be two compact, connected, triangulated 2-manifolds and let D_1 and D_2 be 2-simplices in the triangulations of M_1 and M_2 respectively. Paste $M_1^2 - \operatorname{Int} D_1$ and $M_2^2 - \operatorname{Int} D_2$ along the boundaries of $\operatorname{Bd} D_1$ and $\operatorname{Bd} D_2$. The resulting manifold is called the* connected sum of M_1^2 and M_2^2*, and denoted by $M_1^2 \# M_2^2$. Similarly, define the* connected sum of n 2-manifolds *recursively.*

This definition of connected sum can in fact be generalized to the connected sum of any two n-manifolds. Can you see how to do it?

Exercise 1.10. *Show that $\mathbb{R}\mathrm{P}^2 \# \mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle.*

Exercise 1.11. *Show that $\mathbb{T}^2 \# \mathbb{R}\mathrm{P}^2$, where $\mathbb{T}^2$ is the torus, is homeomorphic $K^2 \# \mathbb{R}\mathrm{P}^2$, where $\mathbb{K}^2$ is the Klein bottle.*

1.4.2 2-manifolds as quotient spaces

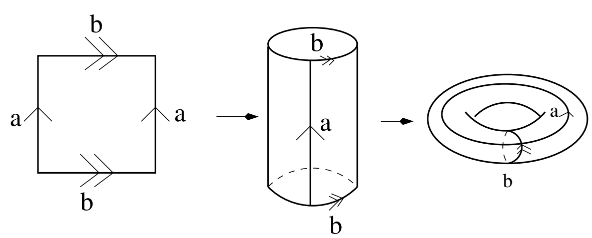
There is another way of thinking of 2-manifolds, as the abstract spaces obtained from a particular kind of quotients (see Appendix A for a review of quotient spaces).

The process of identifying all elements of an equivalence class to a single one is often called a *gluing* when the equivalence classes are mostly small, having 1 or 2 or a finite number of points in each.

In our case, we will be looking at the quotient spaces obtained from polygonal disks, where all points of the interior of the disk are in their own equivalence class, the points on the interior of the edges are in two-point equivalence classes, and the vertices of the polygonal disks are in equivalence classes with any number of other vertices. We think of obtaining the 2-manifold by gluing the edges of the polygonal disk to each other pairwise, in some particular pattern.

Examples 6. *In these examples the kind of arrow indicates which edges are glued together, while the orientations of the arrows indicate how to glue the two edges together. You should convince yourself that any two gluing maps that agree with the given orientations will yield homeomorphic spaces.*

1. *(torus)*



2. *(sphere) (See Figure 1.8)*

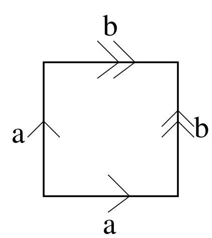


Figure 1.8: The sphere

3. *(sphere) (See Figure 1.9)*
4. *(double torus) (See Figure 1.10)*

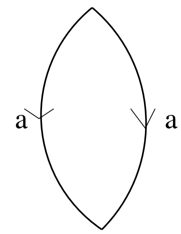


Figure 1.9: Another way to see the sphere

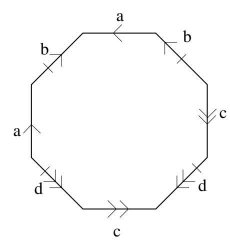


Figure 1.10: The double torus

5. *(Klein bottle) (See Figure 1.11)*

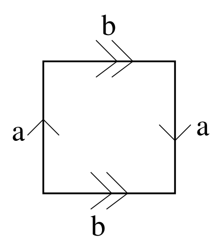


Figure 1.11: The Klein bottle

6. *(projective plane) (See Figure 1.12)*

7. *(projective plane) (See Figure 1.13)*

You should check to see that alternative presentations of the same space are homeomorphic. You should also check that these spaces are homoeomorphic to the triangulable 2-manifolds described in the previous subsection.

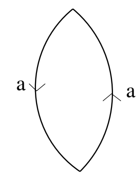


Figure 1.12: The projective plane

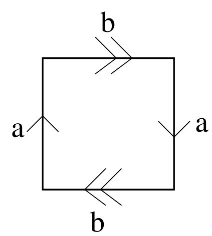


Figure 1.13: Another version of the projective plane

The following theorem will be put off to chapter 4 (and stated in a slightly different but equivalent way). Surprisingly, it is highly non-trivial to prove but not surprisingly it is incredibly useful.

Theorem 1.12 (Jordan Curve Theorem)**.** *Let $h : [0,1] \to \mathbb{D}^2$ be a topological embedding where $h(0), h(1) \in \mathrm{Bd}(\mathbb{D}^2)$. Then $h([0,1])$ separates $\mathbb{D}^2$ into exactly two pieces.*

Theorem 1.13. *Any polygonal disk with edges identified in pairs is homeomorphic to a compact, connected, triangulated* 2*-manifold.*

Theorem 1.14. *Any compact, connected, triangulated* 2*-manifold is homeomorphic to a polygonal disk with edges identified in pairs.*

1.5 Questions

The most fundamental questions in topology are:

Question 1.15. *How are spaces similar and different? Particularly, which are homeomorphic? Which aren't?*

Showing two spaces are homeomorphic means we must construct a homeomorphism between them. But how do we show two spaces are *not* homeomorphic? When we are confronted with the task of trying to explore one space or to specify what is different about two spaces, we must examine the spaces looking for features or properties that are of topological significance.

Question 1.16. *What features of the examples studied are interesting either in their own right or for the purpose of distinguishing one from another?*

Chapter 2

2-manifolds

2.1 Classification of compact 2-manifolds

A surface, or 2-manifold, is locally homeomorphic to $\mathbb{R}^2$, so we know how these spaces look locally. But what are the possibilities for the global character of these spaces? We have seen several examples (the 2-sphere, the torus, the Klein bottle, $\mathbb{R}\mathrm{P}^2$). Now we seek to organize our understanding of the collection of all surfaces, that is, to recognize, describe, and classify each surface as one from a simple list of possible homeomorphism classes. So we need to use the local Euclidean feature of 2-manifolds to help us describe the overall structure of these surfaces.

In working with these compact 2-manifolds, we want to think of them as physical objects made of simple building blocks, namely, triangles. In fact, we will begin by considering just 2-manifolds that reside in $\mathbb{R}^n$ and are made of triangles. This investigation of these simple compact 2-manifolds actually is comprehensive since every compact 2-manifold is homeomorphic to one made of finitely many triangles which is embedded in $\mathbb{R}^n$. The advantage of working with objects made from a finite number of triangles is that we can use inductive procedures moving from triangle to triangle.

The main thing to have in mind at this point is that we should view 2-manifolds as concrete, physical objects that are constructed from a finite number of flat triangles (simplices) that fit together as specified: they overlap, if at all, only along a shared edge or at a vertex of each. This physical view of 2-manifolds will allow us to understand them so clearly that we can describe an effective method for determining the global structure of the object by knowing the local structure.

The goal of the following two sections is to prove (in two different ways) that every compact, triangulated 2-manifold can be constructed by taking the connected sum of simple 2-manifolds, namely the sphere, torus, and projective plane.

In the following section we proceed with a sequence of theorems that show us that after removing one disk, any compact, triangulated 2-manifold is just a disk with strips attached in particularly simple ways. We continually use the local structure of the triangulated 2-manifold to see how the whole thing fits together.

The second proof of the classification theorem views each compact, triangulated 2-manifold as the quotient space of a polygonal disk with its edges identified in pairs.

Definition (regular neighborhood)**.** *Let M^2 be a 2-manifold with triangulation $T = \{\sigma_i\}_{i=1}^k$. Let A be a subcomplex of (M^2, T) . The* regular neighborhood *of A, denoted $N(A)$, equals $\bigcup\{\sigma_j'' \mid \sigma_j'' \in T''$ and $\sigma_j'' \cap A \neq \emptyset\}$.*

Exercise 2.1. *The boundary of a tetrahedron is naturally triangulated with a triangulation T consisting of four 2-simplexes together with their six edges and four vertices. On the boundary of a tetrahedron locate the first and second derived subdivisions of T, the 1-skeleton of T, the regular neighborhood of the 1-skeleton of T, the regular neighborhoods of a vertex and an edge of T, and the dual 1-skeleton of T.*

Exercise 2.2. *On the accompanying pictures of the second derived subdivisions of triangulations of the torus and the Klein bottle, find regular neighborhoods of subsets of the 1-skeleton.*

Exercise 2.3. *Characterize graphs in the 1-skeleton of T for the triangulations of the sphere, torus, and projective plane whose regular neighborhoods are homeomorphic to a disk.*

2.1.1 Classification of compact, connected 2-manifolds, I

The basic idea of this proof is to show that removing an open disk from a compact triangulated 2-manifold gives us a space homeomorphic to a (closed) disk with some number of bands attached to its boundary in a specified way. The number of bands, and how they are attached then gives us the classification of the surface.

Theorem 2.4. *Let M^2 be a compact, triangulated 2-manifold with triangulation T. Let S be a tree whose edges are 1-simplices in the 1-skeleton of T. Then $N(S)$, the regular neighborhood of S, is homeomorphic to $\mathbb{D}^2$.*

Theorem 2.5. *Let M^2 be a compact, triangulated 2-manifold with triangulation T. Let S be a tree equal to a union of edges in the dual 1-skeleton of T. Then $\cup\{\sigma_j'' \mid \sigma_j'' \in T''$ and $\sigma_j'' \cap S \neq \emptyset\}$ is homeomorphic to $\mathbb{D}^2$.*

Theorem 2.6. *Let M^2 be a connected, compact, triangulated 2-manifold with triangulation T. Let S be a tree in the 1-skeleton of T. Let S' be the subgraph of the dual 1-skeleton of T whose edges do not intersect S. Then S' is connected.*

The following two theorems state that M^2 can be divided into two pieces, one a disk D_0, and the other a disk (D_1) with bands (the H_i's) attached to it.

Theorem 2.7. *Let M^2 be a connected, compact, triangulated 2-manifold. Then $M^2 = D_0 \cup D_1 \cup \left(\bigcup_{i=1}^{k} H_i\right)$ where D_0, D_1, and each H_i is homeomorphic to $\mathbb{D}^2$, $\mathrm{Int}\ D_0 \cap D_1 = \emptyset$, the H_i's are disjoint, $\bigcup_{i=1}^{k} \mathrm{Int}\ H_i \cap (D_0 \cup D_1) = \emptyset$, and for each i, $H_i \cap D_1$ equals 2 disjoint arcs each arc on the boundary of each of H_i and D_1.*

Theorem 2.8. *Let M^2 be a connected, compact, triangulated 2-manifold. Then:*

1. *There is a disk D_0 in M^2 such that $M^2 - (\mathrm{Int}\, D_0)$ is homeomorphic to the following subset of $\mathbb{R}^3$: a disk D_1 with a finite number of disjoint strips, H_i for $i \in \{1, \ldots n\}$, attached to boundary of D_1 where each strip has no twist or $1/2$ twist. (See Figure 2.1.)*
2. *Furthermore, the boundary of the disk with strips, $D_1 \cup \left(\bigcup_{i=1}^{k} H_i\right)$, is connected.*

Exercise 2.9. *In the set-up in the previous theorem, any strip H_i divides the boundary of D_0 into two edges e_i^1 and e_i^2, where H_i is* not *attached. Show that if a strip H_j is attached to D_0 with no twists, then there must be a strip H_k that is attached to both e_j^1* and *e_j^2.*

Theorem 2.10. *Let M^2 be a connected, compact, triangulated 2-manifold. Then there is a disk D_0 in M^2 such that $M^2 - \mathrm{Int}\ D_0$ is homeomorphic to a disk D_1 with strips attached as follows: first come a finite number of strips with $1/2$ twist each whose attaching arcs are consecutive along $\mathrm{Bd}\, D_1$, next come a finite number of pairs of untwisted strips, each pair with attaching arcs entwined as pictured with the four arcs from each pair consecutive along $\mathrm{Bd}\, D_1$.*

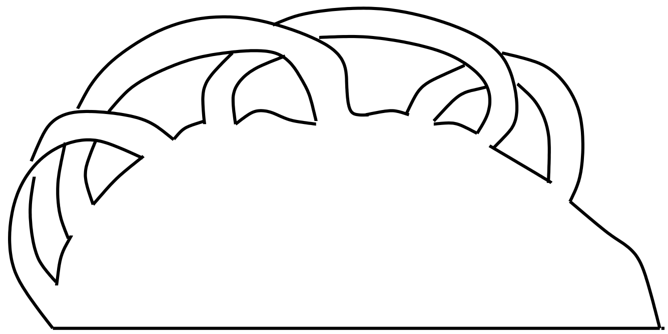


Figure 2.1: A disk with four handles attached.

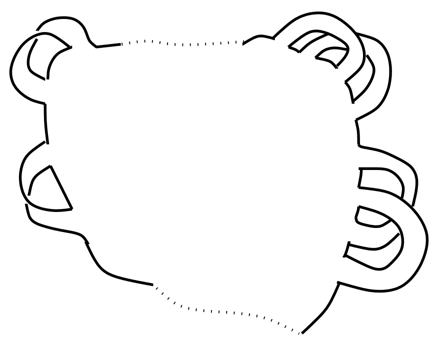


Figure 2.2: Twisted strips and entwined strips

Theorem 2.11. *Let X be a disk D_0 with one strip attached with a $1/2$ twist with its attaching arcs consecutive along* $\operatorname{Bd} D_0$ *and one pair of untwisted strips with attaching arcs entwined as pictured with the four arcs consecutive along* $\operatorname{Bd} D_0$*. Let Y be a disk D_1 with three strips with a $1/2$ twist each whose attaching arcs are consecutive along* $\operatorname{Bd} D_1$*. Then X is homeomorphic to Y.*

Theorem 2.12. *Let M^2 be a connected, compact, triangulated 2-manifold. Then there is a disk D_0 in M^2 such that $M^2 - \operatorname{Int} D_0$ is homeomorphic to one of the following:*

a) a disk D_1,

b) a disk D_1 with k $\frac{1}{2}$-twisted strips with consecutive attaching arcs, or

c) a disk D_1 with k pairs of untwisted strips, each pair in entwining position with the four attaching arcs from each pair consecutive.

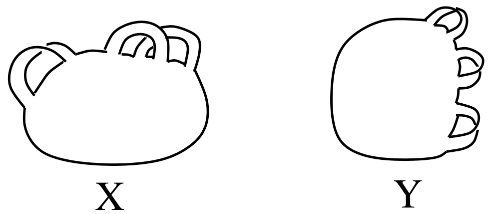


Figure 2.3: These spaces are homeomorphic.

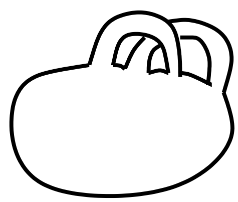


Figure 2.4: Entwining pair of strips

Theorem 2.13 (Classification of compact, connected 2-manifolds)**.** *Any connected, compact, triangulated* 2*-manifold is homeomorphic to the* 2*-sphere* $\mathbb{S}^2$*, a connected sum of tori, or a connected sum of projective planes.*

Notice that at this point we have shown that any compact, connected, triangulated 2-manifold is a sphere, the connected sum of n tori, or the connected sum of n projective planes; however, we have not yet established that those possibilities are all topologically distinct. The classification of 2-manifolds requires us to prove our suspicions that any two different connected sums are indeed not homeomorphic. Before we develop tools for confirming those suspicions, we digress to develop another proof of this first part of the classification theorem.

2.1.2 Classification of compact, connected 2-manifolds, II

We now outline a different approach to proving that any compact, connected, triangulated 2-manifold is a sphere, the connected sum of tori, or the connected sum of projective planes. This approach uses the quotient or identification topology described in the previous chapter.

Suppose that we are gluing the edges of a polygonal disk to create a 2-manifold. If we assign a unique letter to each pair of edges that are glued together, and we read the letters as we follow the edges along the boundary of the disk (starting at a certain edge) going clockwise, we get a "word" made up of these letters. However, to specify the gluing we need to know not only which edges are glued together, but in what orientation. To keep track of that, we will write the letter alone if the orientation given on the edge agrees with the direction we're reading the edges in, and the letter to the -1 power if it disagrees. For example, $abca^{-1}dcb^{-1}d$ represents a gluing of the octagon as indicated, so that the orientations of two identified edges agree:

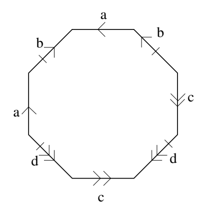


Figure 2.5: The genus two surface

Definition (gluing of a $2n$-gon with edges identified in pairs)**.** *An expression (*word*) of n letters, such as $abca^{-1}dcb^{-1}d$, where each letter appears exactly twice, represents the* 2*-manifold obtained by gluing the edges of a $2n$-gon in pairs as indicated by the sequence of letters. Notice that a pair of edges with the same letter really has two different possible gluings. To determine which gluing, we need to look at the superscript or lack of subscript of each letter. A letter without a subscript is viewed as oriented clockwise around the $2n$-gon, while a superscript -1, as in a^{-1}, indicates that that edge is oriented counterclockwise. Then the identification of the pair of edges respects those directions. So the equivalence classes of the disk specified by such a $2n$ length string of n letters consist of every singleton in the interior of the $2n$-gon, pairs of points one from each interior of the edges with the same label, and then equivalence classes of vertices as come together when the edges are identified as specified. The equivalence classes among vertices might have any number of vertices in them, depending on the string of letters.*

Theorem 2.14.

1. *The bigon with edges identified by aa^{-1} is homeomorphic to $\mathbb{S}^2$.*
2. *The bigon with edges identified by bb is homeomorphic to $\mathbb{RP}^2$.*

3. The square with edges identified by $cdc^{-1}d^{-1}$ is homeomorphic to $\mathbb{T}^2$.

Theorem 2.15 (connected sum relation)**.** *The gluing of a square given by $ccdd$ is homeomorphic to $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ and the gluing of an octagon given by $aba^{-1}b^{-1}cdc^{-1}d^{-1}$ is homeomorphic to $\mathbb{T}^2\#\mathbb{T}^2$.*

Question 2.16. *Generalize the above to the connected sum of any two surfaces.*

The next sequence of theorems will show us how to take a $2n$-gon with edges identified in pairs and modify the gluing prescription to find a canonical representation of the same 2-manifold.

Theorem 2.17. *Let $Abb^{-1}C$ be a string of $2n$ letters where each letter occurs twice, with or without a superscript (so A and C should each be construed as being comprised of many letters). Then the 2-manifold obtained by identifying a $2n$-gon following the gluing $Abb^{-1}C$ is homeomorphic to the 2-manifold which is obtained by identifying a $(2n-2)$-gon following the gluing given by AC.*

Theorem 2.18. *Suppose a 2-manifold M^2 is represented by a $2n$-gon with edges identified in pairs. Then a homeomorphic 2-manifold can be represented by a $2k$-gon with edges identified in pairs where all the vertices are in the same equivalence class, that is, all the vertices are identified to each other.*

Theorem 2.19. *Suppose a 2-manifold $M^2 \not\cong \mathbb{S}^2$ is represented by a $2n$-gon with edges identified in pairs. Then a homeomorphic 2-manifold can be represented by a $2k$-gon with edges identified in pairs where all the vertices are identified and every pair of edges with the same orientation are consecutive.*

Theorem 2.20. *Suppose a 2-manifold $M^2 \not\cong \mathbb{S}^2$ is represented by a $2n$-gon with edges identified in pairs. Then a homeomorphic 2-manifold can be represented by a $2k$-gon with edges identified in pairs where all the vertices are identified, every pair of edges with the same orientation are consecutive, and all other edges are grouped in disjoint sets of two intertwined pairs following the pattern $aba^{-1}b^{-1}$.*

Theorem 2.21. *The 2-manifold represented by $aba^{-1}b^{-1}cc$ is homeomorphic to the 2-manifold represented by $ddeeff$.*

Question 2.22. *Re-state the above theorem in terms of connected sum.*

Theorem 2.23. *Any compact, connected, triangulated* 2*-manifold is homeomorphic to a* $2n$*-gon with edges identified in pairs as specified in one of the three following ways:* aa^{-1}, *or* $a_0a_0a_1a_1 \dots a_na_n$ *(where* $n \geq 0$*) or* $a_0a_1a_0^{-1}a_1^{-1} \dots a_{n-1}a_na_{n-1}^{-1}a_n^{-1}$ *(where* $n \geq 1$ *is odd).*

Theorem 2.24 (Classification of compact, connected 2-manifolds)**.** *Any connected, compact, triangulated 2-manifold is homeomorphic to the 2-sphere* $\mathbb{S}^2$, *a connected sum of tori, or a connected sum of projective planes.*

2.2 PL Homeomorphism

Our goal is to organize connected, compact, triangulated 2-manifolds by homeomorphism type. The concept of topological homeomorphism does not reflect the triangulated structure we have associated with these objects, so here we present a natural way of equating two triangulated 2-manifolds that includes the simplicial structure of them as well as the topological type.

The basic strategy is first to define an equivalence between two triangulated 2-manifolds with triangulations T_1 and T_2 if the simplices of T_1 correspond to the simplices of T_2 in a straightforward 1-1 fashion. Then we describe another idea of equivalence if the two 2-manifolds can be subdivided to find new triangulations that have this 1-1 correspondence.

Definition (simplicial homeomorphism)**.** *We will say that M_1^2 with triangulation T_1 is* simplicially homeomorphic *to M_2^2 with triangulation T_2 if and only if there exists a homeomorphism from M_1^2 to M_2^2 that gives a one-to-one correspondence between T_1 and T_2 in the following way: the homeomorphism maps each simplex in T_1 linearly to a single simplex in T_2. So the vertices of T_1 go to the vertices of T_2 and the rest of the homeomorphism is determined by extending the map on the vertices linearly over each simplex.*

Of course, we have seen that a space can have many different triangulations. Therefore, the concept of a simplicial homeomorphism is too restrictive. An underlying space with a triangulation and the same space with its second derived subdivision triangulation are not simplicially isomorphic. So we can give a broader concept of equivalence:

Definition (PL homeomorphism)**.** *M_1^2 with triangulation T_1 is* PL homeomorphic *to M_2^2 with triangulation T_2 if and only if there exist subdivisions T_1' and T_2' of T_1 and T_2 respectively such that (M_1^2, T_1') is simplicially isomorphic to (M_2^2, T_2').*

The letters "PL" come from *piecewise linear*, as the correspondence described above gives a homeomorphism between M_1^2 and M_2^2 that can be realized as a map that is linear when restricted to each simplex of T_1'.

For 2-manifolds you may assume without proof that a homeomorphism between two manifolds induces a PL-homeomorphism. This however is not true for general n-manifolds.

2.3 Invariants

One of our goals in studying topological spaces is to be able to distinguish non-homeomorphic spaces from one another. A fundamental strategy to tell the difference between two topological spaces is to find some feature of one space that, on the one hand, is preserved under homeomorphism and, on the other hand, is not shared by the other space. In distinguishing spaces in a general topology course, we might look at topological properties such as being normal, compact, or connected. However, since we are now trying to distinguish among spaces all of which are compact, metric spaces, we need to look for different types of features that are invariant under homeomorphisms. We use the word *invariant* to refer to any property of a space that is shared by any homeomorphic space. That is, it is a property that is preserved by homeomorphisms. So compactness, normality, and connectedness are all invariants. The diameter of a 2-manifold embedded in $\mathbb{R}^3$, on the other hand, is not an invariant.

The crux of the whole course is to define and use invariants that are useful for distinguishing one space from another, especially invariants that can help us distinguish rather nice subsets of $\mathbb{R}^n$ that might be constructed from a finite number of simplices. We begin now by defining an invariant that will help us distinguish one compact, connected, triangulated 2-manifold from some others.

2.3.1 Euler characteristic

Definition (Euler characteristic)**.** *Let M^2 be a 2-manifold with triangulation T. Let*

$$\begin{aligned} v &= \textit{number of vertices in } T \\ e &= \textit{number of 1-simplices in } T \\ f &= \textit{number of 2-simplices in } T \end{aligned}$$

and define the Euler characteristic, *$\chi(M^2)$, of M^2 by $\chi(M^2) = v - e + f$.*

Theorem 2.25. *Let M^2 be a connected, compact, triangulated 2-manifold with triangulation T. Let T' be a subdivision of T. Then $\chi(M^2, T) = \chi(M^2, T')$.*

In other words, for a triangulated, compact 2-manifold, the Euler characteristic is preserved under subdivision.

Theorem 2.26. *Let M_1^2 and M_2^2 be connected, compact, triangulated 2-manifolds. If M_1^2 is PL-homeomorphic to M_2^2, then $\chi(M_1^2) = \chi(M_2^2)$.*

Since PL-homeomorphic manifolds must have the same Euler characteristic, Euler characteristic helps to distinguish between 2-manifolds that are *not* PL-homeomorphic.

Theorem 2.27.

1. $\chi(\mathbb{S}^2) = 2$.

2. $\chi(\mathbb{T}^2) = 0$.

3. $\chi(\mathbb{R}\mathrm{P}^2) = 1$.

4. $\chi(\mathbb{K}^2) = 0$.

Theorem 2.28. *Let M_1^2 and M_2^2 be two connected, compact, triangulated 2-manifolds. Then $\chi(M_1^2 \# M_2^2) = \chi(M_1^2) + \chi(M_2^2) - 2$.*

Theorem 2.29. *Let $\mathbb{T}_i^2$ be the torus for $i = 1, \ldots, n$. Then*

$$\chi\left(\mathop{\#}_{i=1}^{n} \mathbb{T}_i^2\right) = 2 - 2n.$$

Definition (genus)**.** *The* genus *of* $\mathbb{S}^2 = 0$. *The* genus *of a 2-manifold* $\Sigma = \mathop{\#}\limits_{i=1}^{n} \mathbb{T}^2$ *is n.*

Theorem 2.30. *Let $\mathbb{R}\mathrm{P}_i^2$ be the projective plane for $i = 1, \ldots, n$. Then*

$$\chi\left(\mathop{\#}_{i=1}^{n} \mathbb{R}\mathrm{P}^2\right) = 2 - n.$$

2.3.2 Orientability

Euler characteristic is a useful invariant, in that it helps to distinguish 2-manifolds. However, it does not distinguish between the torus and the Klein bottle, for example. In fact, for each *even* number ≤ 0 there are *two* non-homeomorphic compact, connected, triangulated 2-manifolds of that Euler characteristic, one a connected sum of tori, and one a connected sum of projective planes. So although Euler characteristic is useful for distinguishing non-homeomorphic surfaces, it does not differentiate *all* different surfaces.

There is a second invariant which, when combined with Euler characteristic, will allow us to distinguish between any two non-homeomorphic, compact, connected 2-manifolds. This invariant is orientability.

A surface is orientable if we can choose an ordered basis for the local Euclidean structure at each point of the surface in such a way that the bases change smoothly as the point moves along a path in the surface.

Note that orientability on its own is a very coarse invariant: a 2-manifold is either orientable or non-orientable. In other words, orientability divides the set of all 2-manifolds into two classes. It turns out that the combination of orientability and Euler characteristic is enough to differentiate any two compact, connected, triangulated 2-manifolds.

We can explore the concept of orientability in triangulated surfaces by considering orderings of the vertices of each simplex.

First let us see what we mean by an orientation of a 0-, 1-, and 2-simplex.

Definitions (oriented simplices)**.** *Let σ^2 be the 2-simplex $\{v_0v_1v_2\}$, σ^1 be the 1-simplex $\{w_0w_1\}$, and σ^0 be the 0-simplex $\{u_0\}$.*

1. *Two orderings of the vertices $v_0,\ v_1, \ldots v_n$ of an n-simplex σ^n are said to be* equivalent *if they differ by an even permutation. Thus $\{v_0, v_1, v_2\} \sim \{v_1, v_2, v_0\}$. However, $\{v_0, v_1, v_2\} \not\sim \{v_1, v_0, v_2\}$ since they differ by a single 2-cycle, which is an odd permutation. Note that this equivalence relation produces precisely two equivalence classes of orderings of vertices of an n-simplex for $n \geq 1$. An equivalence class will be denoted by $[v_0v_1 \ldots v_n]$, where $\{v_0, v_1, \ldots, v_n\}$ is an element of the equivalence class.*

2. *An* orientation *of the 2-simplex σ^2 is a one-to-one and onto function o from the two equivalence classes of the orderings of the vertices of σ^2 to $\{-1, 1\}$. Note that there are two possible such orientations for σ^2. Any vertex ordering that lies in the equivalence class whose image is $+1$ will be called positively oriented or will be said to have a positive orientation, orderings in the other class will be said to be negatively oriented or have a negative orientation. We can indicate the chosen positive orientation for σ^2 by denoting σ^2 as $[v_0v_1v_2]$, where $[v_0v_1v_2]$ is in the positive equivalence class. Note that $-o[v_0v_1v_2] = o[v_1v_0v_2]$. You can draw a circular arrow inside the 2-simplex in the direction indicated by any of the positively oriented orderings. That circular arrow (which will be either clockwise or counterclockwise on the page) indicates the choice of (positive) orientation for that 2-simplex.*

3. *An* orientation *of the* 1*-simplex* σ^1 *is a one-to-one and onto function* o *from the two orderings of the vertices of* σ^1 *to*$\{-1, 1\}$*. The ordering whose image is* $+1$ *has the positive orientation, the other has a negative orientation. As with* σ^2*, note that there are only two possible orientations for* σ^1*, and that* $-o[w_0w_1] = o[w_1w_0]$*. We think of* $[w_0w_1]$ *as being the orientation that "points" from* w_0 *to* w_1*.*

4. *Since* σ^0 *has a single equivalence class of orderings of its vertex, we have a slightly different definition of orientation for a* 0*-simplex. An* orientation *of a* 0*-simplex is a function* o *from* $\{[u_0]\}$ *to* $\{-1, 1\}$*.*

Definition (induced orientation on an edge)**.** *If we choose an orientation of a* 2*-simplex, then there are associated orientations on each of the three edges called the induced orientations on the edges. If* $[v_0v_1v_2]$ *is the positive orientation of a* 2*-simplex, then the* orientations induced on the edges *are*

a. $[v_1v_2]$.

b. $-[v_0v_2] = [v_2v_0]$.

c. $[v_0v_1]$.

respectively.

Note. *The definition of induced orientation is a natural one, since the induced orientations on the edges give a directed cycle of edges (*v_0 *to* v_1 *to* v_2 *and back to* v_0*) which follow the selected positive ordering of the vertices of the* 2*-simplex.*

Exercise 2.31. *Show that the induced orientation on an edge of a* 2*-simplex is well defined; in other words, that it is independent of the choice of positive equivalence class representative.*

Definition (induced orientation on a vertex)**.** *The* orientations induced on the vertices *of* $\sigma^1 = [v_0v_1]$ *are*

a. $-[v_0]$.

b. $[v_1]$.

respectively.

We can now define what we mean by an orientable, triangulated 2-manifold. Intuitively, a triangulated 2-manifold is orientable if it is possible to select orientations for each 2-simplex in such a way that neighboring 2-simplices have compatible orientations. The concept of 'compatible' comes from the following observation. If you draw two triangles in the plane that share an edge and orient them both in a counterclockwise ordering, say, then the shared edge has induced orientations from the two triangles that are opposite to each other. In other words, when the orientations on both triangles are the same, then the induced orientations on a shared edge are opposite. This observation gives rise to the definition of orientability.

Definition (orientablity)**.** *A triangulated* 2*-manifold* M^2 *is* orientable *if and only if an orientation can be assigned to each* 2*-simplex* τ *in the triangulation such that given any* 1*-simplex* $e \subset \tau_1 \cap \tau_2$*, the orientation induced on* e *by* τ_1 *is opposite to the orientation induced by* τ_2*. Otherwise,* M^2 *is* non-orientable*.*

A choice of orientations of the 2*-simplices of a triangulation of* M^2 *satisfying the condition stated above is called an* orientation *of* M^2*.*

Note.

Theorem 2.32. *Suppose* (M^2, T) *is a* 2*-manifold with triangulation* T *and* T' *is a subdivision of* T*. Then if* (M^2, T) *is orientable, so is* (M^2, T')*.*

Theorem 2.33. *Orientability is preserved under PL homeomorphism.*

Theorem 2.34. M^2 *is orientable if and only if it contains no Möbius band.*

Theorem 2.35. *Let* $M = M_1 \# \ldots \# M_n$*. Then* M *is orientable if and only if* M_i *is orientable for each* $i \in \{1, \ldots, n\}$*.*

Compact, connected, triangulated 2-manifolds are determined by orientability and Euler characteristic.

Theorem 2.36 (Classification of compact, connected 2-manifolds)**.** *If* M^2 *is a connected, compact, triangulated 2-manifold then:*

(a) if $\chi(M^2) = 2$*, then* $M^2 \cong \mathbb{S}^2$*.*

(b) if M^2 *is orientable and* $\chi(M^2) = 2 - 2n$*, for* $n \geq 1$*, then*

$$M^2 \cong \left(\mathop{\#}_{i=1}^{n} T_i^2 \right).$$

(c) if M^2 *is non-orientable and* $\chi(M^2) = 2 - n$, *for* $n \geq 1$, *then*

$$M^2 \cong \left(\mathop{\#}_{i=1}^{n} \mathbb{R}\mathrm{P}_i^2 \right) .$$

Notice that orientable connected, compact, triangulated 2-manifolds must have even Euler Characteristic.

Problem 2.37. *Identify the following* 2*-manifolds as a sphere, or a connected sum of* n *tori (specifying* n*), or a connected sum of* n *projective planes (specifying* n*).*

a. $\mathbb{T}\#\mathbb{R}\mathrm{P}$

b. $\mathbb{K}\#\mathbb{R}\mathrm{P}$

c. $\mathbb{R}\mathrm{P}\#\mathbb{T}\#\mathbb{K}\#\mathbb{R}\mathrm{P}$

d. $\mathbb{K}\#\mathbb{T}\#\mathbb{T}\#\mathbb{R}\mathrm{P}\#\mathbb{K}\#\mathbb{T}$

2.4 CW complexes

Triangulating a surface in order to calculate the Euler Characteristic can be quite tedious and time-consuming. However, we don't need to divide a surface into such small pieces in order to compute the Euler Characteristic. We can instead write the 2-manifold as the union of much larger cells that fit together appropriately and use them to compute the Euler Characteristic. Our strategy for discovering an appropriate generalization of triangulation is to start with a triangulated surface and systematically enlarge the triangles and edges to produce other cell decompositions of the surface that will continue to reveal the Euler Characteristic. In a way, this process of enlargement and amalgamation is the opposite of the subdivision process that we saw earlier preserved the Euler Characteristic. We begin by amalgamating two adjacent 2-simplexes of a triangulation.

Theorem 2.38. *Let* (M^2, T) *be a triangulated* 2*-manifold. Suppose* $\sigma = \{uvw\}$ *and* $\sigma' = \{uvw'\}$ *are two distinct* 2*-simplexes in* T *that share the edge* $e = \{uv\}$*. Then we can create a new structure for* M^2 *alternative to* T*, namely,* P *where* $P = T \cup \{\tau\} - \{\sigma, \sigma', e\}$*, where* $\tau = \sigma \cup \sigma'$ *is the polygon formed by the union of the two* 2*-simplices along their shared edge. If* v'*,* e'*,* f' *are the numbers of vertices, edges, and polygons in* P*, then the Euler Characteristic* $\chi(M^2, T) = v' - e' + f'$ *(see Figure 2.6).*

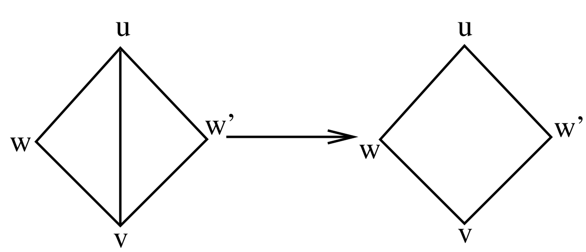


Figure 2.6: The basic idea of CW complexes

The previous theorem amalgamated two triangles together; however, we can continue in that vein by amalgamating polygonal disks together that we may have created.

Theorem 2.39. *Let* (M^2, T) *be compact, triangulated* 2*-manifold with Euler characteristic* $\chi(M^2, T)$*. Suppose we create a polygonal structure* P *on* M^2 *inductively as follows. Let* $P_0 = T$*. Suppose we have created* P_i*. Suppose* *two 2-dimensional objects σ and σ' in P_i share a connected path of edges in the boundary of each from vertex u to w ($v \neq w$). We create P_{i+1} by removing σ and σ' from P_i, removing all the edges in the path from vertex u to w, removing all vertices of the edges in that path except for u and w, and putting in the single two dimensional object $\sigma \cup \sigma'$. Then if v, e, f are the numbers of vertices, edges, and 2-dimensional objects in P_{i+1}, then $\chi(M^2, T) = v - e + f$ (see Figure 2.7).*

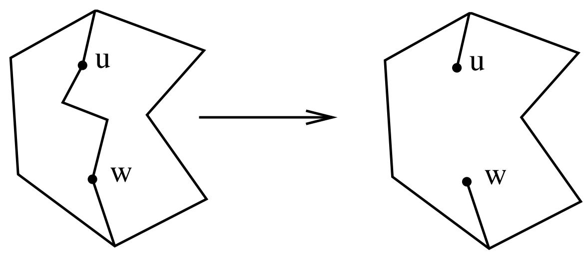


Figure 2.7: Removing a path from a CW complex

Notice that a 2-dimensional object in P_i may no longer be homeomorphic to a disk, but the 'interior' of each is homeomorphic to an open disk. We can continue our inductive definition of our new structure on M^2 by similarly reducing the number of 1-dimensional objects.

Theorem 2.40. *Let (M^2, T) be compact, triangulated 2-manifold with a polygonal structure P as defined inductively in the previous theorem. Suppose we substitute P with a new structure obtained inductively as follows. Let $P = P_0$. If P_i has an edge e with a free vertex v, that is, v is not the boundary of any other edge in P_i, then remove v and e from P_i to create P_{i+1}. If P_i has a vertex v that is one end of an edge e in P_i and one end of an edge f in P_i and v is not on the end of any other edge, then remove v, e, and f from P_i and put in the new 1-dimensional object $e \cup f$ to create P_{i+1}. Then if v', e', f' are the numbers of vertices, 1-dimensional objects, and 2-dimensional objects in an inductively defined P, then $\chi(M^2, T) = v' - e' + f'$.*

Exercise 2.41. *Start with a triangulation of $\mathbb{S}^2$ and carry out the preceding process as far as possible. What "structure" do you get? Confirm that you get the right Euler Characteristic.*

Exercise 2.42. *Start with a triangulation of $\mathbb{T}^2$ and carry out the preceding process as far as possible. What "structure" do you get? Confirm that you get the right Euler characteristic.*

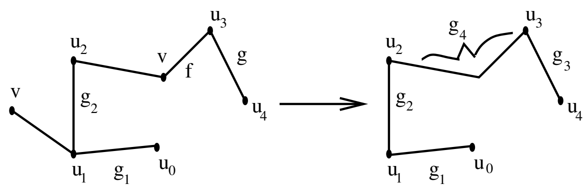


Figure 2.8: Removing edges, vertices, and faces from a CW complex

We will now formalize what we have observed by defining a CW decomposition of a 2-manifold.

Definition (interior of a 0−, 1−, and 2-simplex)**.** *1. For each* 2*-simplex* $\sigma^2 = \{v_0v_1v_2\}$ *let* $\operatorname{Int}\sigma^2 = \{\lambda_0 v_0 + \lambda_1 v_1 + \lambda_0 v_2 | 0 < \lambda_i < 1\}$*.*

2. For each 1*-simplex* $\sigma^1 = \{w_0w_1\}$ *let* $\operatorname{Int}\sigma^1 = \{\lambda_0 w_0 + \lambda_1 w_1 | 0 < \lambda_i < 1\}$*.*

3. For each 0*-simplex* $\sigma^0 = \{u_0\}$ *let* $\operatorname{Int}\sigma^0 = \sigma^0$*.*

Theorem 2.43. *Let* (M^2, T) *be a compact, triangulated* 2*-manifold with triangulation* T*. Then* M^2 *equals the disjoint union of the* $\operatorname{Int}\sigma_i$ *where* $\sigma_i \in T$*.*

Definition (open n-cell from T)**.** *Let* (M^2, T) *be a compact, triangulated* 2*-manifold with triangulation* T*. Suppose* $C = \bigcup\{\operatorname{Int}\sigma_i | \sigma_i \in T\}$ *is homeomorphic to an open* k*-ball (*$k \in \{0, 1, 2\}$*). Then* C *is an* open k-cell from T*.*

Definition (cellular decomposition)**.** *Let* (M^2, T) *be a compact, triangulated* 2*-manifold with triangulation* T*. If* M^2 *is the disjoint union of* C_i^k *(*$k = 0, 1, 2$ *and* $i = 1, \ldots, n_k$*), where each* C_i^k *is an open* k*-cell from* T*, then* $S = \{C_i^k\}$ *is a* cellular decomposition *of* M^2*.*

These decompositions are called *cellular decompositions* or *CW decompositions* because the space can be viewed as constructed from the images of first vertices then 1-cells with their interiors mapped homeomorphically and their boundaries mapped onto 0-cells (points), and then 2-cells with their interiors mapped homeomorphically and their boundaries mapped to the set of images of the lower dimensional cells.

Theorem 2.44. *Let* S *be a cellular decomposition of a compact, triangulated* 2*-manifold* (M^2, T)*. If* v*,* e*, and* f *are the number of* 0, 1 *and* 2 *cells in* S*, then the Euler Characteristic* $\chi(M^2, T) = v - e + f$*.*

Problem 2.45. *Identify the following surfaces:*

a. The surface obtained by identifying the edges of the octagon as indicated:

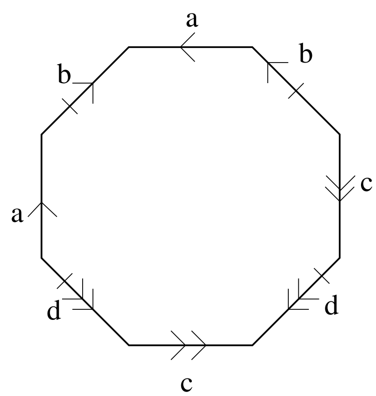


Figure 2.9: The genus two surface

b. The surface obtained by identifying the edges of the decagon as indicated (See Figure 2.10):

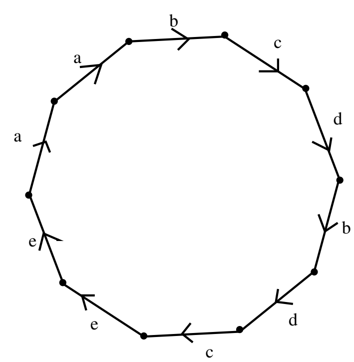


Figure 2.10: The decagon with edges indentified in pairs

2.5 2-manifolds with boundary

Exercise 2.46. *What should be the definition of a connected, compact, triangulated 2-manifold-with-boundary?*

Your definition should be general enough to include the following examples of 2-manifolds-with-boundary:

1. $\mathbb{D}^2$

2. $\mathbb{A}^2 := \{(x_1, x_2) \in \mathbb{R}^2 | \frac{1}{2} \leq x_1^2 + x_2^2 \leq 1\}$, the annulus (See Figure 2.11)

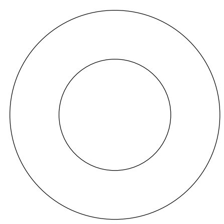


Figure 2.11: The annulus

3. Pair of pants:

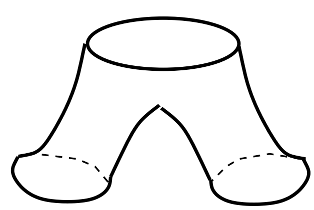


Figure 2.12: The pair of pants

4. A disk with two intertwined handles attached, as shown in Figure 2.13.

5. Möbius band, see Figure 2.14

Exercise 2.47. *Formulate the necessary definitions and theorem statements that classify compact, connected, triangulated 2-manifolds-with-boundary. Prove your theorems.*

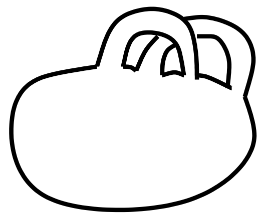


Figure 2.13:

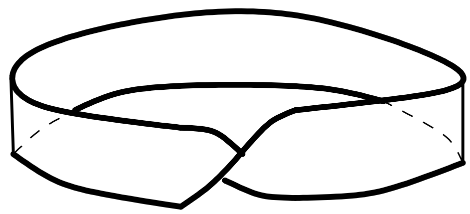


Figure 2.14:

Once you have the above work done, you should be able to completely classify and identify all connected, compact, triangulated 2-manifolds, with and without boundary.

To distinguish a compact manifold with no boundary from one with topological boundary or to emphasize that a compact manifold has no boundary the term "closed manifold" is often used. Beware: this term does not mean topologically closed, as in "the complement of an open set', but rather it means "a manifold that is compact without boundary". Both closed manifolds and compact manifolds-with-boundary are in fact closed subsets (in the topological sense) of $\mathbb{R}^n$ and non-compact manifolds might be embedded as topologically closed subsets of $\mathbb{R}^n$. This unfortunate terminology is one of many examples of the use of a single word to signify several different meanings. Context usually makes the meaning clear.

Problem 2.48. *Identify the following surfaces made by two disks joined by bands as indicated (See Figures 2.15 and 2.16):*

Exercise 2.49. *Fill out the following table, using the connected sum decom-*

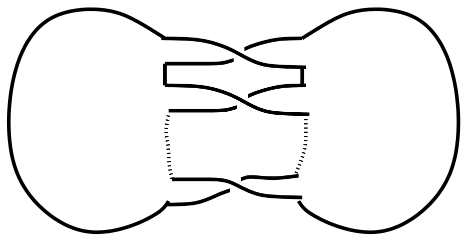


Figure 2.15: a. n twisted bands

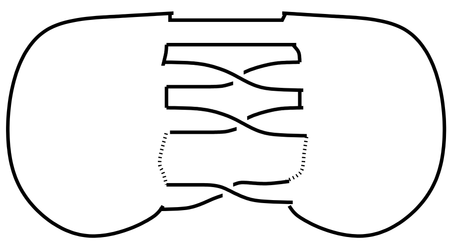


Figure 2.16: b. 1 untwisted band and $n-1$ twisted bands

position. The number of boundary components is denoted by $|\partial|$*.*

$\lvert\partial\rvert$	0		1		2		3	
χ	*orient.*	*non-or.*	*orient.*	*non-or.*	*orient.*	*non-or.*	*orient.*	*non-or.*
2								
1								
0								
-1								
-2								
-3								
-4								
-5								

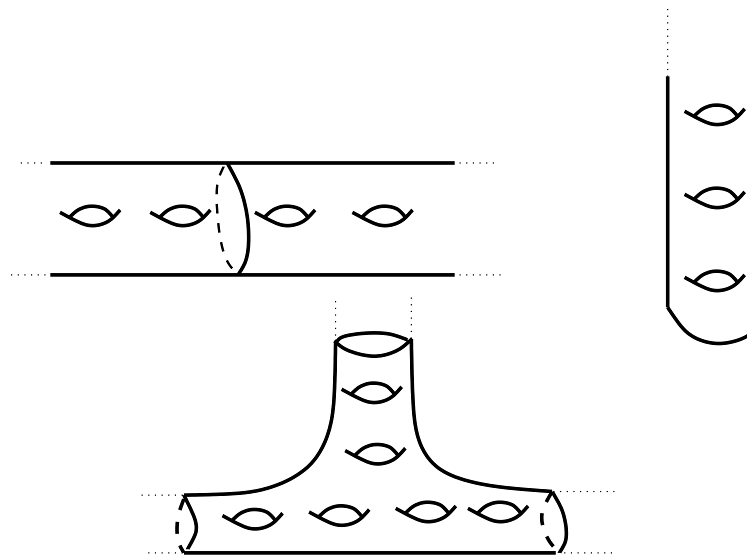


Figure 2.17: Examples of non-compact surfaces with infinite genus

2.6 *Non-compact surfaces

The surfaces studied so far in this chapter are all compact and connected 2-manifolds, with or without boundary. We can also consider non-compact 2-manifolds, but we will not do so in this class. An interesting question to ask yourself is: how do you extend all the concepts learned about compact spaces to non-compact ones?

For example, can you formulate and prove a classification theorem for non-compact, connected, triangulated 2-manifolds? One of the difficulties that arises in the non-compact case is that we no longer have a finite set of simplices in the triangulation.

The following exercises illustrate what some of the complications of classifying non-compact 2-manifolds may be, even when we restrict to the orientable case:

Exercise 2.50. *Below are some non-compact* 2*-manifolds. Are any of these spaces are homeomorphic? (Beware! It may be harder than you think!) Can you prove whether they are or are not homeomorphic? (See Figure 2.17)*

Exercise 2.51. *Let* M *be the non-compact* 2*-manifold made by taking the two parallel planes* $\{(x, y, 1)|x, y \in \mathbb{R}\}$ *and* $\{(x, y, 0)|x, y \in \mathbb{R}\}$*, removing* *disks* $\{(x, y, 1)|(x-a)^2 + (y-b)^{<}\frac{1}{4}, a, b \in \mathbb{N}\}$ *and* $\{(x, y, 0)|(x-a)^2 + (y-b)^{<}\frac{1}{4}, a, b \in \mathbb{N}\}$*, and finally gluing annuli* $\{(x, y, z)|(x-a)^2+(y-b)^{=}\frac{1}{4}, a, b \in \mathbb{N}, 0 \leq z \leq 1\}$*. Is this space homeomorphic to any of the examples shown above?*

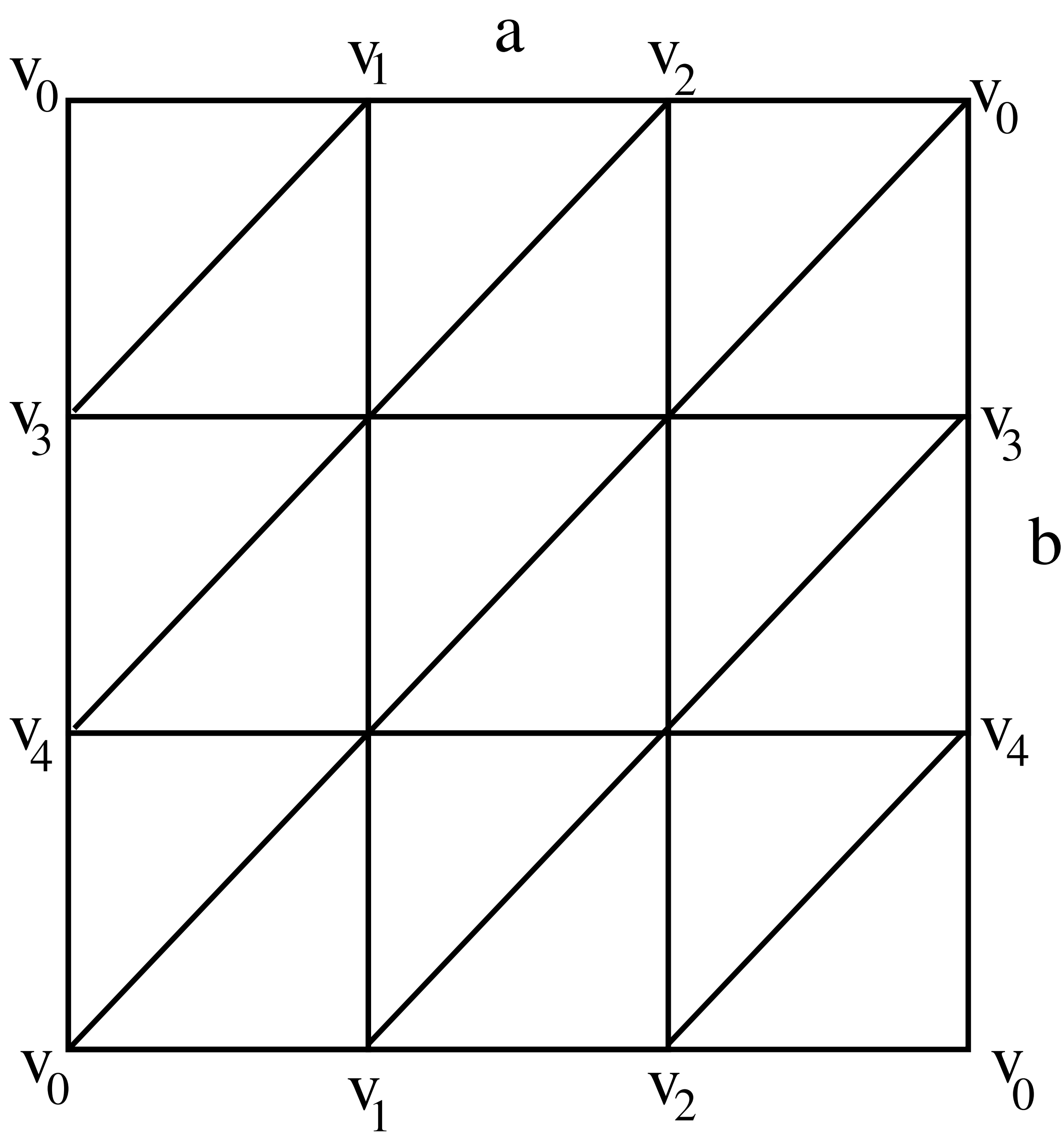


Figure 2.18: The torus triangulated

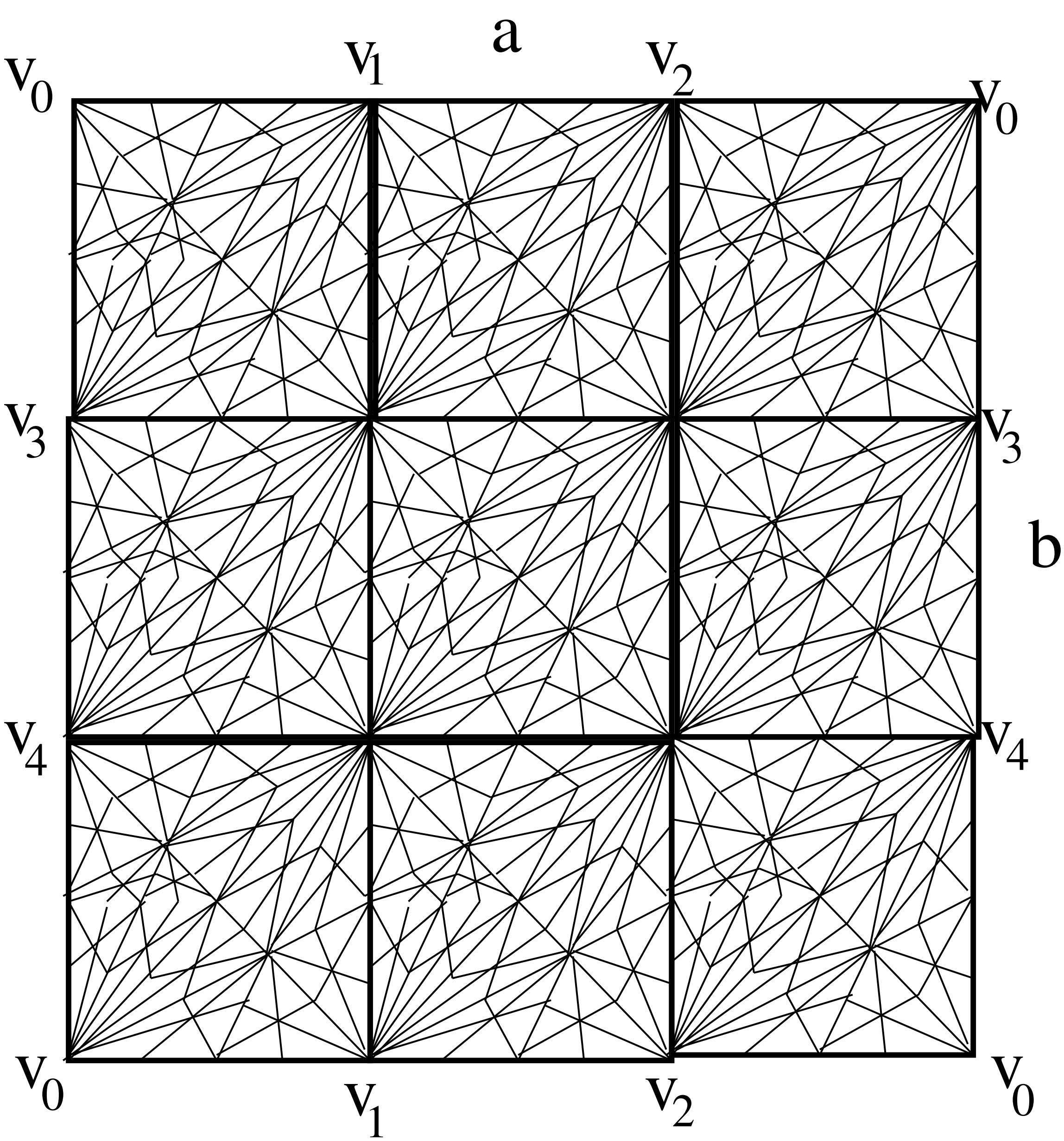


Figure 2.19: The torus triangulated by its second baricentric subdivision

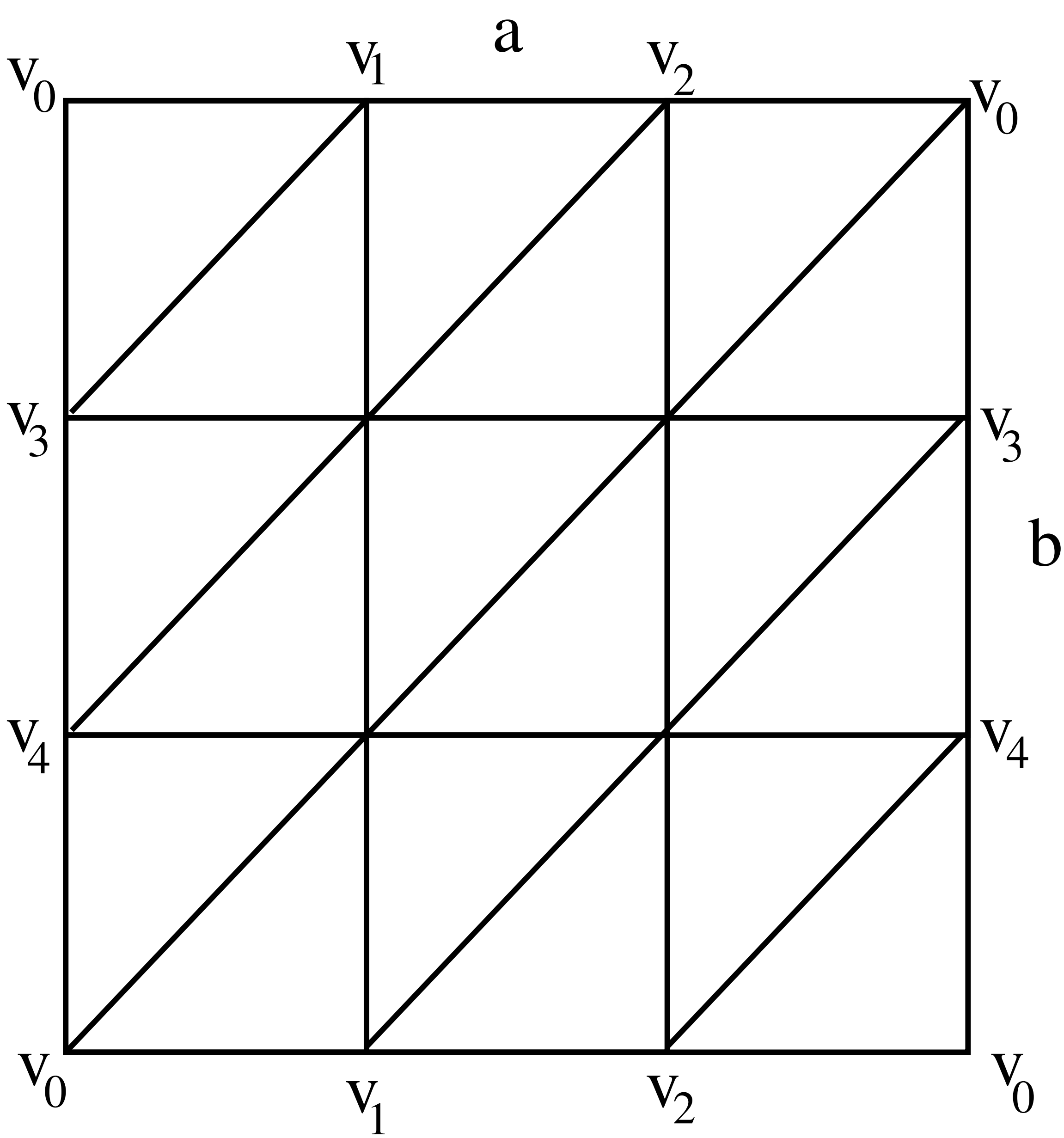


Figure 2.20: The Klein bottle triangulated

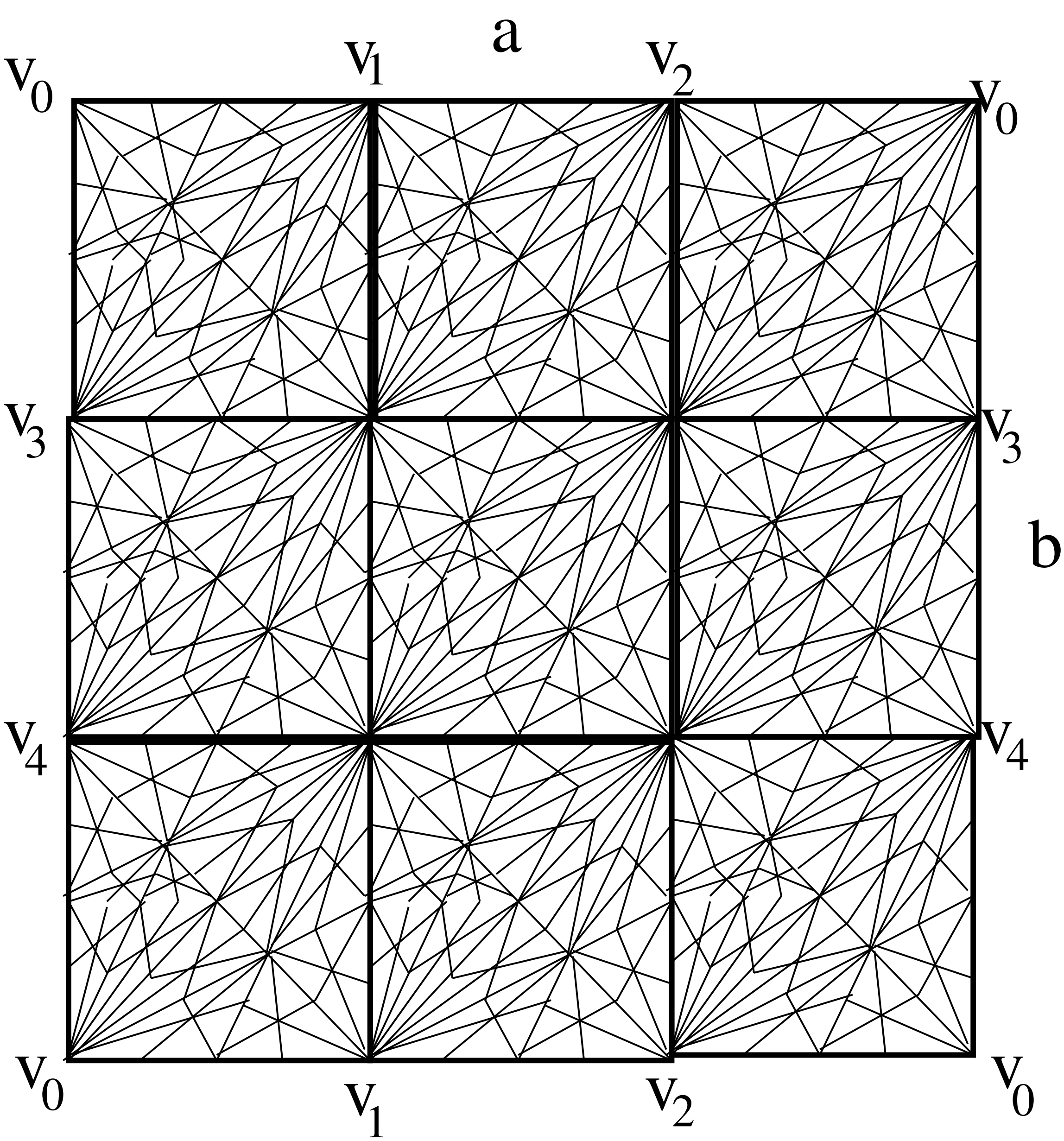


Figure 2.21: The Klein bottle triangled by its second baricentric subdivision

Chapter 3

Fundamental group and covering spaces

We do not need to calculate the Euler characteristic of a torus and a sphere to intuit that they are not homeomorphic. The difference between them (or between them and the surface of a two-holed doughnut) is associated with something we describe as a "holes". We need to make this concept precise. What do we mean by a "hole"?

There are two ways to make this concept definite, both developed by Henri Poincarè at the turn of the 20$^{\text{th}}$ century. The first of these methods to capture the intuitive idea of holes in a space is called the *fundamental group* of a space. Unlike the Euler characteristic, which is a numerical invariant, and orientation, which is a parity (+ or −) invariant, the fundamental group is an algebraic group associated to the space. One would expect this more complex invariant to carry more information about the space, and indeed it often does.

Intuitively, the basic goal of the fundamental group is to recognize holes in a space. If we think about a racetrack, we will have a good idea of how the fundamental group works. From a starting point, a car measures its progress by counting laps. Going once around is different from going twice around. The Indy 500 involves going many times around the track. Going backward around the track is frowned upon in competitive races, but if a car did that, we would know how to count such a feat using negative numbers. Counting number of times around the track is the most basic feature of the fundamental group. But another feature of the racetrack also suggests a basic idea of the fundamental group, namely, when a car has completed a lap, the exact path of the car is not important as long as it stays on the

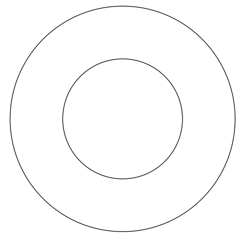


Figure 3.1: The Annulus

track. To make the analogy exact, we will insist that the car does return to the exact point where it started.

Let's become a little more mathematical in our description of the fundamental group of the racetrack. A racetrack is known mathematically as an annulus, which we could describe as $\mathbb{A}^2 = \{(x, y) \in \mathbb{R}^2 | 1/2 \leq x^2 + y^2 \leq 1\}$ (See Figure 3.1).

We choose a point x_0 on the annulus to be the base point. Then consider any continuous function f from a simple closed curve ($\cong \mathbb{S}^1$) into the annulus. We will choose a point on $\mathbb{S}^1$ to start, call it z, and require that $f(z) = x_0$. Intuitively, that map f of $\mathbb{S}^1$ into the annulus 'goes around' the annulus some number of times. Physically, we can think of f as laying a rubber band around the annulus with the point z placed on top of x_0. If we think about the intuitive concept of how many times the map goes around the annulus, we soon see that, in the rubber band model, if we could distort the rubber band to a new position without lifting it out of the annulus, then it would still go around the same number of times. Sliding around and distorting a map are really giving a continuous family of maps of $\mathbb{S}^1$ into the annulus. Such a continuous family of maps is called a homotopy. The fundamental group just makes mathematically precise the idea of putting maps of a circle into a space into equivalence classes via the idea of homotopy. To get to the idea of fundamental group, we'll first develop the idea of a homotopy between maps from any space to any other space and then specialize that idea to define the fundamental group.

3.1 Fundamental group

We all have a familiarity with the idea of a homotopy, because we have all watched movies. Let's think of the physical objects that are being projected onto the screen as the domain and think of the screen as the plane $\mathbb{R}^2$. So, our domain might be two people, a dog, and a table. We will assume that all of these objects are projected at every moment of our movie and that we are thinking of an idealized projection system that projects that image at every instant of time. So, this movie has uncountably many frames per second. We are also thinking of the camera as being fixed throughout. When the scene opens at time 0, the people, dog, and table are shown posed in some no doubt interesting tableau. Every point in each person, dog, and table is mapped to a point on the screen. Notice that this map is not 1–1. Points on the inside of each object certainly are mapped to places where other points map as well. At time 0, the action commences. The people move about, gesticulating animatedly. The dog barks and wags its tail. The table just sits there. At every instant of the film, each point in the people, dog, and table are mapped via projection to a point in $\mathbb{R}^2$, also known as the silver screen. At time 1, the movie is over. It ends with the points of the people, dog, and table being mapped to points on the screen. The first scene of the move, which was a function from the domain (people, dog, table) into the range ($\mathbb{R}^2$), was transformed via a continuous family of maps (the scenes at each moment) into the last scene of the movie, which was another function from the domain (people, dog, table) into the range ($\mathbb{R}^2$). The beginning scene and final scene of our movie illustrate the idea of homotopic maps, which we now define formally.

Definition (homotopic maps)**.** *Let $f,\ g : X \to Y$ be two continuous functions. f and g are said to be* homotopic *if there is a continuous map $F : X \times [0,1] \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. We denote that maps f and g are homotopic by writing $f \simeq g$. The map F is called a* homotopy *between f and g.*

If g is a constant map (mapping all points in X to a single point in Y) and $f \simeq g$, then we say f is null homotopic*.*

Theorem 3.1. *Given topological spaces X and Y, $\simeq$ is an equivalence relation on the set of all continuous functions from X to Y.*

We can think of a homotopy of two maps as a continuous 1-parameter family of maps F_t from X to Y "deforming" f into g (*i.e.*, $F_0 = f$ and $F_1 = g$).

Notice that in the gripping movie that we described earlier, the table remained fixed throughout. So at each moment of the film, the function when restricted to the table was the same for the whole duration of the movie. That consistency on a subset of the domain gives rise to the idea of a relative homotopy.

Definition (relative homotopy)**.** *Given topological spaces X and Y with $S \subset X$, then two continuous functions f, $g : X \to Y$ are* homotopic relative to S *if and only if there is a continuous function $H : X \times [0,1] \to Y$ such that*

$$\begin{array}{rclcll} H(x,0) & = & f(x) & & & \forall x \in X \\ H(x,1) & = & g(x) & & & \forall x \in X \\ H(x,t) & = & f(x) & = & g(x) & \forall\ x \in S \ \&\ t \in [0,1] \end{array}$$

In other words, H is a 1-parameter family of maps $H_t : X \to Y$ ($t \in [0,1]$) which continuously deforms f into g while keeping the images of points in the set S fixed.

Theorem 3.2. *Given topological spaces X and Y with $S \subset X$, being homotopic relative to S is an equivalence relation on the set of all continuous functions from X to Y.*

One of the motivations for studying homotopic maps is to capture the idea of holes in a space. Our method of recognizing a hole is to think about going around the hole. So this focus on marching around holes gives special interest to the paths that we follow in going from place to place in the space and the loops that may go around holes. These paths can be thought of as maps from the interval into a space, so such maps are given a special name.

Definition (paths and loops)**.** *A continuous function $\alpha : [0,1] \to X$ is a* path*. If $\alpha(0) = \alpha(1) = x_0$, then α is a* loop *(or* closed path*)* based at *x_0.*

Notice that a path is a function rather than a subset of the topological space in which the image of the path sits. Two paths from the same starting point to the same ending point are equivalent if, keeping the end points fixed at all times, we can find a continuous family of paths that 'morph' one path into the other. The morphing is formalized as a homotopy relative to the endpoints of $[0,1]$.

Definition (path equivalence)**.** *Two paths α, β are* equivalent*, denoted $\alpha \sim \beta$, if and only if α and β are homotopic relative to $\{0,1\}$. Denote the equivalence class of paths equivalent to α by $[\alpha]$ (See Figure 3.2).*

The concept of path equivalence applies to loops as well—since loops are paths. We will actually be most concerned with equivalences of loops, so we give them special attention. Technically, a loop is a path, that is, a function from $[0,1]$, whose endpoints are mapped to the same place, but intuitively, a loop is a map from $\mathbb{S}^1$ into the space. That intuition is formalized using the following wrapping map.

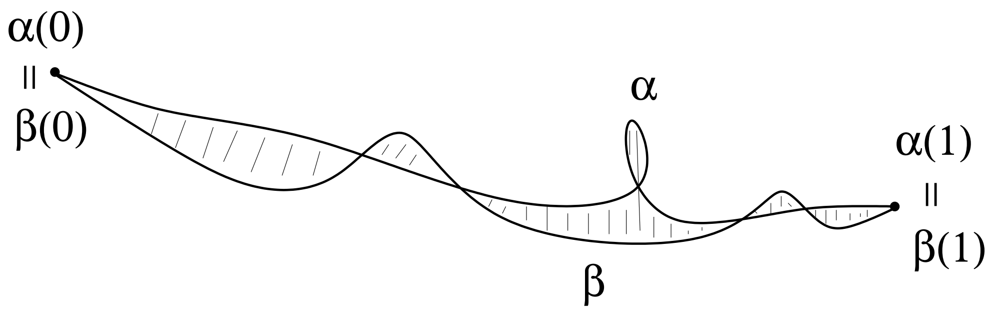


Figure 3.2: Path equivalence

Definition (standard wrapping map)**.** *The map* $\omega : \mathbb{R}^1 \to \mathbb{S}^1 \subset \mathbb{R}^2$ *defined by* $t \mapsto (\cos 2\pi t, \sin 2\pi t)$ *is called the* standard wrapping map *of* $\mathbb{R}^1$ *to* $\mathbb{S}^1$.

Theorem 3.3. *Let* α *be a loop into the topological space* X. *Then* $\alpha = \beta \circ \omega$ *where* ω *is the standard wrapping map and* β *is a continuous function from* $\mathbb{S}^1$ *into* X.

The above theorem allows us to think of a loop as a map from a circle when it is useful for us to do so. This description also allows us to state a useful characterization of triviality of a loop.

Definition (homotopically trivial loop)**.** *Let* X *be a topological space. A loop* α *is* homotopically trivial *or is a* trivial loop *if* α *is equivalent to the* constant path $e_{\alpha(0)}$ *where* $e_{\alpha(0)}$ *takes* $[0, 1]$ *to* $\alpha(0)$.

Theorem 3.4. *Let* X *be a topological space and let* p *be a point in* X. *Then a loop* $\alpha = \beta \circ \omega$ *(where* ω *is the standard wrapping map and* β *is a continuous function from* $\mathbb{S}^1$ *into* X*) is homotopically trivial if and only if* β *can be extended to a continuous function from* $\mathbb{B}^2$ *into* X.

We return now to the exploration of paths. The physical idea of walking from point a to point b and then proceeding from there to point c yields the natural idea of how to combine paths.

Definition (path product)**.** *Let* α, β *be paths with* $\alpha(1) = \beta(0)$. *Then their* product, *denoted* $\alpha \cdot \beta$, *is the path that first moves along* α, *followed by moving along* β. $\alpha \cdot \beta$ *is defined by:*

$$\alpha \cdot \beta(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} < t \leq 1 \end{cases}$$

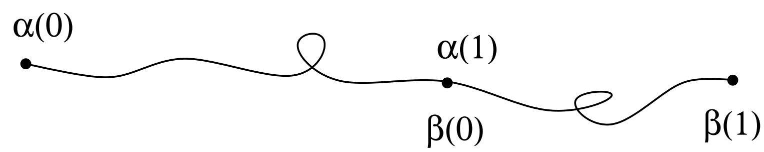


Figure 3.3: Path Product

Notice the need to speed up in order to accomplish both the paths α and β during the prescribed 1 unit of time allotted for a path.

Theorem 3.5. *If $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $\beta \cdot \alpha \sim \beta' \cdot \alpha'$.*

Thus products of paths can be extended to products of equivalence classes by defining $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$. Products of paths and products of equivalence classes of paths enjoy the associative property.

Theorem 3.6. *Given α, β, and γ, then $(\alpha \cdot \beta) \cdot \gamma \sim \alpha \cdot (\beta \cdot \gamma)$ and $([\alpha] \cdot [\beta]) \cdot [\gamma] \sim [\alpha] \cdot ([\beta] \cdot [\gamma])$.*

If we think of a path α as taking us from $\alpha(0)$ to $\alpha(1)$, then traversing that same trail in reverse is the inverse path.

Definition (path inverse)**.** *Let α be a path, then its* path inverse *α^{-1} is the path defined by $\alpha^{-1}(t) = \alpha(1-t)$.*

If we take a path and then take its inverse, that combined path is equivalent to not moving at all.

Theorem 3.7. *Let α be a path with $\alpha(0) = x_0$, then $\alpha \cdot \alpha^{-1} \sim e_{x_0}$, where e_{x_0} is the constant path $e_{x_0} : [0,1] \to x_0$. Stated differently, if α is a path, then $\alpha \cdot \alpha^{-1}$ is homotopically trivial.*

We now have all the ingredients to associate a group with a topological space. This group has been designed to try to capture the idea of holes in the space.

Definition (fundamental group)**.** *Let $x_0 \in X$, a topological space. Then the set of equivalence classes of loops based at x_0 with binary operation $[\alpha][\beta] = [\alpha \cdot \beta]$ is a called the* fundamental group of X based at x_0 *and is denoted $\pi_1(X, x_0)$. The point x_0 is called the* base point *of the fundamental group.*

Theorem 3.8. *The fundamental group $\pi_1(X, x_0)$ is a group. The identity element is the class of homotopically trivial loops based at x_0.*

The fundamental group is defined for a space X with a specified base point selected. However, for many spaces the choice of base point is not significant, because the fundamental group computed using one base point is isomorphic to the fundamental group using any other point. In particular, path connected spaces enjoy this independence of base points.

Theorem 3.9. *If X is path connected, then $\pi_1(X, p) \cong \pi_1(X, q)$ for any points $p, q \in X$.*

Since in path connected spaces the fundamental group is independent of the base point (up to isomorphism), for such spaces X we sometimes just write $\pi_1(X)$ for the fundamental group without specifying the base point.

(Note. A corollary is a theorem whose truth is an immediate consequence of the statement of a preceding theorem. A scholium is a theorem whose truth is an immediate consequence of the proof of a preceding theorem, but does not follow immediately from the statement of the preceding theorem.)

Scholium. Suppose X is a topological space and p, $q \in X$ lie in the same path component. Then $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$.

We have now defined the fundamental group of a space, so let's find the fundamental groups of some spaces. We begin with several example of spaces that have trivial fundamental groups.

Exercise 3.10. *We will use* 1 *to denote the trivial group:*

1. $\pi_1([0, 1]) \cong 1$.
2. $\pi_1(\mathbb{S}^0, 1) \cong 1$ *where* $\mathbb{S}^0$ *is the zero-dimensional sphere* $\{-1, 1\} \subset \mathbb{R}^1$.
3. $\pi_1($*convex set*$) \cong 1$.
4. $\pi_1($*cone*$) \cong 1$.
5. π_1*(cone over Hawaiian earring)* $\cong 1$.
6. $\pi_1(\mathbb{R}^n) \cong 1$ *for* n *for* $n \geq 1$.
7. $\pi_1(\mathbb{S}^2) \cong 1$.

A space whose fundamental group is trivial is called 'simply connected.'

Definition (simply connected)**.** *A path-connected topological space with trivial fundamental group is said to be* simply connected *or* 1-connected.

Of course, the fundamental group would not serve a useful purpose if all spaces were simply connected. The first example we will consider of a space with non-trivial fundamental group is the circle. The following theorem will require some significant work to prove.

Theorem 3.11. *The fundamental group of the circle $\mathbb{S}^1$ is infinite cyclic, that is, $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.*

3.1.1 Cartesian products

To add to the spaces whose fundamental groups we can compute, let us now look at the Cartesian products of spaces and observe that the fundamental group of a product of topological spaces is just the product of the fundamental groups of the factors .

Theorem 3.12. *Let (X, x_0), (Y, y_0) be path connected spaces. Then $\pi_1(X \times Y,\ (x_0, y_0)) \cong \pi_1(X, x_0) \otimes \pi_1(Y, y_0)$.*

Exercise 3.13. *Find:*

1. $\pi_1(\mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1)$;
2. $\pi_1(\mathbb{D}^2 \times \mathbb{S}^1)$;
3. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^1)$;
4. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^2)$;
5. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2)$;
6. $\pi_1(\mathbb{S}^{p_1} \times \ldots \times \mathbb{S}^{p_k})$ *where* $p_j \geq 2$ *for* $1 \leq j \leq k$.

The last 3 sections of the previous exercise present us with many examples of simply-connected spaces. We will see later that Cartesian products of different collections of spheres yield topologically different spaces; however, at this point in the course, it is not obvious how we are going to detect the differences in these spaces. These examples raise the question of what additional ideas beyond the fundamental group we will need to show that locally homeomorphic spaces aren't actually homeomorphic. For now, we leave this tantalizing question, but we will return to it in a later chapter. For the moment we will be content with having established the fundamental groups of quite a few spaces.

3.1.2 Induced homomorphisms

One of the standard techniques of mathematics is to explore the question of how structure on one mathematical object is transported to another mathematical object via a map. We have now defined the fundamental group for a space. Topological spaces are mapped to one another via continuous functions. So we can ask how the fundamental group of one space is carried to a target space via a continuous function.

Definition (induced homomorphism)**.** *Let $f : X \to Y$ be a continuous function. Then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ defined by $f_*([\alpha]) = [f \circ \alpha]$ is called the* induced homomorphism on fundamental groups.

Definition (well-defined function)**.** *Let $f : X \to Y$ and suppose that X has a partition X^* of equivalence classes with equivalence relation $\equiv$. Then $f_* : X^* \to Y$ given by $f_*([x]) = f(x)$ is a* well-defined function *if and only if $f(x_0) = f(x_1)$ for all $x_0 \equiv x_1$.*

In other words a function defined on a set of equivalence classes is well-defined if its image is independent of the choice of representative of the equivalence class.

Exercise 3.14. *Check that for a continuous function $f : X \to Y$, the induced homomorphism on the fundamental group f_* is well-defined.*

Theorem 3.15. *If $g : (X, x_0) \to (Y, y_0)$, $f : (Y, y_0) \to (Z, z_0)$ are continuous functions, then $(f \circ g)_* = f_* \circ g_*$.*

Theorem 3.16. *If f, $g : (X, x_0) \to (Y, y_0)$ are continuous functions and f is homotopic to g relative to x_0, then $f_* := g_*$.*

Theorem 3.17. *If $h : X \to Y$ is a homeomorphism then $h_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a group isomorphism. So homeomorphic spaces have naturally isomorphic fundamental groups.*

The above theorem tells us that the fundamental group of a path connected space is a *topological invariant*, and hence we can establish that two path connected spaces are not homeomorphic if we can show that they have different (non-isomorphic) fundamental groups. Thus, the fundamental group helps to distinguish among spaces, but it does not always detect the differences between spaces, as we have already seen among our examples of spaces with trivial fundamental groups.

The correspondence between a topological space and an algebraic group is extremely useful, because we can use algebra to answer topological questions (*e.g.*, are two spaces not homeomorphic?) and, as we shall see later, we can use topology to answer algebraic questions.

3.2 3-manifolds

In the preceding chapter and in this chapter we have studied 2-manifolds in some detail. In point of fact, we only need Euler characteristic and orientability to completely classify compact 2-manifolds. But what happens when we look at higher-dimensional manifolds? There, χ is no longer a useful invariant (orientability is always of interest). For any closed, compact, oriented 3-manifold M^3, $\chi(M^3) = 0$ (it's zero for any odd-dimensional closed, compact, oriented manifold). 3-manifolds are not as well understood as 2-manifolds, but the fundamental group can be a useful invariant in the study of 3-manifolds.

Let us look at two specific examples: lens spaces, which are compact 3-manifolds with no boundary, and knot exteriors, which are compact 3-manifolds with boundary.

3.2.1 Lens spaces

Every compact, connected 3-manifold can be constructed by taking familiar 3-manifolds with boundaries (such as solid tori or solid double tori, or, in general, solid n-tori) and gluing a pair of them together along their boundaries. The 3-manifolds that can be obtained by gluing a pair of solid tori together along their boundaries are called lens spaces. These relatively simple 3-manifolds are completely classified.

Let p and q be relatively prime natural numbers. A (p, q)-lens space, denoted $L(p, q)$, indexed by $p/q \in \mathbb{Q}$, can be defined in several different ways. $L(p, q)$ was first defined as an identification space that started with a 3-ball drawn in the shape of a lens (hence the name). The top and bottom hemispheres of this lens are each divided into p triangle-like wedges. Each triangle from the top hemisphere is identified with a triangle in the bottom hemisphere that is a certain specified number (relatively prime to p) of triangles around the equator. The resulting quotient space is a lens space (See Figure 3.4).

Definition (isotopy)**.** *A homotopy $H_t : X \to Y$ $(t \in I)$ is an* isotopy *if and only if for every t in I, H_t is an embedding.*

Exercise 3.18. *Show that isotopies form an equivalence relation on the set of all embeddings of X into Y.*

Definition (meridian)**.** *Let $V \cong \mathbb{D}^2 \times \mathbb{S}^1$ be a solid torus. A simple closed curve J on* $\mathrm{Bd}\, V$ *is a* meridian *if and only if it bounds a disk in V (See Figure 3.5).*

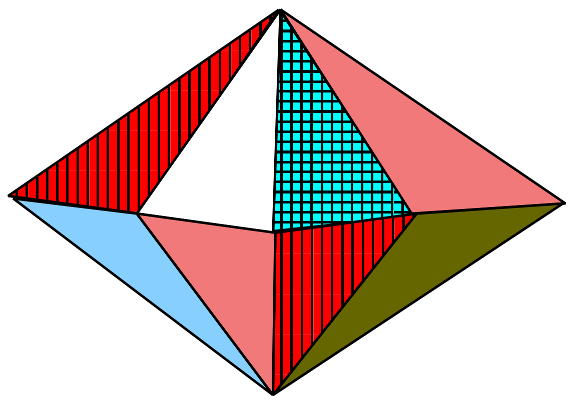


Figure 3.4: Lens space as a quotient of a lens

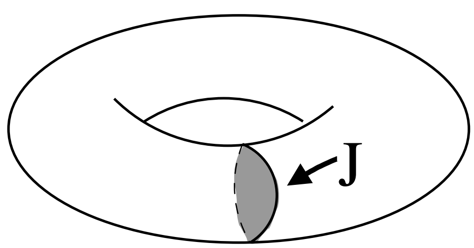


Figure 3.5: Solid torus with meridian

Definition (longitude)**.** *Let $V \cong \mathbb{D}^2 \times \mathbb{S}^1$. A simple closed curve K on* $\mathrm{Bd}\, V$ *is a* longitude *or* longitudinal curve *if and only if K represents the generator of $\pi_1(V)$, that is, K goes once around V. A longitude can be isotoped to intersect a meridianal curve once.*

The meridian of a solid torus is unique up to isotopy, but the longitude is not, since a longitude can spiral around the torus a number of full turns as it goes around. A choice of longitude is called a *framing*.

Lemma 3.19. *Two simple closed curves α and β in $\mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$ are homotopic if and only if they are isotopic.*

Lemma 3.20. *Given a meridian μ and longitude λ, $\{[\mu], [\lambda]\}$ forms a basis for $\pi_1(\mathrm{Bd}(\mathbb{D}^2 \times \mathbb{S}^1))$.*

Lemma 3.21. *Let $\{[\mu], [\lambda]\}$ be a basis for $\pi_1(\mathrm{Bd}(\mathbb{D}^2 \times \mathbb{S}^1)) \cong \mathbb{Z} \times \mathbb{Z}$. Then $q[\mu] + p[\lambda]$ has a simple closed curve representative if and only if p and q are relatively prime.*

We can therefore use $q[\mu] + p[\lambda]$ (where p and q are relatively prime) to mean a simple closed curve representative of that class. Two different representatives that are simple will be isotopic.

Definition (lens space)**.** *Let V_1 and V_2 be two solid tori with meridians μ_i and chosen longitudes λ_i respectively. Let $h : \mathrm{Bd}(V_1) \to \mathrm{Bd}(V_2)$ be a homeomorphism such that $h(\mu_1)$ goes to a curve in the isotopy class $q\mu_2 + p\lambda_2$. Then the quotient space $V_1 \cup_h V_2$ is the (p, q)-*lens space *$L(p, q)$.*

Lens spaces can also be defined other ways, for example, as quotient spaces of $\mathbb{S}^3$ under certain group actions. Beware that some authors use $L(p, q)$ to mean $L(q, p)$.

Exercise 3.22. *Use Van Kampen's Theorem to explicitly calculate a group presentation of $\pi_1(L(p, q))$.*

3.2.2 Knots in $\mathbb{S}^3$

Definition (knot)**.** *Let $i : \mathbb{D}^2 \times \mathbb{S}^1 \to \mathbb{S}^3$ be a PL embedding, that is, an injective continuous function that is linear on each simplex of a triangulation of $\mathbb{D}^2 \times \mathbb{S}^1$. Then $K = i(\mathbf{0} \times \mathbb{S}^1) \cong \mathbb{S}^1$ is a* knot *in $\mathbb{S}^3$, and $N(K) = (\mathrm{Int}(D^2 \times \mathbb{S}^1))$ is an open regular neighborhood of K. The* knot complement *or* knot exterior *of K is $M_K = \mathbb{S}^3 - N(K)$.*

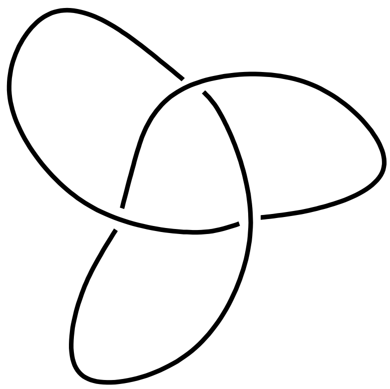


Figure 3.6: The trefoil knot

If we look at a knot from above, we see a curve with crossings where it goes over or under itself. For example, this picture is a picture of a trefoil knot (See Figure 3.6):

If we are given a picture of a projection of a knot K into $\mathbb{R}^2$ where gaps indicate where undercrossings occur and where all crossings are transverse crossings of two arcs, then we can use the pictures, along Van Kampen's Theorem to produce a presentation of $\pi_1(M_K)$. Roughly speaking, each arc on the picture gives a generator and each crossing represents a relation.

For each arc in a knot projection, draw a labeled perpendicular arrow as shown (See Figure 3.7):

The arrow a_i, for example, represents the loop in M_K obtained by starting well above the knot (at the base point chosen for $\pi_1(M_K)$), going straight down to the tail of a_i, then going along a_i under the knot, and finally returning to the starting point going straight from the head of a_i.

Lemma 3.23. *Every loop in M_K is homotopic in M_K to a product of a_i's. In other words, the loops $\{a_i\}$ generate $\pi_1(M_K)$.*

Lemma 3.24. *At every crossing, such as that illustrated in Figure 3.8, the following relation holds: $acb^{-1} = c$ or $acb^{-1}c^{-1} = 1$.*

Theorem 3.25. *Let K be a knot in $\mathbb{S}^3$ and let $\{a_i\}$ be the set of loops consisting of one loop for each arc in a knot projection of K as described above. Then $\pi_1(M_K) = \{a_1, a_2, \ldots, a_n | a_i a_j a_k^{-1} a_j^{-1}$ where there is one relation of the form $a_i a_j a_k^{-1} a_j^{-1}$ for each crossing in the knot projection$\}$.*

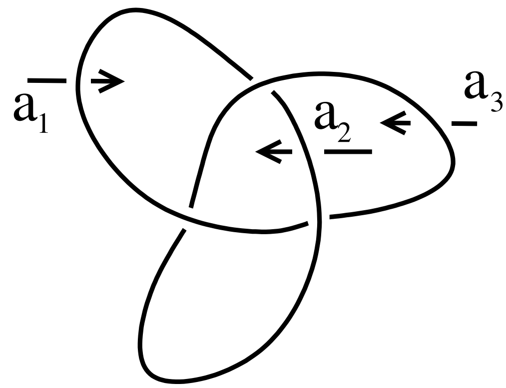


Figure 3.7: The arrows for the arcs of a trefoil knot

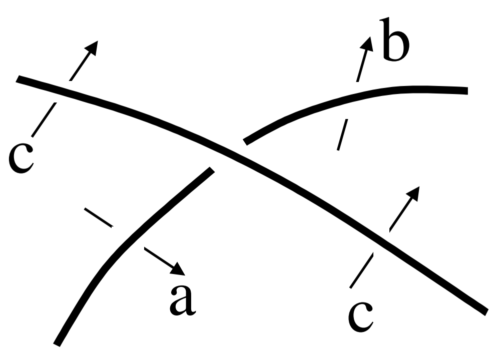


Figure 3.8: The arrows around a crossing

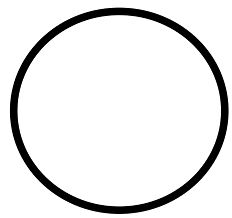


Figure 3.9: The unknot

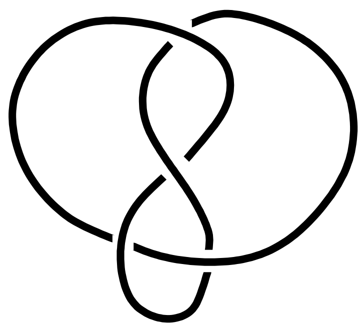


Figure 3.10: The figure-8 knot

Exercise 3.26. *Find the fundamental group of the complement of the unknot (See Figure 3.9).*

Exercise 3.27. *Find the fundamental group of the complement of the trefoil knot.*

Exercise 3.28. *Find the fundamental group of the complement of the figure-8 knot (See Figure 3.10).*

3.3 Homotopy equivalence of spaces

Definition (homotopy equivalence)**.** *Two spaces X and Y are said to be* homotopy equivalent *or to have the same* homotopy type *if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq i_X$ and $f \circ g \simeq i_Y$, where i_X denotes the identity on X and i_Y denotes the identity on Y. We write $X \sim Y$ to mean that X and Y are homotopy equivalent.*

Theorem 3.29. *If X is a strong deformation retract of Y, then X and Y are homotopy equivalent.*

Theorem 3.30. *If $X \sim Y$ then $\pi_1(X) \cong \pi_1(Y)$.*

In other words π_1 doesn't distinguish spaces that have the same homotopy type.

3.4 Higher homotopy groups

The fundamental group of a space X is the set of homotopy equivalence classes of maps of $\mathbb{S}^1$ into X with certain constraints. This idea can be generalized to maps of $\mathbb{S}^n$ into X, giving the *higher homotopy groups*.

Recall that a loop can be thought of as either a map α from $[0,1] = \mathbb{D}^1$ to X where $\alpha(\operatorname{Bd}\mathbb{D}^1) = x_0$ for some point $x_0 \in X$, or as a map $\mathbb{S}^1 \to X$, and in fact we used the two ways interchangeably. We will use the second way of looking at the higher-dimensional analogues of paths:

Definition (product of homotopy classes)**.** *Let X be a topological space and $\mathbf{x_0} \in X$. Let f, $g : (\mathbb{D}^n, \partial\mathbb{D}^n) \to (X, \mathbf{x_0})$, that is f and g are continuous maps that take $\partial\mathbb{D}^n \mapsto \mathbf{x_0}$. Let $[f]$ and $[g]$ denote the respective homotopy classes of these maps rel $\partial\mathbb{D}^n$. Then we define $[f]\cdot[g]$ to be the homotopy class of:*

$$f \cdot g(x_1, x_2, \ldots, x_n) = \begin{cases} \alpha(2x_1, x_2, \ldots, x_n), & 0 \le t \le \frac{1}{2} \\ \beta(2x_1 - 1, x_2, \ldots, x_n), & \frac{1}{2} < t \le 1 \end{cases}$$

where $(x_1, x_2, \ldots, x_n) \in \mathbb{D}^n$

Exercise 3.31. *The set of homotopy classes of maps $f : (\mathbb{D}^n, \partial\mathbb{D}^n) \to (X, \mathbf{x_0})$, where $\partial\mathbb{D}^n \mapsto \mathbf{x_0}$, with the product defined above, forms a group.*

Definition (higher homotopy groups)**.** *The above mentioned group is called the n^{th}* homotopy group of X based at x_0 *and is denoted $\pi_n(X, x_0)$. The point x_0 is called the* base point *of the homotopy group.*

Theorem 3.32. *Homotopy equivalent spaces have the same homotopy groups.*

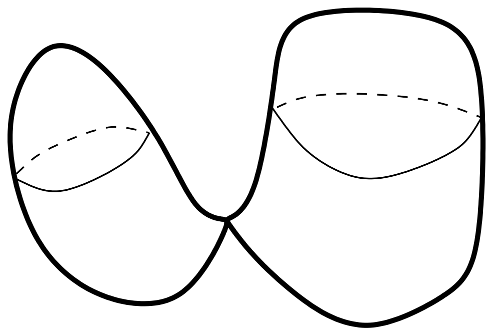


Figure 3.11: Two spheres with the same basepoint

Homotopy groups are generally hard to compute (even for $\mathbb{S}^n$). In the next chapter we will develop the study of homology groups, which turn out to be easier to compute and, hence, are generally more useful than the higher homotopy groups in distinguishing higher dimensional topological spaces from one another.

3.5 Covering spaces

Theorem 3.33.

1. *Any loop* $\alpha : I \to \mathbb{S}^1$ *can be written* $\alpha = \omega \circ \widetilde{\alpha}$, *where* $\widetilde{\alpha} : I \to \mathbb{R}^1$, $\widetilde{\alpha}(0) = 0$ *and* ω *is the standard wrapping map.*

2. $\widetilde{\alpha}(1)$ *is an integer.*

3. α_1 *and* α_2 *are equivalent loops in* $\mathbb{S}^1$ *if and only if* $\widetilde{\alpha_1}(1) = \widetilde{\alpha_2}(1)$.

4. $\pi_1(\mathbb{S}^1) = \mathbb{Z}$.

The preceding theorem outlines a method to compute $\pi_1(\mathbb{S}^1)$. The preimage of any small open arc in $\mathbb{S}^1$ under the standard wrapping map ω is a collection of open intervals in $\mathbb{R}^1$ and ω is a homeomorphism when restricted to any single one of these open intervals. This map ω and its use in the above theorem can be generalized to create the concept that is the subject of this section. From the idea that the real line is covering the simple closed curved via the wrapping map, the term "covering space" is used to refer to these generalized wrapping maps. Just as the wrapping map was useful in our method of computing the fundamental group of the circle, covering spaces in general are useful for understanding the structure of the fundamental groups of spaces.

Definition (covering space)**.** *Let* X, $\widetilde{X}$ *be connected, locally path connected spaces and let* $p : \widetilde{X} \to X$ *be a continuous function. Then the pair* $(\widetilde{X}, p)$ *is a* covering space *of* X *if and only if for each* $x \in X$ *there exists an open set* U *containing* x *such that* p *restricted to each component of* $p^{-1}(U)$ *is a homeomorphism onto* U. *When* $(\widetilde{X}, p)$ *is a covering space of* X, *we refer to the space* $\widetilde{X}$ *as a cover of* X *and* p *as a covering map.*

Example 1. *Let* $X = \mathbb{S}^1$, $\widetilde{X} = \mathbb{R}^1$, *and* $p : \mathbb{R}^1 \to \mathbb{S}^1$ *be defined by* $p(t) = (\cos t, \sin t)$. *Then* $(\mathbb{R}^1, p)$ *is a covering space of* $\mathbb{S}^1$.

Example 2. *Let* $X = \mathbb{S}^1$, $\widetilde{X} = \mathbb{S}^1$, *and* $p : \mathbb{S}^1 \to \mathbb{S}^1$ *be defined by* $p(z) = z^n$, *where* $z \in \mathbb{C}$ *is a complex number with* $|z| = 1$. *Then* $(\mathbb{S}^1, p)$ *is a covering space of* $\mathbb{S}^1$.

Example 3. *If* $X \cong$ *the wedge of two circles,* $\widetilde{X}$*'s are as Figure 3.12, and* p*'s are the maps indicated, then in each case,* $(\widetilde{X}, p)$ *is a covering space of* X.

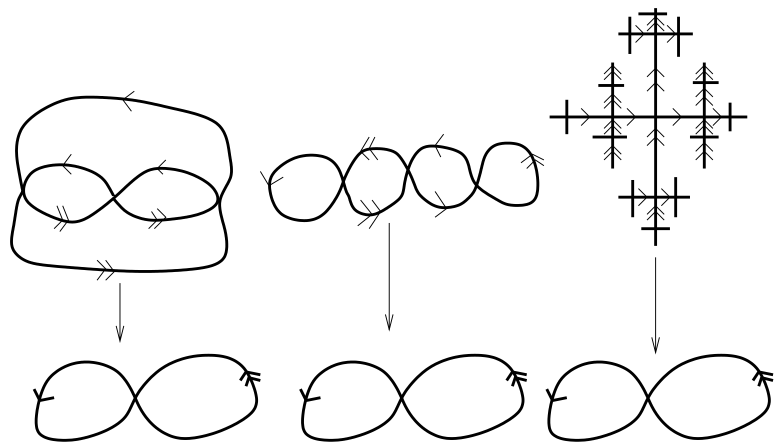


Figure 3.12: Several coverings of the wedge of two circles

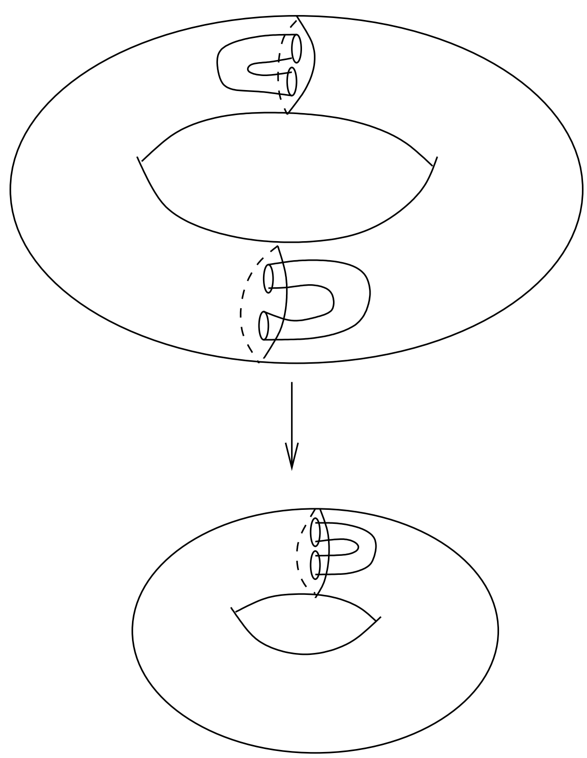


Figure 3.13:

Example 4. *Let Σ^2 be a PL non-separating, two-sided, properly embedded surface in a connected 3-manifold M^3. Gluing two copies of $M^3 - N(\Sigma^2)$ together gives a covering space of M^3. See Figure 3.13.*

Theorem 3.34. *Let $(\widetilde{X}, p)$ be a covering space of X. If x, $y \in X$, then $|p^{-1}(x)| = |p^{-1}(y)|$.*

Definition (n-fold covering)**.** *If $(\widetilde{X}, p)$ is a covering space of a space X and $n = |p^{-1}(x)|$ for some $x \in X$, then $(\widetilde{X}, p)$ is called an n-*fold *covering of X. We also say $\widetilde{X}$ is a cover of* degree n.

Example 5. *The example above, namely, $p : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p(z) = z^n$, where $z \in \mathbb{C}$, is an n-fold covering of $\mathbb{S}^1$ by itself.*

Exercise 3.35.

1. *Describe two non-homeomorphic 2-fold covers of the Klein bottle.*
2. *Describe all non-homeomorphic 2-fold covers of the wedge of two circles.*
3. *Describe all non-homeomorphic 3-fold covers of the wedge of two circles.*

There is a quick way of eliminating many possibilities from the potential n-fold coverings of a surface.

Theorem 3.36. *Let F be a compact connected surface and $p_n : \widetilde{F} \to F$ be an n-fold covering of F. Then F is a compact surface and $\chi(\widetilde{F}) = n\chi(F)$.*

Theorem 3.37. *Let F be a compact connected orientable manifold and $p : \widetilde{F} \to F$ be an n-fold covering of F. Then $\widetilde{F}$ is orientable.*

Exercise 3.38.

1. *Describe all non-homeomorphic 3-fold covers of the Klein bottle.*
2. *Describe all non-homeomorphic 2-fold covers of $\mathbb{T}^2 \# \mathbb{T}^2$.*
3. *Describe all non-homeomorphic 3-fold covers of $\mathbb{T}^2 \# \mathbb{T}^2 \# \mathbb{T}^2$.*
4. *Describe all non-homeomorphic 3-fold covers of $\mathbb{R}\mathrm{P}^2$.*

Definition (lift of a function). *Given a covering space $(\widetilde{X}, p)$ of X and a continuous function $f : Y \to X$, then a continuous function $\widetilde{f} : Y \to \widetilde{X}$ is called a* lift of f *if* $p \circ \widetilde{f} = f$.

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Theorem 3.39. *Let $(\mathbb{R}^1, \omega)$ be the standard wrapping map covering of $\mathbb{S}^1$. Then any path $f : [0, 1] \to \mathbb{S}^1$ has a lift.*

Theorem 3.40. *If $(\widetilde{X}, p)$ is a cover of X, Y is connected, and f, $g : Y \to \widetilde{X}$ are continuous functions such that $p \circ f = p \circ g$, then $\{y \mid f(y) = g(y)\}$ is empty or all of Y.*

Theorem 3.41. *Let $(\widetilde{X}, p)$ be a cover of X and let f be a path in X. Then for each $x_0 \in \widetilde{X}$ such that $p(x_0) = f(0)$, there exists a unique lift $\widetilde{f}$ of f satisfying $\widetilde{f}(0) = x_0$.*

Question 3.42. *Let p be a k-fold covering of $\mathbb{S}^1$ by itself and α a loop in $\mathbb{S}^1$ which when lifted to $\mathbb{R}^1$ by the standard lift has $\widetilde{\alpha}(0) = 0$ and $\widetilde{\alpha}(1) = n$. For which integers n does α lift to a loop in the k-fold covering?*

Theorem 3.43 (Homotopy Lifting Lemma). *Let $(\widetilde{X}, p)$ be a cover of X and α, β be two paths in X. If $\widetilde{\alpha}$, $\widetilde{\beta}$ are lifts of α, β satisfying $\widetilde{\alpha}(0) = \widetilde{\beta}(0)$, then $\widetilde{\alpha} \sim \widetilde{\beta}$ if and only if $\alpha \sim \beta$.*

Theorem 3.44. *If $(\widetilde{X}, p)$ is a cover of X, then p_* is a monomorphism (*i.e.*, 1–1 or injective) from $\pi_1(\widetilde{X})$ into $\pi_1(X)$.*

The previous theorem implies that the fundamental group of a cover of X is isomorphic to a subgroup of the fundamental group of the space X.

Theorem 3.45. *Let $(\widetilde{X}, p)$ be a cover of X, α a loop in X, and $\widetilde{x}_0 \in \widetilde{X}$ such that $p(\widetilde{x}_0) = \alpha(0)$. Then α lifts to a loop based at $\widetilde{x}_0$ if and only if $[\alpha] \in p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

Exercise 3.46. *Restate the proof of the fact that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ in terms of covering spaces.*

Theorem 3.47. *Let $(\widetilde{X}, p)$ be a covering space of X and let $x_0 \in X$. Fix $\widetilde{x}_0 \in p^{-1}(x_0)$. Then a subgroup H of $\pi_1(X, x_0)$ is in $\{p_*(\pi_1(\widetilde{X}, \widetilde{x}))\}_{p(\widetilde{x})=x_0}$ if and only if H is a conjugate of $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$.*

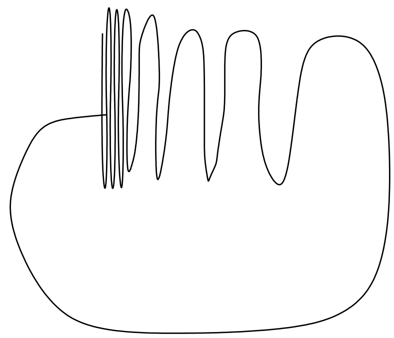


Figure 3.14: Consider this picture

Theorem 3.48. *Let $(\widetilde{X}, p)$ be a covering space of X. Choose $x \in X$, then $|p^{-1}(x)| = [\pi_1(X) : p_*(\pi_1(\widetilde{X}))]$.*

So, the index of the subgroup of $\pi_1(X)$ corresponding to a finite covering $(\widetilde{X}, p)$ equals the degree of the covering.

Exercise 3.49. *Describe a 3-fold cover $(\widetilde{X}, p)$ of $\mathbb{S}^1$ and the subgroup $p_*(\pi_1(\widetilde{X}))$ of $\pi_1(\mathbb{S}^1)$.*

Theorem 3.50. *Let $(\widetilde{X}, p)$ be a covering space of X and $\widetilde{x}_0 \in \widetilde{X}$, $x_0 \in X$ with $p(\widetilde{x}_0) = x_0$. Also let $f : Y \to X$ be continuous where Y is connected and locally path connected and $y_0 \in Y$ such that $f(y_0) = x_0$. Then there is a lift $\widetilde{f} : Y \to \widetilde{X}$ such that $p \circ \widetilde{f} = f$ and $\widetilde{f}(y_0) = \widetilde{x}_0$ if and only if $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$. Furthermore, $\widetilde{f}$ is unique.*

Exercise 3.51. *Let $X = \mathbb{S}^1$, $\widetilde{X} = \mathbb{R}$, $(\widetilde{X}, \omega)$ be the covering space of X given by the standard wrapping map, and Y as in Figure 3.15. When does a map $f : Y \to X$ not have a lift? Why is this example here?*

Definition (cover isomorphism)**.** *Let $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ be covering spaces of X. Then a map $f : \widetilde{X}_1 \to \widetilde{X}_2$ such that $p_2 \circ f = p_1$ is called a* cover isomorphism.

Theorem 3.52. *Let $(\widetilde{X}_1, p_1)$ and $(\widetilde{X}_2, p_2)$ be covering spaces of X. Let $\widetilde{x}_1 \in \widetilde{X}_1$ and $\widetilde{x}_2 \in \widetilde{X}_2$ such that $p_1(\widetilde{x}_1) = p_2(\widetilde{x}_2)$. Then there is cover isomorphism $f : \widetilde{X}_1 \to \widetilde{X}_2$ with $f(\widetilde{x}_1) = \widetilde{x}_2$ if and only if $p_*(\pi_1(\widetilde{X}_1, \widetilde{x}_1)) = p_*(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$.*

Definition (covering transformation)**.** *Let $(\widetilde{X}, p)$ be a covering space. Then a cover isomorphism from $\widetilde{X}$ to itself is called a* covering transformation.

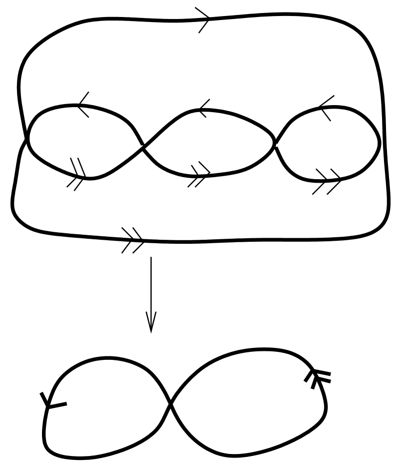


Figure 3.15: A covering of the figure 8 or wedge of two circles

The set of covering transformations, denoted $\mathcal{C}(\widetilde{X}, p)$*, is a group where the group operation is composition.*

Exercise 3.53. *What is* $\mathcal{C}(\widetilde{X}, p)$ *for the covering space of the figure eight shown in Figure 3.15.*

Theorem 3.54. *If* $(\widetilde{X}, p)$ *is a covering space of* X *and* $f \in \mathcal{C}(\widetilde{X}, p)$*, then* $f = Id_{\widetilde{X}}$ *if and only if* f *has a fixed point.*

Definition (regular cover)**.** *Let* $(\widetilde{X}, p)$ *be a covering space of* X*. If* $p_*(\pi_1(\widetilde{X})) \lhd \pi_1(X)$*, then* $(\widetilde{X}, p)$ *is a* regular *covering space.*

Question 3.55. *Consider the second three-fold covering space of the figure eight in Example 3. Find an element of* $p_*(\pi_1(\widetilde{X}))$ *which, when conjugated, is not in* $p_*(\pi_1(\widetilde{X}))$*.*

Theorem 3.56. *If* $(\widetilde{X}, p)$ *is a regular covering space of* X *and* $x_1, x_2 \in \widetilde{X}$ *such that* $p(x_1) = p(x_2)$*, then there exists a unique* $h \in \mathcal{C}(\widetilde{X}, p)$ *such that* $h(x_1) = x_2$*.*

Question 3.57. *The preceding theorem tells us that for a regular covering space, there is a (unique) covering transformation carrying any point in the set* $p^{-1}(x)$ *to any other point in the same set. Is this true of an irregular covering space?*

Theorem 3.58. *A covering space is regular if and only if for every loop either all its lifts are loops or all its lifts are paths that are not loops.*

Exercise 3.59.

1. *Describe all regular 3-fold covering spaces of a figure eight.*
2. *Describe all irregular 3-fold covering spaces of a figure eight.*
3. *Describe all regular 4-fold covering spaces of a figure eight.*
4. *Describe all irregular 4-fold covering spaces of a figure eight.*
5. *Describe all regular 3-fold covering spaces of a wedge of 3 circles.*
6. *Describe all regular 4-fold covering spaces of a wedge of 3 circles.*

There is an important correspondence between the covering transformations of regular covers of X and the normal subgroups of $\pi_1(X)$.

Theorem 3.60. *Let $(\widetilde{X}, p)$ be a regular covering space of X. Then $\mathcal{C}(\widetilde{X}, p) \cong \pi_1(X)/p_*(\pi_1(\widetilde{X}))$. In particular, $\mathcal{C}(\widetilde{X}, p) \cong \pi_1(X)$ if $\widetilde{X}$ is simply connected.*

Exercise 3.61. *Observe that the standard wrap map is a regular covering map of $\mathbb{S}^1$ by $\mathbb{R}^1$. Describe the covering transformations for this covering space. Describe the covering map that maps $\mathbb{R}^2$ to the torus $\mathbb{T}^2$ and describe the covering transformations for this covering space.*

Definition (semi-locally simply connected)**.** *A space X is called* semi-locally simply connected *if and only if every $x \in X$ is contained in an open set U such that every loop in U based at x is homotopically trivial in X.*

Note that U need not be simply connected itself.

Theorem 3.62 (Existence of covering spaces)**.** *Let X be connected, locally path connected, and semi-locally simply connected. Then for every $G < \pi_1(X, x_0)$ there is a covering space $(\widetilde{X}, p)$ of X and $\widetilde{x}_0 \in \widetilde{X}$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = G$. Furthermore, $(\widetilde{X}, p)$ is unique up to isomorphism.*

Definition (universal cover)**.** *A connected, locally path connected cover is called* universal *if and only if its fundamental group is trivial.*

Corollary 3.63. *Every connected, locally path connected, semi-locally simply connected space has a universal covering space.*

Exercise 3.64. *Find universal covers for the Klein bottle, torus, and projective plane. Then show explicitly that $\mathcal{C}(\widetilde{X}, p) \cong \pi_1(X)$.*

3.6 Theorems about groups

The algebra of a fundamental group tells us something about the topology of its corresponding space. It is also possible to use topology to study algebraic groups. One can construct a complex that represents any finitely generated group, and use the fundamental group and covering spaces to deduce properties of the group.

Theorem 3.65. *Every tree is simply connected.*

Theorem 3.66. *Let G be a graph, and T be a maximal tree in G. Then if $\{e_1, \ldots e_n\}$ is the set of edges that are not in T, $\pi_1(G) = F_n$, the free group on n generators; and there is a system of generators that are in one-to-one correspondence with the edges $\{e_1, \ldots e_n\}$.*

Corollary 3.67. *A subgroup H of a free group F_n is always a free group.*

Question 3.68. *Describe a regular k-fold cover $\widetilde{X}$ of a wedge of n-circles. What is the number of generators of $\pi_1(\widetilde{X})$, given k and n? What does this tell us about the normal subgroups of finite index of the free group with n generators?*

Exercise 3.69.

1. *Let F be a free group on n letters. Let $G \subset F$ be of finite index k and contain 7 free generators. What can the value of n be?*

2. *Let F be a free group on n letters. Let $G \subset F$ be of finite index k and contain 4 free generators. What can the value of n be?*

3. *Let F be a free group on n letters. Let $G \subset F$ be of finite index k and contain 24 free generators. What can the value of n be?*

Exercise 3.70. *Let S_k be the set of all k-fold covers of $\mathbb{K}$, the Klein Bottle.*

1. *Describe S_k.*

2. *Describe all subgroups of $\pi_1(\mathbb{K})$ of finite index.*

Exercise 3.71. *Let F be the closed orientable surface of genus 2, and $G = \pi_1(F)$. Show that all subgroups of G of finite index k are isomorphic.*

Chapter 4

Homology

We have seen that the fundamental group of a space uses *loops* that are not null-homotopic to understand the "holeyness" of a space. For distinguishing spaces, the fundamental group is a valuable tool, but has some challenges associated with it. For one thing, the fundamental group is in general not an abelian group. This feature makes it difficult to determine in general whether two fundamental groups might be the same if they have different presentations. The Classification of Abelian Groups allows us to determine which abelian groups are isomorphic and which are not, so it would be nice to associate meaningful abelian groups with spaces. Of course, the fundamental group of a space may be giving us more refined information. However, in many cases, we don't need that much information to capture some of the "holeyness" of a space. For example, recall that when we used the fundamental group to distinguish surfaces from one another, the abelianizations of the fundamental groups were sufficient to make the distinctions–the whole fundamental group was not used.

The second shortcoming of the fundamental group for measuring "holeyness" of spaces is that it captures only holes that are surrounded by loops. It does not measure holes surrounded by spheres, for example. A 3-ball with a sub-ball removed still has trivial fundamental group. In this chapter, then, we will introduce the concept of *homology*, which associates with spaces abelian groups that measure holes in all dimensions and are often easy to compute.

We will study the homology of simplicial complexes. The ideas we develop here can be generalized to apply to more comprehensive classes of spaces, but we will not study those generalizations in this course.

4.1 $\mathbb{Z}_2$ homology

In the fundamental group, we concerned ourselves with loops. In homology we will be concerned with *cycles*, which will help us quantify in some way the number of "holes" in a surface or space. Cycles will come in all dimensions.

Intuitively, a 1-cycle is a sum of edges in a simplicial complex that makes a loop or loops. Consider the space made up of two 2-simplexes (σ_1, σ_2), seven edges (e_i, $i = 1, 2, ..., 7$), and five vertices (v_i, $i = 1, 2, ..., 5$) as shown.

Then there are several cycles (loops of edges) in this example. For notational consistency that will become clear in our later development of $\mathbb{Z}$ homology, we will write each loop as a sum of edges, but you should think of the loop as a physical loop sitting inside our space and should be wondering, "Why don't we just use a union symbol?" $e_1 + e_2 + e_4 + e_5$ is one loop, $e_1 + e_2 + e_6 + e_7 + e_5$ is another loop, and $e_6 + e_7 + e_4$ is a third loop. Which ones represent "holes" in our space? Well, $e_1 + e_2 + e_4 + e_5$ bounds a filled in piece comprised of the two 2-simplices, so it doesn't represent a hole, and it should be considered equivalent to 0. $e_1 + e_2 + e_6 + e_7 + e_5$ and $e_6 + e_7 + e_4$ both go around the same hole, so they should be viewed as the same. In fact, we know that these loops are homotopic to each other by a homotopy that takes the edges $e_5 + e_1 + e_2$ to e_3, so if we were looking at π_1 then we'd have two equivalent loops in the sense of the fundamental group. However, we want to keep things as simple as possible, so we will give a simple criterion for declaring that these two cycles are $\mathbb{Z}_2$ equivalent. Consider the edges that the two cycles $e_1 + e_2 + e_6 + e_7 + e_5$ and $e_6 + e_7 + e_4$ differ by, namely, $e_1 + e_2 + e_4 + e_5$. That difference is a cycle that bounds an object composed of two 2-simplices. So the intuitive concept that we will formalize below is that two cycles will be declared to be equivalent if their difference is the boundary of some collection of simplices in our space.

4.1.1 Simplicial $\mathbb{Z}_2$ homology

Recall the definitions of a simplex, a face of a simplex, and a finite simplicial complex. We now define the objects used to define homology–chains, cycles, and equivalence classes of cycles. When dealing with $\mathbb{Z}_2$ homology, we could use slightly simpler notation; however, instead, we will use notation in this section that will later be valuable when we deal with $\mathbb{Z}$ homology where orientation comes into play.

Definition ($\mathbb{Z}_2$ n-chain)**.** *The $\mathbb{Z}_2$-sum of k n-simplices $\sum_{i=1}^{k} \sigma_i^n$ (where the sum is mod 2) is called a ($\mathbb{Z}_2$) n-*chain*.*

Definition ($\mathbb{Z}_2$-boundary of a simplex)**.** *The* $\mathbb{Z}_2$-boundary *of any n-simplex σ^n is*

$$\partial(\sigma^n) = \sum_{i=0}^{n} \sigma_i^{n-1}$$

where the σ_i^{n-1}'s are the $(n-1)$-dimensional faces of σ^n.

Definition ($\mathbb{Z}_2$-boundary of an n-chain)**.** *The $\mathbb{Z}_2$-boundary of the n-chain $\sum_{i=1}^{k} \sigma_i^n$ is*

$$\partial(\sum_{i=1}^{k} \sigma_i^n) = \sum_{i=1}^{k} \partial(\sigma_i^n)$$

where the sum is mod 2; i.e., *if a simplex appears an even number of times it cancels.*

For example, $\sigma_1 + \sigma_2 + \sigma_2 + \sigma_3 + \sigma_3 + \sigma_3 = \sigma_1 + \sigma_3$.
Notice that boundary of an n-chain is an $(n-1)$-chain.

Definition ($\mathbb{Z}_2$ n-cycle)**.** *A* $\mathbb{Z}_2$ n-cycle *is an n-chain whose $\mathbb{Z}_2$-boundary is zero.*

Theorem 4.1. *For any $\mathbb{Z}_2$ n-chain C, $\partial(\partial(C)) = 0$, that is, the boundary of any n-chain is an $(n-1)$-cycle.*

Definition ($\mathbb{Z}_2$-equivalence of cycles)**.** *In a complex (K, T) n-cycles A^n and B^n are* $\mathbb{Z}_2$-equivalent *if and only if $A^n - B^n$ $(= A^n + B^n) = \partial(C^{n+1})$, where C^{n+1} is an $(n+1)$-chain. The equivalence class that A^n is a member of will be denoted by $[A^n]$.*

[Note: We use the minus sign in $A^n - B^n$ in anticipation of $\mathbb{Z}$ homology to come.]

Question 4.2. *List all the equivalence classes of* 0-, 1- *and* 2-*cycles in a triangulated sphere, torus, projective plane, Klein bottle, and then each compact, connected, triangulated* 2-*manifold.*

Definition ($\mathbb{Z}_2$ n^{th}-simplicial homology)**.** *The* $\mathbb{Z}_2$ n^{th}-homology *of a finite simplicial complex (K, T), denoted $H_n((K, T); \mathbb{Z}_2)$ is the additive group whose elements are equivalence classes of cycles under the $\mathbb{Z}_2$-equivalence defined above, where $[A^n] + [B^n] := [A^n + B^n]$.*

Exercise 4.3. *Show that the addition for $H_n((K,T);\mathbb{Z}_2)$ defined above is well-defined.*

Theorem 4.4. *Let (K,T) be a finite simplicial complex with triangulation T, then $H_n((K,T);\mathbb{Z}_2)$ is an abelian group.*

Theorem 4.5. *Let K be connected and let (K,T) be a finite simplicial complex with triangulation T, then $H_0((K,T);\mathbb{Z}_2)$ is $\mathbb{Z}_2$.*

Homology would be an uninteresting concept if it depended on the particular triangulation selected; however, in fact, the homology groups of a space are independent of which triangulation is used. For now, let's accept the following two theorems, which are the basis of that fact, and later we'll look at an outline of how they are proved. The first of these theorems simply states that subdividing the triangulation of a complex does not change the homology that we compute.

Theorem 4.6. *Let (K,T) be a finite simplicial complex and let T' be a subdivision of T. Then $H_n((K,T);\mathbb{Z}_2)$ is isomorphic to $H_n((K,T');\mathbb{Z}_2)$.*

Theorem 4.7. *Let K be a subset of $\mathbb{R}^n$ and let T and T' be triangulations of K that make (K,T) and (K,T') finite simplicial complexes. Then there is a triangulation T'' that is a subdivision of both T and T'.*

Corollary 4.8. *Let K be the underlying subset of a finite simplicial complex. Then $H_n((K,T);\mathbb{Z}_2) \cong H_n((K,T');\mathbb{Z}_2)$ for any triangulations T and T' of K.*

This corollary means that the homology of a set is independent of the particular triangulation chosen for it.

4.1.2 CW $\mathbb{Z}_2$-homology

The difficulty with simplicial complexes is that they frequently have a lot of simplices. To compute H_n, we would need to consider the boundaries of all the possible subsets of simplices in the triangulation, which is a laborious undertaking. But, on the bright side, we saw in the previous section that $H_n((K,T);\mathbb{Z}_2)$ is the same for any triangulation T of K. We want to further simplify the task of computing homology groups by considering *CW* decompositions, which we encountered earlier when we discussed simpler ways to compute the Euler Characteristic.

As we did before, let's look at the triangulation subdivision result a bit backwards. We can view the subdivision result as saying that grouping simplices into bigger simplices does not affect the $\mathbb{Z}_2$-homology of the simplicial complex. Let's push that strategy even further by showing that we can compute the same homology groups by breaking K up into cells that may or may not be simplices rather than sticking to triangulations. Our definition of CW complex of any dimension captures the idea of 'cellulating' K rather than triangulating K, as we saw before for 2-complexes.

First, let's note that a simplicial complex of any dimension can be written as the disjoint union of the interiors of the simplices in its triangulation. (Recall that the boundary of a vertex is the empty set.)

Definition (interior of a simplex)**.** *For each simplex σ, let the interior of σ, denoted by* $\mathrm{Int}(\sigma)$ *or $\overset{\circ}{\sigma}$, be $\overset{\circ}{\sigma} = \sigma - \partial\sigma$.*

Theorem 4.9. *Let (K, T) be a finite simplicial complex where $T = \{\sigma_i\}_{i=1,..k}$. Then K*

$$K = \bigsqcup_{1}^{k} \overset{\circ}{\sigma}_i,$$

where $\sqcup$ denotes the disjoint union.

To simplify our computation of homology groups, it is often useful to view K as a union of open cells other than interiors of simplices, because then we can use many fewer cells. We accomplish such a decomposition of K by grouping simplices to create larger cells.

Definition (open cell complex, open cell decomposition, and CW decomposition)**.** *Let (K, T) be a finite simplicial complex. Suppose we write K as the disjoint union of $A_0^{k_0}, A_1^{k_1}, A_2^{k_2}, \ldots, A_n^{k_3}$ such that each $A_i^{k_i}$ is homeomorphic to the interior of a k_i-cell, each $A_i^{k_i}$ is the disjoint union of interiors of simplices in T, and the closure of each $A_i^{k_i}$ minus $A_i^{k_i}$ is the union of $A_j^{k_j}$ (each of dimension less than k_i). Let S be the set of $A_i^{k_i}$'s. Then (K, S) will be called an open cell complex and S will be called an open cell decomposition of K. S is equivalent to what is sometimes called a CW decomposition of K.*

Examples 1.

1. *Let (K, T) be a simple closed curve with triangulation shown (a square). Then one vertex and one open cell would form a CW decomposition of*

K.

2. *Let (K, T) be a 3-simplex with triangulation shown (a tetrahedron). Then one vertex, one open 2-cell, and one open 3-cell would form a CW decomposition of K. This example shows that it is not necessary to have every dimension represented.*

3. *Let (K, T) be the 2-complex created by starting with a triangle and attaching a disk whose boundary goes around it twice. We could look at this complex as a quotient space of a 6-sided triangulated disk with opposite edges identified with arrows all going clockwise around the hexagon. It could exist in R^4. Then a CW decomposition of K could consist of one open 2-cell, one open 1-cell, and one vertex, as shown.*

The following two exercises probably should have occurred in the fundamental group section.

Exercise 4.10. *Let K be the space described in Example 3 above. What is $\pi_1(K)$?*

Exercise 4.11. *For any finitely presented group G, describe a 2-complex K such that $\pi_1(K) = G$.*

Back to homology. We can define $\mathbb{Z}_2$-homology for a CW complex exactly as we defined it for a triangulated complex. As expected we will soon see that the homology groups are exactly the same as they are when we compute the $\mathbb{Z}_2$-homology groups using the triangulation rather than the open cell decomposition. Here then are the relevant definitions.

Definition (boundary of open cell). *Let (K,T) be a simplicial complex. Let A be an element of an open cell decomposition (K,S) of (K,T). So A is homeomorphic to the interior of a k-cell and $\overline{A}$ = union of σ_i^k where each σ_i^k is a k-simplex in T. Then $\partial A :=$ the sum of all $(k-1)$-cells in S that are contained in $\sum \partial \sigma_i^k$.*

Definition (boundary of an n-chain). *Let $B = \sum A_i$ where each A_i is an open n-cell in a CW complex (K,S). Then B is an n-chain and the boundary of B is $\sum \partial A_i$.*

Recall that the sum is done mod 2.

Definition (n-cycle). *An n-cycle in an open cell complex (K,S) is a sum of n-cells in S, that is, an n-chain, with empty boundary.*

Definition (equivalence of cycles). *A k-cycle A^k is $\mathbb{Z}_2$-equivalent to a k-cycle B^k if and only if there exists a $(k+1)$-chain C^{k+1} such that $\partial C^{k+1} = A^k - B^k (= A^k + B^k$ because we are summing mod 2).*

Definition ($\mathbb{Z}_2$ n^{th}-CW homology). *The $\mathbb{Z}_2$ n^{th} homology group of a CW complex (K,S), denoted $H_n((K,S);\mathbb{Z}_2)$, is the group whose elements are the $\mathbb{Z}_2$-equivalence classes of n-cycles with addition defined by $[A^n]+[B^n] = [A^n + B^n]$.*

Finally, we confirm our goal that the CW homology is the same as simplicial homology, and once again we will accept this theorem and its corollary as true and later outline proofs of them.

Theorem 4.12. *Let (K,S) be a CW decomposition of the finite simplicial complex (K,T). Then for each n, $H_n((K,S);\mathbb{Z}_2) = H_n((K,T);\mathbb{Z}_2)$.*

Corollary 4.13. *$H_n((K,T);\mathbb{Z}_2)$ does not depend on the triangulation or CW decomposition of K used to compute H_n.*

Since any triangulation or CW decomposition of K yields the same homology groups, we often suppress the triangulation or CW decomposition in the notation and just refer to $H_n(K;\mathbb{Z}_2)$.

Question 4.14. *For each space K below, describe a CW decomposition of it and describe $H_n(K;\mathbb{Z}_2)$ for $n = 0,1,2,3,\ldots$:*

1. *the sphere.*

2. *the torus.*
3. *the projective plane.*
4. *the Klein bottle.*
5. *the double torus.*
6. *any compact, connected, triangulated* 2*-manifold.*
7. *the Möbius band.*
8. *the annulus.*
9. *Two (hollow) triangles joined at a vertex.*

Question 4.15. *What is $H_n(K;\mathbb{Z}_2)$ if dim(K)$< n$?*

Question 4.16. *What are $H_n(G;\mathbb{Z}_2)$ for $n = 0, 1, 2, \ldots$ for a graph G?*

Question 4.17. *What are $H_n(\mathbb{S}^k;\mathbb{Z}_2)$ for $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$?*

Question 4.18. *For any n, what is $H_n(M^n;\mathbb{Z}_2)$ where M^n is a connected n-manifold?*

Question 4.19. *For any n, what is $H_n(M^n;\mathbb{Z}_2)$ where M^n is a connected n-manifold with non-empty boundary?*

Question 4.20. *What are $H_n(T;\mathbb{Z}_2)$ for $n = 0, 1, 2, \ldots$ for a solid torus T?*

4.2 Homology from parts, special cases

One strategy for computing homology is to divide the space into parts and see how the homology groups of the whole are related to the homology groups of its parts. The next several theorems should remind the reader of special cases of Van Kampen's theorem from the theory of fundamental groups.

Theorem 4.21. *Let $K \vee L$ denote the wedge of finite simplicial complexes K and L. Then $H_n(K \vee L;\mathbb{Z}_2) \cong H_n(K;\mathbb{Z}_2) \oplus H_n(L;\mathbb{Z}_2)$ for $n > 0$.*

Theorem 4.22. *Suppose M is a finite simplicial complex with subcomplexes K and L such that $K \cup L = M$. If $H_n(K \cap L;\mathbb{Z}_2) = 0$, then $H_n(M;\mathbb{Z}_2) \cong H_n(K;\mathbb{Z}_2) \oplus H_n(L;\mathbb{Z}_2)$.*

Exercise 4.23. *Compute $H_n(K;\mathbb{Z}_2)$ ($n = 0, 1, 2, \ldots$) for each complex K below.*

1. *A wedge of k circles.*

2. *A wedge of a 2-sphere and a circle.*

3. *A 2-sphere union its equatorial disk.*

4. *A double solid torus.*

Theorem 4.24. *Let M be a finite simplicial complex with subcomplexes K and L such that $K \cup L = M$. Suppose $n \geq 2$ and that for every n-cycle Z in $K \cap L$, $Z \sim_{\mathbb{Z}_2} 0$ in K and $Z \sim_{\mathbb{Z}_2} 0$ in L, and $H_{n-1}(K \cap L;\mathbb{Z}_2) = 0$. Then $H_n(M;\mathbb{Z}_2) \cong H_n(K;\mathbb{Z}_2) \oplus H_n(L;\mathbb{Z}_2)$.*

The case $n = 1$ is slightly different, namely, suppose for every 1-cycle Z in $K \cap L$, $Z \sim_{\mathbb{Z}_2} 0$ in K and $Z \sim_{\mathbb{Z}_2} 0$ in L, and $H_0(K \cap L;\mathbb{Z}_2) = \mathbb{Z}_2$. Then $H_1(M;\mathbb{Z}_2) \cong H_1(K;\mathbb{Z}_2) \oplus H_1(L;\mathbb{Z}_2)$.

Again, the previous theorem might remind the reader of a a special case of Van Kampen's Theorem.

Question 4.25. *State and prove a theorem giving the* $\mathbb{Z}_2$*-homology groups for connected compact* 2*-manifolds, using the theorem above and the connected sum decomposition of each* 2*-manifold.*

4.3 Chain groups and induced homomorphisms

Definition (chain group). *Let K be a finite simplicial complex. The* chain group $C_n(K;\mathbb{Z}_2)$ *is the group whose elements are the n-chains of K where the group operation is $\mathbb{Z}_2$ chain addition. So each n-simplex of K is a generator for the abelian group $C_n(K;\mathbb{Z}_2)$, which is therefore the direct sum of k copies of $\mathbb{Z}_2$ where k is the number of n-simplexes in K.*

We will now investigate how simplicial maps from one complex to another induce homomorphisms on homology groups.

Definition. *Let (K,T) and (L,S) be finite simplicial complexes. A map $g:(K,T)\to(L,S)$ is a* simplicial map *if and only if for any $\sigma^n = \{v_0v_1v_2\dots v_n\}\in T$, there is a simplex $\tau=\{w_0w_1w_2\dots w_k\}\in S$ such that for each $i=0,1,2,\dots n$, $g(v_i)=w_j$ for some $j\in\{0,1,2,\dots,k\}$ and for each point $x=\sum_{i=0}^n\lambda_iv_i$ where $0\le\lambda_i\le 1$ and $\sum_{i=0}^n\lambda_i=1$, then $g(x)=\sum_{i=0}^n\lambda_ig(v_i)$. Note that $g(v_i)$ is not necessarily distinct from $g(v_k)$, and that g is linear on each simplex of T.*

Question 4.26. *Let (K,T) and (L,S) be finite simplicial complexes. Let $f:(K,T)\to(L,S)$ be a simplicial map. Find a "natural" definition for the induced map $f_\#:C_n(K;\mathbb{Z}_2)\to C_n(L;\mathbb{Z}_2)$. Notice that the map $f_\#$ is a homomorphism of n-chains of K to n-chains of L.*

Recall that the boundary of an n-chain is an $(n-1)$-cycle.

Theorem 4.27. *The map $\partial: C_n(K;\mathbb{Z}_2)\to C_{n-1}(K;\mathbb{Z}_2)$ is a group homomorphism.*

Theorem 4.28. *Let $f:K\to L$ be a simplicial map, and let $f_\#$ be the induced map $f_\#:C_n(K;\mathbb{Z}_2)\to C_n(L;\mathbb{Z}_2)$. Then for any $C\in C_n(K;\mathbb{Z}_2)$, $\partial(f_\#(C))=f_\#(\partial(C))$. In other words, the diagram:*

$$\begin{array}{ccc} C_n(K;\mathbb{Z}_2) & \xrightarrow{\partial} & C_{n-1}(K;\mathbb{Z}_2) \\ {\scriptstyle f_\#}\big\downarrow & & \big\downarrow{\scriptstyle f_\#} \\ C_n(L;\mathbb{Z}_2) & \xrightarrow{\partial} & C_{n-1}(L;\mathbb{Z}_2) \end{array}$$

commutes.

Definition. *Let (K,T) and (L,S) be finite simplicial complexes. Let $f: K\to L$ be a simplicial map. The* induced homomorphism $f_*: H_n(K;\mathbb{Z}_2)\to H_n(L;\mathbb{Z}_2)$ *is defined by $f_*([A])=[f_\#(A)]$.*

Note that at this point, it is not clear that f_* takes cycles to cycles. The following theorem asserts that all is well.

Theorem 4.29. *Let (K,T) and (L,S) be finite simplicial complexes. Let $f : K \to L$ be a simplicial map. Then the induced homomorphism $f_* : H_n(K;\mathbb{Z}_2) \to H_n(L;\mathbb{Z}_2)$ is a well-defined homomorphism.*

In order to deal with continuous functions rather than simply with simplicial maps, we sketch a proof below that any continuous function can be approximated by a simplicial map that is homotopic to it.

Theorem 4.30. *Let (K,T) and (L,S) be finite simplicial complexes and $f : K \to L$ be a continuous function. Then there exists a subdivision T' of T and a simplicial map $g : (K,T') \to (L,S)$ such that f is homotopic to g and g is a simplicial map.*

The proof of this theorem uses the following.

Definition (star of a vertex)**.** *Let (K,T) be a simplicial complex. Then the* star *of v, where v is a vertex of T, is defined as* $\mathrm{St}(v,T) = \cup\{\tau \in T | v \text{ is a vertex of } \tau\}$.

Sketch of Proof. Subdivide T to obtain T' such that for every vertex v of T', $f(\mathrm{St}(v,T')) \subseteq \mathrm{Int}(\mathrm{St}(w,S))$ for some vertex w of S. For each v define $g(v) = w$ where w is any vertex in S for which $f(\mathrm{St}(v,T')) \subseteq \mathrm{Int}(\mathrm{St}(w,S))$. Extend g linearly over each simplex. Show that g is a simplicial map and g is homotopic to f. □.

Theorem 4.31. *Let T and T' be two different triangulations of a finite simplicial complex K and let (L,S) be a finite simplicial complex. Suppose $f : (K,T) \to (L,S)$ and $g : (K,T') \to (L,S)$ are simplicial maps such that f is homotopic to g. Then the induced homomorphism $f_* : H_n(K;\mathbb{Z}_2) \to H_n(L;\mathbb{Z}_2)$ is the same homomorphism as g_*.*

Sketch of Proof. Find a simplicial approximation of the homotopy H between f and g that is linear with respect to a common subdivision T" of T and T'. For any n-cycle A in $(K,T")$, show that $[f_\#(A)] = [g_\#(A)]$ by using the $(n+1)$-chain $H_\#(A \times [0,1])$. □.

These results allow us to extend our definition of the induced homomorphism f_* to apply to continuous functions $f : K \to L$ (rather than just applying to simplicial maps) and to know that f_* is well-defined.

Theorem 4.32. *Let K and L be finite simplicial complexes such that K is a strong deformation retract of L. Then $H_n(K;\mathbb{Z}_2) \cong H_n(L;\mathbb{Z}_2)$.*

Theorem 4.33. *If A, B are finite simplicial complexes and A is homotopy equivalent to B, then $H_n(A;\mathbb{Z}_2) \cong H_n(B;\mathbb{Z}_2)$.*

In other words homology does not distinguish between a space and a strong deformation retract of it, or between homotopy equivalent spaces in general. So, once again, homology is not a complete invariant. It is however, very useful, and, in general, easier to compute than homotopy.

Theorem 4.34. *$H_n(\text{Dunce's hat}) \cong 0$ for all $n \neq 0$.*

Theorem 4.35. *$H_n(\text{House with 2 rooms}; \mathbb{Z}_2) \cong 0$ for all $n \neq 0$.*

Theorem 4.36. *If $f : K \to L$ is a homeomorphism between finite simplicial complexes K and L, then $f_* : H_n(K;\mathbb{Z}_2) \to H_n(L;\mathbb{Z}_2)$ is an isomorphism.*

In other words, homology groups are topological invariants.

4.4 Applications of $\mathbb{Z}_2$ homology

Recall the definition of a retraction.

Definition (retraction)**.** *Let $K \subset L$, then a continuous function $r : L \to K$ is a* retraction *if and only if $r(x) = x$ for all $x \in K$.*

Theorem 4.37 (No Retraction Theorem)**.** *Let M^n be an n-manifold with $\partial M^n \neq \emptyset$. Then there is no retraction $r : M^n \to \partial M^n$.*

Theorem 4.38 (n-dimensional Brouwer Fixed Point Theorem)**.** *For every continuous function $f : \mathbb{D}^n \to \mathbb{D}^n$ there exists a point $x \in \mathbb{D}^n$ such that $f(x) = x$.*

We will develop a proof of the Borsuk-Ulam Theorem below through a sequence of preliminary theorems.

Lemma 4.39. *Let M^n be a finite, triangulated, connected n-manifold without boundary. Let $f : M^n \to M^n$ be a simplicial map. Then $f_* : H_n(M^n; \mathbb{Z}_2) \to H_n(M^n; \mathbb{Z}_2)$ is onto if and only if $f_\#(M^n) = M^n$.*

Theorem 4.40. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be an antipode preserving map (that is, for every $x \in \mathbb{S}^1$, $f(-x) = -f(x)$). Then $f_* : H_1(\mathbb{S}^1; \mathbb{Z}_2) \to H_1(\mathbb{S}^1; \mathbb{Z}_2)$ is onto.*

Theorem 4.41 (Borsuk-Ulam for $\mathbb{S}^2$)**.** *Let $f : \mathbb{S}^2 \to \mathbb{R}^2$ be a continuous map. Then there is an $x \in \mathbb{S}^2$ such that $f(-x) = f(x)$.*

Hint. If there existed a counterexample to the Borsuk-Ulam Theorem, then there would be a continuous function from the equator to itself that is antipode preserving where the induced homomorphism on the $\mathbb{Z}_2$ first homology is not onto.

The above strategy for proving the Borsuk-Ulam Theorem in dimension 2 can be extended to work in all dimensions.

Theorem 4.42. *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be an antipode preserving map (that is, for every $x \in \mathbb{S}^n$, $f(-x) = -f(x)$). Then $f_* : H_n(\mathbb{S}^n; \mathbb{Z}_2) \to H_n(\mathbb{S}^n; \mathbb{Z}_2)$ is onto.*

Theorem 4.43 (Borsuk-Ulam)**.** *Let $f : \mathbb{S}^n \to \mathbb{R}^n$ be a continuous function. Then there is an $x \in \mathbb{S}^n$ such that $f(-x) = f(x)$.*

The following lemmas will allow us to prove the n-dimensional Jordan Curve Theorem.

Lemma 4.44. *Let $f : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a simplicial map. Then there exist $\mathbb{Z}_2$ n-chains A^n and C^n such that $\partial(A^n) = \partial(C^n) = f_\#(\mathbb{S}^{n-1})$ and $A^n \cup C^n = \mathbb{S}^n$.*

Lemma 4.45. *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding. Then there exists an $\epsilon > 0$ such that if $f : \mathbb{S}^{n-1} \to \mathbb{S}^n$ is a simplicial map such that $d(f(x), h(x)) < \epsilon$ for all $x \in \mathbb{S}^{n-1}$, then $f_\#(\mathbb{S}^{n-1})$ does not bound an n-chain in the ϵ-neighborhood of $h(\mathbb{S}^{n-1})$.*

Theorem 4.46 (n-dimensional Jordan Curve Theorem)**.** *Let $h : \mathbb{S}^{n-1} \to \mathbb{S}^n$ be a topological embedding. Then $h(\mathbb{S}^{n-1})$ separates $\mathbb{S}^n$ into two components and is the boundary of each.*

Theorem 4.47. *A topologically embedded compact, triangulated, connected $(n-1)$-dimensional manifold separates $\mathbb{R}^n$.*

Theorem 4.48 (Invariance of Domain or Invariance of Dimension Theorem)**.** *n-manifolds are not homeomorphic to m-manifolds if $n \neq m$.*

These are just some of the fundamental theorems of topology that we can prove using $\mathbb{Z}_2$ homology.

4.5 $\mathbb{Z}_2$ Mayer-Vietoris Theorem

The goal of this section is to describe how the homology groups of a complex are related to the homology groups of pieces of the complex. We begin with some theorems that relate cycles in the whole complex to cycles and chains in its parts.

Theorem 4.49. *Let K be a finite simplicial complex and K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. If A_0, A_1 are $(n-1)$-cycles in K_0 and K_1 respectively and if $A_0 \sim_{\mathbb{Z}_2} A_1$ in K, then there is a $(n-1)$-cycle C in $K_0 \cap K_1$ such that $A_0 \sim_{\mathbb{Z}_2} C$ in K_0 and $A_1 \sim_{\mathbb{Z}_2} C$ in K_1.*

Theorem 4.50. *Let K be a finite simplicial complex and K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. Let Z be a $\mathbb{Z}_2$ n-cycle on K. Then there exist $\mathbb{Z}_2$ n-chains W_0 and W_1 in K_0 and K_1 respectively such that:*

1. *$Z = W_0 + W_1$ and*
2. *$\partial(W_0) = \partial(W_1)$ is an $(n-1)$-cycle C in $K_0 \cap K_1$.*

Furthermore, if $Z = W_0' + W_1'$ where W_i' is an n-chain in K_i, and $C' = \partial(W_0') = \partial(W_1')$ is an $(n-1)$-cycle, then C' is $\mathbb{Z}_2$-equivalent to C in $K_0 \cap K_1$.

Question 4.51. *Let K be a simplicial complex and K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. Describe the natural homomorphisms below, and verify that they are homomorphisms:*

1. $\phi : H_n(K_0 \cap K_1; \mathbb{Z}_2) \to H_n(K_0; \mathbb{Z}_2) \oplus H_n(K_1; \mathbb{Z}_2)$.
2. $\psi : H_n(K_0; \mathbb{Z}_2) \oplus H_n(K_1; \mathbb{Z}_2) \to H_n(K; \mathbb{Z}_2)$
3. $\delta : H_n(K; \mathbb{Z}_2) \to H_{n-1}(K_0 \cap K_1; \mathbb{Z}_2)$

Van Kampen's Theorem was phrased in terms of quotient groups of free products. The analogous theorem in homology couches its result in terms of exact sequences.

Definition (exact sequences)**.**

1. *Given a sequence (finite or infinite) of groups and homomorphisms:*

$$\dots G_1 \xrightarrow{\phi_1} G_2 \xrightarrow{\phi_2} G_3 \dots$$

   then the sequence is exact at G_2 *if and only if* $\operatorname{im} \phi_1 = \ker \phi_2$.

2. *The sequence is called an* exact sequence *if it is everywhere exact (except at the first and last groups if they exist).*

Theorem 4.52.

1. $0 \to A \xrightarrow{\phi} B$ *is exact at A if and only if ϕ is one-to-one.*
2. $B \xrightarrow{\psi} C \to 0$ *is exact at B if and only if ψ is onto.*
3. $0 \to A \xrightarrow{\phi} B \to 0$ *is exact if and only if ϕ is an isomorphism.*
4. $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ *is exact if and only if $C \cong B/\phi(A)$.*

Theorem 4.53 ($\mathbb{Z}_2$ Mayer-Vietoris)**.** *Let K be a finite simplicial complex and K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. The sequence $\dots \to H_n(K_0 \cap K_1; \mathbb{Z}_2) \to H_n(K_0; \mathbb{Z}_2) \oplus H_n(K_1; \mathbb{Z}_2) \to H_n(K; \mathbb{Z}_2) \to H_{n-1}(K_0 \cap K_1; \mathbb{Z}_2) \to \dots$ using the natural homomorphisms above, is exact.*

Question 4.54. *Use the Mayer-Vietoris Theorem to compute $H_n(M; \mathbb{Z}_2)$ for every compact, triangulated 2-manifold M^2.*

Question 4.55. *What compact, triangulated 2-manifolds are not distinguished from one another by $\mathbb{Z}_2$-homology?*

Question 4.56. *Use the Mayer-Vietoris Theorem to compute $H_n(K;\mathbb{Z}_2)$ for the complexes K pictured below.*

Exercise 4.57. *Use the Mayer-Vietoris Theorem to find the $\mathbb{Z}_2$ homology of the following:*

1. *$\mathbb{S}^n$.*

2. *a cone over a finite simplicial complex (K, T) (that is, the finite simplicial complex $(K\,\mathrm{St}\,v, T\,\mathrm{St}\,v)$ created by adding a vertex v in a higher dimension and creating from each simplex in T, a new simplex with one more vertex, v.).*

3. *a suspension over a finite simplical complex (K, T) (that is, the finite simplicial complex created by gluing two cones over K along K).*

4. *$\mathbb{R}\mathrm{P}^n$ ($= \mathbb{S}^n$ with antipodal points identified).*

Question 4.58. *What are $H_n(L(p,q);\mathbb{Z}_2)$ for $n = 0, 1, 2, \ldots$?*

4.6 Introduction to simplicial $\mathbb{Z}$-homology

In this section, we will refine the concept of homology to include the idea of the orientation of simplices. In $\mathbb{Z}_2$-homology, orientation of simplices was not mentioned. For example, $\mathbb{Z}_2$-homology made no distinction between the edge $[v_0v_1]$ and the edge $[v_1v_0]$. That edge was either present or absent. Failing to consider the orientation of simplices caused $\mathbb{Z}_2$-homology to fail to detect distinctions among some 2-manifolds. Recall that $\mathbb{Z}_2$-homology does not distinguish between $\mathbb{T}^2$, the torus, and $\mathbb{K}^2$, the Klein bottle, for example. Note that orientability is a basic difference between those two spaces.

So our strategy now is to repeat the same development that we used for $\mathbb{Z}_2$-homology, but now taking orientation into account. The effect of this shift in perspective is that now our basic objects will be oriented simplices (in effect, the order of the vertices matters) and they will come with an integer coefficient. These coefficients make the objects of $\mathbb{Z}$-homology become abstractions of the directly geometrical interpretation available when we were talking about $\mathbb{Z}_2$-homology. What does five times a cycle mean geometrically? What does negative two times a simplex mean geometrically? From one point of view, these are just abstract generalizations, but in another sense, the coefficients retain some geometrical meaning. In particular, the negative of a simplex means the same simplex with the opposite orientation. Including these coefficients will allow us to draw finer distinctions among spaces. For example, we will find that $\mathbb{T}^2$ and $\mathbb{K}^2$ have different second homology groups (that is, second $\mathbb{Z}$-homology groups), and, in fact, we will see that the $\mathbb{Z}$-homology groups will successfully distinguish all compact, connected surfaces from one another.

$\mathbb{Z}$-homology is the most common type of homology used, so when people refer to the homology of a space without specifying what type of homology, they are referring to $\mathbb{Z}$-homology.

4.6.1 Chains, boundaries, and definition of simplicial $\mathbb{Z}$-homology

The definition of homology groups (that is, $\mathbb{Z}$-homology groups) of a finite simplicial complex (K, T) involves chains, boundaries, cycles, and the concept of when cycles are equivalent. The building blocks for these ideas are the simplices of (K, T) just as in $\mathbb{Z}_2$-homology; however, in this case the ordering of the vertices makes a difference, so we will be working with ordered vertices of each simplex.

Definition (oriented simplex)**.** *Let (K,T) be a simplicial complex. An* oriented n-simplex $\sigma^n = [v_0v_1 \dots v_n]$ *is an n-simplex $\{v_0v_1 \dots v_n\}$ in T along with a particular ordering of its vertices* up to even permutation. *That is, the* orientation *of an n-simplex is the choice of an equivalence class of all possible ordering of its vertices, where two orderings are equivalent if and only if they differ by an even permutation.*

In other words, the oriented n-simplex with underlying (unoriented) n-simplex $\{v_0v_1 \dots v_n\}$ and whose chosen ordering of the vertices differs from $[v_0v_1 \cdots v_n]$ by an even permutation represents the same oriented n-simplex σ^n. Any ordering of those same vertices that differs from $[v_0v_1 \cdots v_n]$ by an odd permutation is the negative of that oriented n-simplex, and is denoted by $-\sigma^n$.

Definition (n-chain group)**.** *In order to define the n-chain group $C_n(K,T)$, we need to say what the elements are and what the addition operation is. The n-chain group $C_n(K,T)$ is the free abelian group whose set of generators consists of one oriented n-simplex for each n-simplex in (K,T).*

Example 2. *Let (K,T) be the simplicial complex in the plane where*

$$\begin{aligned} T &= \{\, \{(0,0)(0,1)(1,0)\}, \{(0,0)(0,-1)\}, \{(0,-1)(1,0)\}, \\ &\quad \{(0,0)(0,1)\}, \{(0,1)(1,0)\}, \{(1,0)(0,0)\}, \\ &\quad \{(0,0)\}, \{(0,1)\}, \{(1,0)\}, \{(0,-1)\}\} \,. \end{aligned}$$

So K is a filled in triangle and a hollow triangle as pictured.

Then $C_2(K,T)$ has one generator which is an oriented 2-simplex, perhaps, $[(0,1)(0,0)(1,0)]$. So $C_2(K,T)$ is isomorphic to $\mathbb{Z}$. We are free to choose any ordering of the vertices in selecting this generator of $C_2(K,T)$.

$C_1(K,T)$ is a free abelian group on 5 generators, which could be the oriented 1-simplices $[(0,0)(0,-1)]$, $[(1,0)(0,-1)]$, $[(0,1)(0,0)]$, $[(0,1)(1,0)]$, and $[(1,0)(0,0)]$. We are free to choose any ordering of the vertices that we wish for each 1-simplex.

$C_0(K,T)$ is a free abelian group on 4 generators, namely, $[(0,0)]$, $[(0,1)]$, $[(1,0)]$, and $[(0,-1)]$. Of course, in the case of vertices, there is no choice of order involved.

Definition ($\mathbb{Z}$ n-chain)**.** *Each element of $C_n(K,T)$ is called an n-chain. It is the formal sum of oriented n-simplices with coefficients in $\mathbb{Z}$, $\sum_{i=1}^{k} c_i\sigma_i^n$.*

The definition of $C_n(K,T)$ already tells us what the sums of n-chains are, but let's just record it here.

Definition ($\mathbb{Z}$ n-chain sum)**.** *The $\mathbb{Z}$-sum of two n-chains $C = \sum_{i=1}^{k} c_i\sigma_i^n$ and $D = \sum_{j=1}^{m} d_i\nu_j^n$ is given by $C + D = \sum_{i=1}^{k} c_i\sigma_i^n + \sum_{j=1}^{m} d_i\nu_j^n$.*

For example, $3\sigma_1^n + \sigma_2^n + \sigma_2^n - 5\sigma_3^n = 3\sigma_1^n + 2\sigma_2^n - 5\sigma_3^n$ and $\sigma_1^n + (-\sigma_1^n) = 0$, the empty n-chain.

Definition (induced orientation on a subsimplex)**.** *A non-oriented $(n-1)$-subsimplex $\{v_0 \dots \widehat{v_j} \dots v_n\}$ of an oriented n-simplex $\sigma = [v_0 \dots v_n]$ will have the* orientation induced by σ *if it is oriented as $\nu_j = (-1)^j[v_0 \dots \widehat{v_j} \dots v_n]$.*

Definition (boundary of a simplex)**.** *The $\mathbb{Z}$-boundary of an oriented n-simplex $\sigma^n = [v_0 \dots v_n]$ is*

$$\partial(\sigma^n) = \sum_{k=0}^{n} \sigma_k^{n-1} = \sum_{k=0}^{n} (-1)^k [v_0 \dots \widehat{v_k} \dots v_n]$$

That is, the σ_k^{n-1}'s are oriented $(n-1)$-dimensional boundary simplices of σ^n with their orientations induced from σ^n.

Exercise 4.59. *Find the boundary of the oriented 2-simplex $\tau = [v_0v_1v_2]$ and the boundary of the oriented 3-simplex $\sigma = [w_0w_1w_2w_3]$.*

Repeat the procedure for $-\tau$ and $-\sigma$. What is the relationship between the boundary of τ and the boundary of $-\tau$? What is the relationship between the boundary of σ and the boundary of $-\sigma$?

Theorem 4.60. *For any n-simplex σ*

$$\partial(-\sigma) = -\partial(\sigma).$$

Definition ($\mathbb{Z}$-boundary of an n-chain)**.** *The boundary (implicitly $\mathbb{Z}$-boundary) of the n-chain $\sum_{i=1}^{k} c_i\sigma_i^n$ is*

$$\partial\left(\sum_{i=1}^{k} c_i\sigma_i^n\right) = \sum_{i=1}^{k} c_i\partial(\sigma_i^n).$$

Note that boundary of an n-chain is an $(n-1)$-chain.

Definition ($\mathbb{Z}$ n-cycle)**.** *An* n-cycle *(implicitly a $\mathbb{Z}$-n-cycle) is an n-chain whose boundary is zero.*

Theorem 4.61. *For any n-chain C, $\partial(\partial(C)) = 0$, that is, the boundary of any n-chain is an $(n-1)$-cycle.*

Definition (equivalence of cycles)**.** *In a complex (K,T), n-cycles A^n and B^n are* equivalent *if and only if $A^n - B^n = \partial(C^{n+1})$, where C^{n+1} is an $(n+1)$-chain. The equivalence class that A^n is a member of will be denoted by $[A^n]$.*

Definition (n^{th}-simplicial homology)**.** *The* n^{th}-homology *of a finite simplicial complex (K,T), denoted $H_n((K,T);\mathbb{Z})$ or just $H_n(K)$, is the additive group whose elements are equivalence classes of cycles under the equivalence defined above where $[A^n] + [B^n] := [A^n + B^n]$.*

Exercise 4.62. *Show that the addition for $H_n((K,T);\mathbb{Z})$ defined above is well-defined.*

Theorem 4.63. *Let (K,T) be a finite simplicial complex with triangulation T, then $H_n((K,T);\mathbb{Z})$ is an abelian group.*

Theorem 4.64. *Let K be connected and let (K,T) be a finite simplicial complex with triangulation T, then $H_0((K,T);\mathbb{Z})$ is $\mathbb{Z}$. If K has r components, then $H_0((K,T);\mathbb{Z})$ is a free abelian group whose rank is r.*

Exercise 4.65. *Give an oriented triangulation of a Möbius band such that the central circle forms a 1-cycle Z. Show that the boundary 1-cycle Z' is equivalent to either $2Z$ or $-2Z$.*

Definition (∂ operator)**.** *Let (K,T) be a finite simplicial complex. The boundary operator restricted to the n-chain group $C_n(K,T)$ gives a homomorphism $\partial : C_n(K,T) \to C_{n-1}(K,T)$. We sometimes will write ∂_n to emphasize what n we are restricting ourselves to.*

Theorem 4.66. *$\partial^2 = 0$. In other words, for $n \geq 2$ the composition of $\partial : C_n(K,T) \to C_{n-1}(K,T)$ and $\partial : C_{n-1}(K,T) \to C_{n-2}(K,T)$ is the trivial homomorphism. To extend the statement to $n = 1$, we can define $C_{-1}(K,T) = 0$*

Elements of the kernel of the boundary operator are the cycles.

Definition (Z_n)**.** *Let (K,T) be a finite simplicial complex. Then the group* $\ker\partial : C_n(K,T) \to C_{n-1}(K,T)$ *is denoted by $Z_n(K,T)$. Its elements are called cycles.*

Elements of the image of the boundary operator are the cycles that bound.

Definition (B_n)**.** *Let (K,T) be a finite simplicial complex. Then the group $\operatorname{Im}\partial : C_{n+1}(K,T) \to C_n(K,T)$ is denoted by $B_n(K,T)$. Its elements are called bounding cycles.*

These allow us to give an algebraic definition of the homology groups.

Definition (algebraic definition of H_n)**.** *Let (K,T) be a finite simplicial complex. Then $H_n(K;\mathbb{Z}) \cong Z_n(K;\mathbb{Z})/B_n(K;\mathbb{Z})$.*

We refer to this fact as "homology is cycles mod boundaries". You should verify that this definition is equivalent to the first definition of homology given.

As with $\mathbb{Z}_2$ homology, integer homology would be an uninteresting concept if it depended on the particular triangulation selected. Fortunately, as in the $\mathbb{Z}_2$ case, one can show that integer homology is independent of triangulation. So we will write $H_n(K;\mathbb{Z})$ or $H_n(K)$ instead of $H_n((K,T);\mathbb{Z})$ to denote the n^{th}-homology group of the complex K. As with $\mathbb{Z}_2$-homology, $\mathbb{Z}$-homology can be computed from a CW-decomposition. Also, just as in the case of $\mathbb{Z}_2$-homology, $\mathbb{Z}$-homology does not distinguish between homotopy equivalent spaces. Feel free to use these facts in doing the following exercises.

Question 4.67. *For each space K below, describe $H_n(K;\mathbb{Z})$ for $n = 0, 1, 2, 3, \ldots$:*

1. *the sphere.*
2. *the torus.*
3. *the projective plane.*
4. *the Klein bottle.*
5. *the double torus.*
6. *any compact, connected, triangulated 2-manifold.*
7. *the Möbius band.*
8. *the annulus.*
9. *Two (hollow) triangles joined at a vertex.*

Question 4.68. *What is $H_n(K)$ if dim(K)$< n$?*

Question 4.69. *What are $H_n(G)$ for $n = 0, 1, 2, \ldots$ for a graph G?*

Question 4.70. *What are $H_n(\mathbb{S}^k)$ for $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$?*

Question 4.71. *For any n, what is $H_n(M^n)$ where M^n is a connected n-manifold?*

Question 4.72. *For any n, what is $H_n(M^n)$ where M^n is a connected n-manifold with non-empty boundary?*

Question 4.73. *What are $H_n(T)$ for $n = 0, 1, 2, \ldots$ for a solid torus T?*

4.7 Chain groups and induced homomorphisms

The ideas in this section are small variations of those that occurred in the $\mathbb{Z}_2$-homology section.

We will now investigate how simplicial maps from one complex to another induce homomorphisms on homology groups.

Definition. *Let (K, T) and (L, S) be finite simplicial complexes. Let $f : (K, T) \to (L, S)$ be a simplicial map. Suppose that $C = \sum_{i=1}^{k} c_i \sigma_i^n \in C_n(K; \mathbb{Z})$. To get a "natural" definition for the induced map $f_\# : C_n(K; \mathbb{Z}) \to C_n(L; \mathbb{Z})$ we first define $f_\#([v_0 \ldots v_n]) = 0$ if the images of the v_i's are not distinct, and otherwise $f_\#([v_0 \ldots v_n]) = [f(v_0) \ldots f(v_n)]$. We can then extend $f_\#$ to all elements of $C_n(K; \mathbb{Z})$ by $f_\#(C) = \sum_{i=1}^{k} c_i f_\#(\sigma_i^n)$. Notice that the map $f_\#$ is a homomorphism of n-chains of K to n-chains of L.*

Theorem 4.74. *Let $f : K \to L$ be a simplicial map, and let $f_\#$ be the induced map $f_\# : C_n(K; \mathbb{Z}) \to C_n(L; \mathbb{Z})$. Then for any $C \in C_n(K; \mathbb{Z})$, $\partial(f_\#(C)) = f_\#(\partial(C))$. In other words, the diagram:*

$$\begin{array}{ccc}
C_n(K;\mathbb{Z}) & \xrightarrow{\partial} & C_{n-1}(K;\mathbb{Z}) \\
{\scriptstyle \mathrm{f}_\#}\downarrow & & \downarrow{\scriptstyle \mathrm{f}_\#} \\
C_n(L;\mathbb{Z}) & \xrightarrow{\partial} & C_{n-1}(L;\mathbb{Z})
\end{array}$$

commutes.

Definition. *Let (K, T) and (L, S) be finite simplicial complexes. Let $f : K \to L$ be a simplicial map. The* induced homomorphism *$f_* : H_n(K) \to H_n(L)$ is defined by $f_*([A]) = [f_\#(A)]$.*

As before, the induced map can be defined for continuous functions by using the simplicial approximation theorems we proved before. Specifically the induced map for a continuous function f is defined to be the induced map for any simplicial approximation that is homotopic to f.

Theorem 4.75. *Let (K,T) and (L,S) be finite simplicial complexes. Let $f : K \to L$ be a continuous function. Then the induced map $f_* : H_n(K;\mathbb{Z}) \to H_n(L;\mathbb{Z})$ is a well-defined homomorphism.*

The essence of the fact that homotopic continuous functions induce the same homomorphism on homology was used in proving the preceding theorem, but we record the fact below for clarity.

Theorem 4.76. *Let (K,T) and (L,S) be finite simplicial complexes. Let $f, g : K \to L$ be homotopic continuous functions. Then $f_* = g_*$.*

Theorem 4.77. *Let K and L be finite simplicial complexes such that K is a strong deformation retract of L. Then $H_n(K;\mathbb{Z}) \cong H_n(L;\mathbb{Z})$.*

Theorem 4.78. *If A, B are finite simplicial complexes and A is homotopy equivalent to B, then $H_n(A) \cong H_n(B)$.*

Theorem 4.79. *If $f : K \to L$ is a homeomorphism between finite simplicial complexes K and L, then $f_* : H_n(K;\mathbb{Z}) \to H_n(L;\mathbb{Z})$ is an isomorphism.*

In other words, homology groups are topological invariants.

4.8 Relationship between fundamental group and first homology

There is a close connection between the fundamental group of a space and its first homology group.

Theorem 4.80. *Suppose that K is a finite, connected simplicial complex. Then $H_1(K;\mathbb{Z}) \simeq (\pi_1(K))/[\pi_1(K), \pi_1(K)]$, that is, the first homology group of K is isomorphic to the abelianization of the fundamental group of K.*

Sketch of proof. Let $\phi : \pi_1(K) \to H_1((K,T);\mathbb{Z})$ be the map that takes an element $[\alpha]$ of $\pi_1(K)$ to the element $[\alpha_{\#}(\mathbb{S}^1)]$ of $H_1((K,T);\mathbb{Z})$ where α is understood to be a simplicial map from $\mathbb{S}^1$ into (K,T). First note that ϕ is a well-defined, surjective homomorphism.

It remains to show that $\mathrm{Ker}(\phi)$ is the commutator subgroup of $\pi_1(K)$. For this purpose, let $[\alpha] \in \mathrm{Ker}(\phi)$. Let $C^2 = \sum_{i=1}^{k} c_i\sigma_i^2$ be a 2-chain such that $\partial(C^2) = \alpha_\#(\mathbb{S}^1)$. Since $\partial(C^2) = \alpha_\#(\mathbb{S}^1)$, for each edge of any σ_i^2 that is not in $\alpha_\#(\mathbb{S}^1)$, that edge must be cancelled out when computing $\partial(C^2)$. So we can create an abstract 2-manifold whose 2-simplexes are the σ_i^2's where we take several copies of a simplex depending on its coefficient. Using the Classification Theorem of oriented 2-manifolds, we can recognize that α is in the commutator subgroup of $\pi_1(K)$.

4.9 Mayer-Vietoris Theorem

The goal of this section is to describe how the homology groups of a complex are related to the homology groups of pieces of the complex. We begin with some theorems that relate cycles in the whole complex to cycles and chains in its parts.

Theorem 4.81. *Let M be a finite simplicial complex with subcomplexes K and L such that $K \cup L = M$. Suppose $n \geq 2$ and that for every n-cycle Z in $K \cap L$, $Z \sim_{\mathbb{Z}} 0$ in K and $Z \sim_{\mathbb{Z}} 0$ in L, and $H_{n-1}(K \cap L) = 0$. Then $H_n(M) \cong H_n(K) \oplus H_n(L)$.*

The case $n = 1$ is slightly different, namely, suppose for every 1-cycle Z in $K \cap L$, $Z \sim_{\mathbb{Z}} 0$ in K and $Z \sim_{\mathbb{Z}} 0$ in L, and $H_0(K \cap L) = \mathbb{Z}$. Then $H_1(M) \cong H_1(K) \oplus H_1(L)$.

Theorem 4.82. *Let K be a finite simplicial complex and K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. If A_0, A_1 are $(n-1)$-cycles in K_0 and K_1 respectively and if $A_0 \sim A_1$ in K, then there is an $(n-1)$-cycle C in $K_0 \cap K_1$ such that $A_0 \sim C$ in K_0 and $A_1 \sim C$ in K_1.*

Theorem 4.83. *Let K be a finite simplicial complex and K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. Let Z be an n-cycle on K. Then there exist n-chains W_0 and W_1 in K_0 and K_1 respectively such that:*

1. *$Z = W_0 - W_1$ and*

2. *$\partial(W_0) = \partial(W_1)$, and $\partial(W_0)$ is an $(n-1)$-cycle C in $K_0 \cap K_1$.*

Furthermore, if $Z = W_0' - W_1'$ where W_i' is an n-chain in K_i, and $C' = \partial(W_0') = \partial(W_1')$ is an $(n-1)$-cycle, then C' is equivalent to C in $K_0 \cap K_1$.

Question 4.84. *Let K be a simplicial complex and K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. Describe the natural homomorphisms below, and verify that they are homomorphisms:*

1. *$\phi : H_n(K_0 \cap K_1) \to H_n(K_0) \oplus H_n(K_1)$.*

2. $\psi : H_n(K_0) \oplus H_n(K_1) \to H_n(K)$

3. $\delta : H_n(K) \to H_{n-1}(K_0 \cap K_1)$

Theorem 4.85 (Mayer-Vietoris)**.** *Let K be a finite simplicial complex and K_0 and K_1 be subcomplexes such that $K = K_0 \cup K_1$. The sequence $\cdots \to H_n(K_0 \cap K_1) \to H_n(K_0) \oplus H_n(K_1) \to H_n(K) \to H_{n-1}(K_0 \cap K_1) \to \ldots$ using the natural homomorphisms above, is exact.*

Question 4.86. *Use the Mayer-Vietoris Theorem to compute $H_n(M^2)$ for every compact, triangulated 2-manifold M^2.*

Question 4.87. *Use the Mayer-Vietoris Theorem to compute $H_n(K)$ for the complexes K pictured below.*

Exercise 4.88. *Use the Mayer-Vietoris Theorem to find the homology of the following:*

1. $\mathbb{S}^n$.

2. *a cone over a finite simplicial complex (K,T) (that is, the finite simplicial complex $(v * K, v * T)$ created by adding a vertex v in a higher dimension and creating from each simplex in T, a new simplex with one more vertex, v.).*

3. *a suspension over a finite simplical complex (K,T) (that is, the finite simplicial complex created by gluing two cones over K along K).*

4. $\mathbb{R}\mathrm{P}^n$ *(= $\mathbb{S}^n$ with antipodal points identified).*

Question 4.89. *What are $H_n(L(p,q))$ for $n = 0, 1, 2, \ldots$?*

4.10 Degrees of Maps

We are now going to apply homology to maps of spheres.

Definition. *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. Recall that $H_n(\mathbb{S}^n, \mathbb{Z}) \simeq \mathbb{Z}$. Let Z be a generator of $H_n(\mathbb{S}^n, \mathbb{Z})$, then $f_*(z) = kZ$. Then k is called the* degree of the map f *and is denoted* $\deg(f)$.

Theorem 4.90. *Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. Then $\deg(f)$ is well-defined.*

Theorem 4.91. *Let f and $g : \mathbb{S}^n \to \mathbb{S}^n$ be homotopic continuous maps. Then $\deg(f) = \deg(g)$.*

The following theorem describes the degrees of various familiar maps.

Theorem 4.92.

1. *If f is the identity map, then $\deg(f) = 1$.*
2. *If f is the constant map, then $\deg(f) = 0$.*
3. *If f is a homeomorphism, then $\deg(f) = \pm 1$.*
4. *If f is the antipodal map (where $f(x) = -x$ for all x), then $\deg(f) = (-1)^{n+1}$.*

Suppose that we have a map $f : K \to K$, a finite simplicial complex of dimension n. Then $f_{q*} : H_q(K, \mathbb{Z}) \to H_q(K, \mathbb{Z})$ is a linear map, and we can define the following:

Definition (Lefschetz number)**.**

$$\Lambda_f = \sum_{q=0}^{n} (-1)^q trace f_{q*}$$

- Do we need the homology to be over $\mathbb{Q}$ here, as in Armstrong?

Theorem 4.93 (Lefschetz fixed point theorem)**.** *If f has no fixed point then $\Lambda_f = 0$.*

Appendix A

Review of Point-Set Topology

Definition (topology)**.** *A* topology *on a set X is a non-empty collection $\mathcal{T}$ of subsets of X such that:*

1. $\emptyset \in \mathcal{T}$
2. $X \in \mathcal{T}$
3. $\displaystyle\bigcup_{\alpha \in J} U_\alpha \in \mathcal{T}$ for any $\{U_\alpha\}_{\alpha \in J} \subset \mathcal{T}$
4. $\displaystyle\bigcap_{1}^{n} U_i \in \mathcal{T}$ for any finite collection$\{U_1, \ldots, U_n\} \subset \mathcal{T}$.

The set X with a given topology $\mathcal{T}$ is called a topological space*, and is denoted by $(X, \mathcal{T})$ or simply X if the topology is implicitly understood. Elements of X are called* points *of X.*

Definition (open set, neighborhood)**.** *Given a topological space X with a topology $\mathcal{T}$, the sets in $\mathcal{T}$ are called* open sets*. If x contains X and U is an open set containing x, we say U is a* neighborhood *of x.*

Note that the definition of neighborhood includes the fact that it is open. Not all authors use that definition, some say a set $N \ni x$ is a neighborhood of x if there is an open set U such that $x \in U \subset N$. In this case N need not be open.

Example 1.

1. *For any set X, $\mathcal{T} = \{\emptyset, X\}$ is called the* trivial topology *on X.*

2. $\mathbb{R}$ *with*

$$\mathcal{T} = \{\ U = \cup_{i\in J}(a_i, b_i) | a_i < b_i, a_i, b_i \in \mathbb{R}\} \cup \{\emptyset\}\,,$$

called the standard topology *on* $\mathbb{R}$. *All non-empty open subsets of* $\mathbb{R}$ *consist of unions of open intervals.*

The second example above suggests that we can define a topology by giving a subcollection of open sets that generate the toplogy:

Definition (basis for a topology)**.** *A* basis *of a topology is a sub-collection* $\mathcal{B} \subset \mathcal{T}$ *such that for every* $U \in \mathcal{T}$, U *can be written as the union of an arbitrary collection of sets in* $\mathcal{B}$.

Equivalently, $\mathcal{B} \subset \mathcal{T}$ is a basis for $\mathcal{T}$ if for every $U \in \mathcal{T}$, and every $x \in U$, there is some $B \in \mathcal{B}$ such that $x \in B \in \mathcal{B}$.

Definition (limit point)**.** *Given a toplogical space* $(X, \mathcal{T})$*, and* $A \subset X$, $x \in X$ *is a* limit point *of* A *if any neighborhood of* x *contains a point in* A *different from* x. *The set of all limit points of* A *is denoted by* A'.

Note that if $x \in A'$, then x may or may not be in A itself. Also, if $x \in A$, it may or may not be in A'.

Definition (closed set)**.** *Given a toplogical space* $(X, \mathcal{T})$*, then if* $C \subset X$ *contains all of its limit points,* C *is called a* closed set.

Theorem A.1. *Given a toplogical space* $(X, \mathcal{T})$*, then if* $C \subset X$ *closed, then* $C = X - U$ *for some* $U \in \mathcal{T}$.

Note that there can be (and often are) sets that are neither open nor closed, and sets that are both open and closed.

Exercise A.2. *Define a topology in terms of the closed sets. What conditions must a collection of closed sets possess so that their complements form a topology?*

Definition (closure of a set)**.** *Let* $(X, \mathcal{T})$ *be a topological space. Then the* closure *of* $A \in X$*, denoted by* $\overline{A}$ *or* $\operatorname{Cl} A$*, is* $A \cup A'$.

Theorem A.3. *Let* $(X, \mathcal{T})$ *be a topological space. Then the closure of* $A \in X$, $\overline{A}$ *is the smallest closed set containing* A*, in other words, it is the intersection of all closed sets of* $(X, \mathcal{T})$ *that contain* A.

Definition (interior of a set)**.** *Let $(X, \mathcal{T})$ be a set X endowed with the topology $\mathcal{T}$. Then the* interior *of $A \in X$ denoted by $\overset{\circ}{A}$ or $\operatorname{Int} A$, is the largest open set contained in A, in other words, the union of all open sets of $\mathcal{T}$ that are contained in A.*

Definition (boundary of a set)**.** *Let $(X, \mathcal{T})$ be a set X endowed with the topology $\mathcal{T}$. Then the* boundary *of $A \in X$ denoted by* $\operatorname{Bd} A$ *or ∂A, is* $\operatorname{Cl} A - \operatorname{Int} A$.

Definition (subspace topology)**.** *Let $A \subset X$, a topological space with topology $\mathcal{T}$. Then A inherits a topology from X in a natural way: $\mathcal{T}_A = \{U \cap A | U \in \mathcal{T}\}$. A with this topology is called a (topological)* subspace *of X.*

Definition (continuity)**.** *Let X and Y be two sets with their respective topologies, and $f : X \to Y$ a function such that for each open set V in Y, $f^{-1}(V)$ is open in X. Then f is called a* continuous function *from X to Y.*

In other words $f : X \to Y$ is continuous if and only if the *preimage* of every open set in Y is open in X. Note that the image of an open set under a continuous function need not be open.

Definition (quotient topology)**.** *Let X be a topological space and Y be a set and let $f : X \to Y$ be a surjective (onto) function. Then the* quotient topology *on Y is defined by saying that a subset U is open in Y if and only if $f^{-1}(U)$ is open in X.*

Theorem A.4. *Let X be a topological space, $f : X \to Y$ a surjective function onto the space Y that has the quotient topology. Then f is continuous and, furthermore, if the topology on Y included any more sets in addition to those in the quotient topology, then f would fail to be continuous.*

Definition (identification or quotient space)**.** *Let X be a topological space and X^* be a partition of X. Let $f : X \to X^*$ be the map taking each point $x \in X$ to $[x]$, the element of X^* containing x ($[x]$ is called the* equivalence class *of x). Then X^* with the quotient topology is called the* identification space *of X under the partition X^*. The quotient topology applied in this manner to a partition of X is sometimes called the* identification topology.

Definition (homeomorphism)**.** *Let X and Y be two sets with their respective topologies, and $f : X \to Y$ a one-to-one and onto (or surjective) function which is continuous, and such that its inverse function f^{-1} is continuous. Then f is called a* homeomorphism*, and we say that X and Y are* homeomorphic.

Definition (Hausdorff)**.** *A topological space X is* Hausdorff *if for all $x \neq y$ points in X, there are disjoint open sets U and V such that $x \in U$ and $y \in V$.*

Definition (cover)**.** *Let A be a subset of X, a topological space. A* cover *of A is a collection $\mathcal{U}$ of subsets of X such that A is contained (as a subset) in their union. An* open cover *is a cover in which all the sets of the cover are open. A* subcover *is a subcollection of $\mathcal{U}$ that is a cover of A.*

Definition (compactness)**.** *Let A be a subset of X, a topological space. A is* compact *if every open cover of A have a finite subcover.*

Definition (connectedness)**.** *Let X be a topological space. Then X is* connected *if whenever X is decomposed into the union of two disjoint open sets U and V (so $X = U \cup V$ and $U \cap V = \emptyset$), either U or V must be the empty set.*

Definition (path)**.** *Let $p : I \to X$ be a continuous function, where $I = [0, 1]$ with the subspace topology inherited from the standard topology on $\mathbb{R}$ and X a topological space. Suppose $p(0) = x$ and $p(1) = y$. Then p is called a* path *in X from x to y.*

Definition (path connectedness)**.** *A space X is* path connected *if for any two points x and y in X there is a path from x to y*

Definition (local path connectedness)**.** *A space X is* locally path connected *if for every $x \in X$ and neighborhood U of x, there is a path connected open set V containing x such that $V \subset U$.*

The following example is a space with infinitely many circles all tangent at a point. This "Hawaiian earring" sometimes is useful in seeing the limits of generality of some of our theorems

Example 2 (Hawaiian earring)**.**

$$\bigcup_{n=1}^{\infty} \left\{ S_n \in \mathbb{R}^2 \,\middle|\, S_n = \left\{ (x, y) \,\middle|\, \left(x - \frac{1}{2n} \right)^2 + y^2 = \frac{1}{4n^2} \right\} \right\}$$

Definition (metric)**.** *Let X be a set and $d : X \times X \to \mathbb{R}_{0,+}$, a function to the non-negative real numbers, satisfying, for all x, y, and z in X:*

1. *$d(x, y) = 0$ if and only if $x = y$*
2. *$d(x, y) = d(y, x)$*

3. $d(x,z) \leq d(x,y) + d(y,z)$ *(triangle inequality)*

is called a d-metric *on X.*

Definition (metric space)**.** *Let X be a space with a metric d. Then* $\mathcal{B} = \{B_d(x,r) = \{y \in X | d(x,y) < r\}\}$ *is a basis for a topology on X, called the* metric topology. $B_d(x,r)$ *is called the* open ball of radius r centered at x.

Appendix B

Review of Group Theory

Definition (group)**.** *A* group *is a set G along with a binary operation $G \times G \to G$, denoted by $\cdot$ (multiplicative notation) or $+$ (additive notation) satisfying the following three conditions:*

1. *There exists an element $1 \in G$, called the* identity element *such that $g \cdot 1 = 1 \cdot g$ for all $g \in G$. When we are dealing with multiplicative groups we will write 1_G to denote the identity of G.*

2. *For every $g \in G$ there exists an element $g^{-1} \in G$, called the* inverse *of G, such that $g \cdot g^{-1} = g^{-1} \cdot g = 1$.*

3. *For all $g_1, g_2, g_3 \in G$ we have $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ (*associativity*)*

In additive notation the identity element is denoted by 0: $g+0 = 0+g = g$ $\forall g \in G$, the inverse by $-g$ so that we write $g+(-g) = (-g)+g = 0$. Additive notation is usually reserved for *commutative* or *abelian* groups:

Definition (abelian or commutative group)**.** *A group G is* commutative *or* abelian *if $g_1 \cdot g_2 = g_2 \cdot g_1$ for all $g_1, g_2 \in G$.*

Often instead of using $\cdot$ to denote the group operation, we use juxtaposition. In other words, $xy = x \cdot y$. We often use the verb "to multiply" to indicate the group operation.

Definition (trivial group)**.** *The* trivial group *is the group that contains only one element, namely the identity. In other words $G = \{1\}$ or $= \{0\}$ (depending on which notation is being used).*

Definition (permutation)**.** *Let A be a set of n elements. Then a* permutaion *is a bijective function from A to itself. Usually we use positive integers to* *describe A, that is $A = \{1, \ldots, n\}$. Let $\{a_1, \ldots, a_m\} \subseteq A$, then we use $(a_1 a_2 \ldots a_m)$ to represent the function that takes a_i to a_{i+1} for $1 \leq i \leq m-1$ and a_m to a_1. Such a permutation is called an m-*cycle*. A 2-cycle is called a* transposition.

Exercise B.1.

1. *Show that the set of all permutations on n elements forms a group with the group operation of function composition.*
2. *Show that any permutation can be written as a composition of disjoint cycles.*
3. *Show that any m-cycle can be written as a composition of transpositions.*

Definition (order of a group)**.** *Let G be a finite group. Then the group's cardinality $|G|$ is called the* order *of G.*

Definition (symmetric group)**.** *The group of all permutations on the first n positive integers is called the* symmetric group, *denoted by S_n.*

Question B.2. *What is the order of S_n?*

Note that S_n is not an abelian group for $n \geq 3$.

Definition (even and odd permutations)**.** *A permutation is* even *if it can be written as the composition of an even number of transpositions and* odd *otherwise.*

Exercise B.3.

1. *Show that an n-cycle can be written as the composition of $n-1$ transpositions. Thus a 3-cycle is an even permutation and a 4-cycle is an odd permutation!*
2. *Show that the group of even permutations is a subgroup of S_n.*

Definition (alternating group)**.** *The group of even permutations is called the* alternating group, *denoted by A_n.*

Question B.4. *What is the order of A_n?*

Definition (dihedral group)**.** *The symmetry group of a regular n-sided polygon (under composition) is called the dihedral group.*

Exercise B.5. *Show that if we let a represent a reflection along a line passing through the polygon's center and a vertex, and b a rotation of $2\pi/n$ around its center, then the elements of D_n are the set $\{1, b, \ldots, b^{n-1}, ab, \ldots, ab^{n-1}\}$*

Exercise B.6. *Show that in D_n as above, we have $ab = b^{n-1}a$, and thus D_n is not abelian for $n > 2$.*

Definition (subgroup)**.** *A subgroup H of a group G is a subset of G such that H is a group with the binary operation of G.*

Exercise B.7. *Show that D_n is isomorphic to a proper subgroup of S_n.*

Question B.8. *Under what conditions, if ever, is D_n is isomorphic to a subgroup of A_n?*

Since the symmetries of a polygon induce permutations on its vertices, it is easy to see that $D_n \cong H \subset S_n$, and $H \neq S_n$.

Definition (left coset)**.** *Let $g \in G$, a group, and H be a subgroup of G. Then the* left coset *of H by g is*

$$gH := \{gh | h \in H\}.$$

We can define the *right coset Hg* similarly.

Exercise B.9. *Let $g, g' \in G$. Then either $gH = g'H$ or $gH \cap g'H = \emptyset$.*

Definition (index of a subgroup)**.** *Let H be a subgroup of G, then the* index *of H in G, denoted $[G : H]$, is the number of left cosets of H in G.*

Theorem B.10 (Lagrange's Theorem)**.** *Let G be a finite group, and H a subgroup. Then the cardinality $|H|$ of H divides the cardinality $|G|$of G and*

$$[G : H] = \frac{|G|}{|H|}$$

Definition (normal subgroup)**.** *A subgroup H of G is called a* normal subgroup *of G (denoted $H \lhd G$) if $gHg^{-1} = H$, where $aHb := \{ahb | h \in H\}$.*

Multiplying a group or an element on the left by one element and on the right by its inverse is called *conjugation*, so a normal subgroup is one which is unchanged (set-wise) by conjugation.

Theorem B.11. *Let $H \lhd G$ be a normal subgroup. Then its left and right cosets coincide for all $g \in G$, in other words $gH = Hg$ for all $g \in G$.*

Definition (direct product, direct sum)**.** *The* direct product $G \otimes H$ *of two groups G and H is the set $G \times H$ with the group operation defined by $(g, h) \cdot (g', h') = (gg', hh')$. When the groups are additive we call this* direct sum *and write $G \oplus H$.*

Definition (homomorphism)**.** *A function $f : G \to H$ is a (group)* homomorphism *if $f(g \cdot g') = f(g) \cdot f(g')$ for all $g, g' \in G$.*

In other words f preserves the group structure in the image of G.

Definition (isomorphism)**.** *A bijective homomorphism $f : G \to H$ is an* isomorphism*, in which case we say G is isomorphic to H and write $G \cong H$.*

But what about when the homomorphism is not bijective?

Definition (kernel)**.** *The* kernel *of a homomorphism $f : G \to H$ is*

$$\operatorname{Ker} f := \{g \in G | f(g) = 1_H\}$$

Theorem B.12. *An onto homomorphism $f : G \to H$ is an isomorphism if and only if* $\operatorname{Ker} f = \{1_G\}$.

Theorem B.13. *Let $f : G \to H$ be a homomorphism from a group G to a group H, then* $\operatorname{Ker} f \lhd G$.

Definition (quotient group)**.** *A normal subgroup $N \lhd G$ has its left cosets equal its right cosets: $gN = Ng$. Therefore the set $G/N := \{gN | g \in G\}$ of all left cosets of N is a group with the group operation*

$$(gN) \cdot (g'N) := gg'N$$

This group is called the quotient group *of G by N.*

Definition (normalizer of a subgroup)**.** *Let H be a subgroup of G. Then the* normalizer *of H in G is $N(H) = \left\{g \in G | gHg^{-1} = H\right\}$.*

Note that $N(H)$ is a subgroup of G, $H \lhd N(H)$, and that it is the largest subgroup of G in which H is normal, meaning that any subgroup of G containing H in which H is normal must be contained in $N(H)$.

Theorem B.14 (First isomorphism theorem)**.** *Let $f : G \to H$ be an onto (or surjective) homomorphism with* $\operatorname{Ker} f = N$. *Then $H \cong G/N$.*

Definition (cyclic subgroup). *Let $g \in G$. Then $\langle g \rangle$ the* cyclic subgroup generated by g *is the subgroup formed by all powers of g:*

$$\langle g \rangle := \{g^n | n \in \mathbb{Z}\}$$

where $g^n = \overbrace{g \cdot g \cdots g}^{n \text{ times}}$ *if* $n > 0$, $g^0 = 1$, *and* $g^{-n} = \overbrace{g^{-1} \cdot g^{-1} \cdots g^{-1}}^{n \text{ times}}$ *for* $n \in \mathbb{N}$.

Note that with additive notation $\langle g \rangle = \{ng | g \in G, n \in \mathbb{Z}\}$, where $ng = \overbrace{g + g + \cdots + g}^{n \text{ times}}$ for $n \in \mathbb{N}$, $g^0 = 0$, and $-ng = \overbrace{g + g + \cdots + g}^{n \text{ times}}$ for $n \in \mathbb{N}$.

Definition (cyclic group). *If $G = \langle g \rangle$ for some $g \in G$ we say G is a* cyclic group *with* generator g.

Note that cyclic groups are abelian.

Definition (finite cyclic group of order n). *If $G = \langle g \rangle$ and there exists $n \in \mathbb{Z}$ such that $g^n = 1$, then there exists a least $n \in \mathbb{N}$ such that $g^n = 1$. G is said to have* order n, $|G| = n$.

Theorem B.15. *A cyclic group that is non-finite must be isomorphic to $\mathbb{Z}$.*

Theorem B.16. *A finite cyclic group of order n is isomorphic to $\mathbb{Z}_n$, the integers with addition* $\mod n$.

Definition (free abelian group of rank n). *A group* $G \cong \overbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}}^{n \text{ times}}$ *is called the* free abelian group of rank n. *G has a generating set of n elements of infinite order, one for each $\mathbb{Z}$ factor.*

Definition (generators). *Let G be a group and $S \subseteq G$. Then the smallest subgroup H of G containing S is called the* subgroup generated by S. *If $H = G$ then we say G is generated by S, or that S generates G.*

Note that the set of generators of a group is by no means necessarily unique. We can view the subgroup H generated by S as the set of all possible products $g_1 g_2 \ldots g_n$ where $g_i \in S$ or $g_i^{-1} \in S$. We can also view H as the intersection of all subgroups of G that contain S.

Exercise B.17.

1. *Verify that the dihedral group $D_n = \{1, b, \ldots, b^{n-1}, ab, \ldots, ab^{n-1}\}$ is generated by $\{a, b\}$.*

2. *Show that the symmetric group S_n, for $n \geq 2$, is generated by the set of 2-cycles: $\{(12), (23), \ldots, (n-1, n)\}$.*

3. *Show that the symmetric group S_n, for $n \geq 2$, is generated by the pair of cycles (12) and $(12 \ldots n)$.*

Definition (finitely generated group)**.** *A group is* finitely generated *if there exists a finite subset S of G that generates G.*

Theorem B.18 (Classification of Finitely Generated Abelian Groups)**.** *Let G be a finitely generated abelian group. Then G is isomorphic to:*

$$H_0 \oplus H_1 \oplus \ldots \oplus H_m$$

where H_0 is a free abelian group, and $H_i \cong \mathbb{Z}_{p_i}$ $(i = 1, \ldots, n)$ where p_i is a prime. The rank of H_0 is unique, and the orders $p_1, \ldots, p_m$ are unique up to reordering.

Definition (commutator, commutator subgroup)**.** *A* commutator *in a group G is an element of the form $ghg^{-1}h^{-1}$. The* commutator subgroup *G' is the subgroup generated by the commutators of G.*

Theorem B.19. *$G' \lhd G$, and is the smallest subgroup for which G/G' is abelian. In other words, if there is a subgroup $N \lhd G$ such that G/N is abelian, then $G' \subset N$.*

There is a useful notation for groups that, roughly speaking, uses the fact that if we know a set of generators and the "rules" (called "relations") to tell when two elements are the same, then the group (up to isomorphism) is determined by a list of these generators and relations. What follows is a very non-technical description of the generator-relation notation for groups.

For example, in D_n (as described above) it is enough to know that there are two generators a and b, of order 2 and n respectively, and that they satisfy $ab = b^{n-1}a$. These facts determine a complete list of elements $1, b, \ldots, b^{n-1}, ab, \ldots, ab^{n-1}$. The expressioin $ab = b^{n-1}a$ can be written as $aba^{-1}b = 1$, and the word $aba^{-1}b$ is called a *relation*. By a and b's order, we also know that a and b satisfy $a^2 = 1$ and $b^n = 1$. The two letters a and b, together with the above relations completely determine the group D_n, and thus we write:

$$D_n = \langle a, b | a^2, b^n, aba^{-1}b \rangle$$

Similarly, we can write the cyclic group of order n as:

$$C_n = \langle a | a^n \rangle.$$

We can write the infinite cyclic group as:

$$C_\infty = \langle a| \quad \rangle.$$

We can denote the free abelian group of rank n as:

$$F_n^{\text{ab}} = \langle a_1, \ldots, a_n | a_i a_j a_i^{-1} a_j^{-1}, i \neq j \in \{1, \ldots, n\} \rangle$$

Exercise B.20. *Confirm that the lists of generators and relations given above completely determine the groups.*

We should note that since the relations $g \cdot g^{-1} = 1$, $g \cdot 1 = g$ and $1 \cdot g = g$ hold for any $g \in G$, as they are implicit in the definition of a group, such relations are not included in the list of relations. In general a group G can be written as $G = \langle \textit{generators} | \textit{relations} \rangle$. This is called a *group presentation* G. This notation is very useful, especially when dealing with fundamental group and Van Kampen's theorem. The problem with this notation, however, is that it is *very* difficult, in general, given two groups with this notation, to tell if the groups are isomorphic or not, or even if two words represent the same group element.

Exercise B.21. *What is a group presentation for an arbitrary finitely generated abelian group? for the symmetric group?*

Appendix C

Review of Graph Theory

Although a graph is an abstract object (a pair (V, E), where $E = \{v, w\}$, and unordered pair of elements of V, we will look at *embedded* graphs formed of 1-simplices in $\mathbb{R}^n$.

Definition (graph)**.** *A* graph *G is the union of* 1*-simplices* $\{\sigma_i\}_{i=1}^k$ *in* $\mathbb{R}^n$ *such that for $i \neq j$, $\sigma_i \cap \sigma_j$ is empty or an endpoint of each of σ_i and σ_j. The σ_i's are the* edges *of G.*

Definition (walk)**.** *A* walk *is a finite sequence of oriented edges, with the vertices starting at a vertex v_0 and ending at a vertex v_n:*

$$[v_0v_1], [v_1v_2], \ldots, [v_iv_{i+1}], \ldots, [v_{n-1}v_n].$$

Note that any two successive edges share a vertex. We say the walk is a walk from v_0 to v_n*.*

Definition (trail)**.** *A* trail *is a path from v_0 to v_n where no edge is repeated, in other words it is finite sequence of edges vertices starting at vertex v_0 and ending at vertex v_n:*

$$[v_0v_1], [v_1v_2], \ldots, [v_iv_{i+1}], \ldots, [v_{n-1}v_n],$$

and $[v_iv_{i+1}] \neq [v_jv_{j+1}]$ *for* $i \neq j$ *and* $0 \leq i, j \leq n-1$*.*

Definition (circuit)**.** *A* circuit *is a trail from v_0 back to v_0 (also called a closed trail), in other words it is finite sequence of edges vertices starting and ending at a vertex v_0:*

$$[v_0v_1], [v_1v_2], \ldots, [v_iv_{i+1}], \ldots, [v_{n-1}v_0],$$

and no edge is repeated, that is: $[v_iv_{i+1}] \neq [v_jv_{j+1}]$ *for* $i \neq j$ *and* $1 \leq i, j \leq n-1$ *(where the addition in the indices is done modulo n).*

Definition (path)**.** *A* path *is a walk from v_0 to v_n:*

$$[v_0v_1], [v_1v_2], \ldots, [v_iv_{i+1}], \ldots, [v_{n-1}v_n].$$

where $v_i \neq v_j$ when $i \neq j$. We say the walk is a path from v_0 to v_n*.*

Definition (cycle)**.** *A* cycle *is a path from v_0 to v_0 where $v_i \neq v_j$ whenever $i \neq j$, for $0 \leq i, j \leq n$, in other words, it is finite sequence of edges vertices starting and ending at a vertex v_0:*

$$[v_0v_1], [v_1v_2], \ldots, [v_iv_{i+1}], \ldots, [v_{n-1}v_0],$$

where no vertex (other than the starting vertex) is repeated, except for the ending vertex and beginning vertex respectively of two successive edges.

Definition (tree)**.** *A* tree *is a connected graph with no circuits.*

Definition (maximal tree)**.** *Given a connected graph G with edges $\{\sigma_i\}_{i=1}^k$, a subgraph T of G is a* maximal tree *if and only if T is a tree and for any edge e of G not in T, $T \cup e$ has a circuit.*

Theorem C.1. *Let G be a connected graph. Then G contains a maximal tree and every maximal tree for G contains every vertex of G.*

Michael Starbird
Department of Mathematics
RLM 8.100
The University of Texas at Austin
Austin, TX 78712
starbird@math.utexas.edu

Cynthia Verjovsky Marcotte
Department of Mathematics
St. Edward's University
3001 South Congress Ave.
Austin, TX 78704
cynthm@admin.stedwards.edu