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ERIC CLAPHAM
ÅKE GUNNELIN

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Stockholm Institute for Financial Research, Saltmätargatan 19A 11, SE-113 59 Stockholm, Sweden
Phone: +46-8-728 51 20, Fax: +46-8-728 51 30, E-mail: info@sifr.org, Web: www.sifr.org

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Executive Summary

This paper analyzes the relationship between rental expectations and the term structure of lease rates. An important question is whether the term structure contains information regarding expectations of future rents. That is, can we expect an expectations hypothesis similar to that of interest rates to hold? In a rigorous form the expectations hypothesis states that forward rents are unbiased estimates of future spot rents. We examine the conditions under which this holds true and what factors that bias such a hypothesis.

We use a continuous time lease valuation framework, in which both the short lease rate and the short interest rate are stochastic. Since the setting is non-parametric, we are able to distinguish among relationships that hold in general and those that arise from specific models. We show that the same factors that bias the expectations for interest rates (stochastic interest rates and risk aversion) also bias the expectations hypothesis for lease rates. That is, the shape of the term structure of lease rates is not directly related to objective rental expectations. Instead it depends on risk-neutral rental expectations, the correlation between interest rates and rents and the volatility of interest rates. The term structure is upward-sloping if risk-neutral rental expectations are positive and vice versa (ignoring the effect of interest rate uncertainty). The volatility of the interest rate matters in two ways. Firstly, it has a direct effect if there is correlation between the short rent and short interest rate. The sign of this effect is opposite to that of the correlation coefficient, i.e. an increase in the correlation makes the term structure more downward-sloping. Secondly, increasing interest rate volatility changes the term structure of interest rates and thereby indirectly the term structure of rents. With a constant drift rate in the short rent, the effect is such that the absolute value of the slope of the term structure of rents increases

By parameterizing the model we show that the magnitude of the bias of the expectations hypothesis can be quite large. For example, a small increase in the risk aversion parameter can make an upward-sloping term structure turn downward sloping holding the objective expectation constant. Similar effects are obtained when changing the correlation between interest rates and rents or changing the interest rate volatility. We conclude that it is important to carefully analyze the effect of risk aversion and interest rate uncertainty before trying to infer objective rental expectations from the term structure of lease rates.

Rental Expectations and the Term Structure of Lease Rates

Eric Clapham and Åke Gunnelin*

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Abstract

We consider the term structure of lease rates in a general setting where both the interest rate and the short rent are stochastic. Our framework is applicable to any leasing market, but we focus on real estate. We find that the "expectations hypothesis" of lease rates, i.e. that the forward rent is an unbiased estimator of the future short rent, requires similar assumptions as in interest rate theory to hold. To study the magnitude of the bias we parameterize our general framework. The simulations show that different realistic parameter values for risk aversion and interest rate stochastics can generate widely different shapes of the rental term structure, holding the objective rental expectations constant. As a result, an expected increase in rent may very well be consistent with a downward-sloping term structure and vice versa.

There has recently been an increasing interest in lease valuation. Particular attention has been given to the term structure of lease rates, that is, the determination of equilibrium lease rates for different contractual terms. One important issue is the extent to which the shape of the term structure is related to expectations of future market rents.

This paper uses a continuous time lease valuation framework in which both the short lease rate and the short interest rate are stochastic. It is applicable to any valid rent and interest rate process, as well as any leasing object, although we mainly discuss real estate. By using a non-parametric setting it is possible to distinguish among relationships that hold in general and those that arise from specific models. Previous work has also used a deterministic interest rate. Several general expressions relating to the equilibrium term structure of lease rates are derived. Among other things, this allows us to analyze the effect on the term structure of interest rate uncertainty, risk aversion and objective market expectations.

Our work builds on the earlier literature on lease and term structure analysis. Although this strand of the finance literature is far smaller than that dealing with the term structure of interest rates, several papers do exist. Miller and Upton (1976) provide an early analysis based on the equilibrium condition that the present value of lease payments should equal the present value of the service flow from the asset. McConnell and Schallenheim (1983) as well as Schallenheim and McConnell (1985) extend the analysis and value several common types of lease contracts in a parameterized discrete time model.

*Stockholm School of Economics (Clapham) and Stockholm Institute for Financial Research and Royal Institute of Technology (Gunnelin). E-mail: finec@hhs.se and gunnelin@infra.kth.se. We are grateful for valuable comments from Tomas Björk, Peter Englund and John M. Quigley. All errors are ours.

An important contribution is made by Grenadier (1995). He explicitly considers the term structure in a continuous time setting. Grenadier derives the term structure in a competitive industry equilibrium where the short rent process is endogenously determined from expectations of future demand and supply. Grenadier (2002) extends the perfect competition equilibrium by analyzing the term structure of lease rates in an oligopolistic property market. Grenadier (1995, 2002) also provide valuation formulas for many common leasing arrangements, such as forward leases, leases with cancellation or renewal options and indexed leases.

The shape of the term structure in Grenadier's models is determined by expectations of future short term lease rates. In a market in which the short rate is expected to increase, Grenadier argues that the term structure should be upward-sloping since lessors otherwise would prefer to roll over short term leases to take advantage of increasing short lease rates. The opposite will be the case when the short rate is expected to decrease. In an intermediate case, in which the short rate is expected to increase in the short run but thereafter come down again, the term structure can be expected to be single humped (that is, at first upward- and then downward-sloping).

The equilibrium term structure in Grenadier (1995, 2002) thus suggests an "expectations hypothesis" similar to that of interest rates. It is well known that the expectations hypothesis for interest rates does not hold in general, i.e. the forward rate is not an unbiased estimator of the future short rent (for a thorough discussion, see Cox *et al.* [1981]; a more recent contribution is Frachot [1996]). We show that the same factors that bias the expectations hypothesis for interest rates, namely stochastic interest rates and risk aversion, also bias the expectations hypothesis for lease rates. Furthermore, by parameterizing the framework and undertaking simulations we show that the magnitude of the bias can be large. Different sets of realistic parameter values for the interest rate and risk aversion can generate widely different term structures even though the rent expectation under the objective probability measure is kept constant. For example, the slope of the term structure is very sensitive to risk aversion towards rent. A small increase in risk aversion can make an upward-sloping term structure turn downward-sloping and vice versa. Similarly, changing the correlation between interest rates and rents or changing the interest rate volatility can invert the term structure. Hence, our simulations show that careful analysis of the risk aversion prevailing in the market and of interest rate- and rent stochastics is required before trying to infer objective rental expectations from the term structure.

The paper is organized as follows: In section one, we present the continuous time framework used and review forward rental agreements. In section two we derive and analyze the term structure of lease rates. In particular we consider the expectations hypothesis and the relationship between the rental and bond market term structure. Further, section three applies the framework and presents a simple parameterized model. Section four presents a brief conclusion.

1 The model

We consider a frictionless market containing bonds and contracts on the service flow from a standardized asset. We will think of the asset mainly as property, but it could be any asset. The basic objects of study are two stochastic processes:

- The instantaneous spot rent per unit time, denoted X_t. The rental income over an infinitesimally short time period dt is thus $X_t dt$.
- The short interest rate, denoted r_t.

For ease of presentation we assume that the asset corresponds to only one leasing unit and we also abstract from operating costs. Hence, the value of the asset H_t equals the present value of all future lease payments from the single leasing unit.

The spot rent may be modelled as an exogenous process, or be endogenously determined through some form of equilibrium condition. For instance, several papers model X_t as an exogenous geometric Brownian motion with constant drift rate (e.g. Stanton and Wallace, 2002), while in the model of Grenadier (1995), the spot rent is a function of current demand and supply. We do not at this stage parameterize X_t, as we wish to study properties that hold for any valid spot rent process. For the rent process to be valid it must be non-negative and such that $H_t < \infty$, for all t.

The technical setup is the general continuous time framework, as presented in e.g. Björk (1998). We thus use the "risk-neutral pricing" method where processes are adjusted to reflect risk aversion in the market and discounting is done with the risk free short interest rate. Originally the use of risk-neutral pricing was motivated using arbitrage arguments for a traded asset. However, as argued by Rubinstein (1976) and commonly applied in e.g. the real options literature, risk-neutral drift rates can also be inferred from general equilibrium arguments. This makes it possible to use risk-neutral pricing even in the presence of market imperfections. Risk-neutral expectations are denoted using notation $E_t^Q[\cdot]$.

The asset value is defined as the present value of future spot rents,

$$H_t = E_t^Q\left[\int_t^\infty e^{-\int_t^s r_u du} X_s ds\right]. \tag{1}$$

We also introduce notation $\delta_t = X_t/H_t$. The quantity δ_t is thus the continuous dividend yield, or payout ratio. The local drift rate of the asset will as usual be equal to the risk free rate less the dividend yield, or $r_t - \delta_t$, under the risk-neutral measure.

Recently the technique known as change of numeraire has seen increased use (for a textbook treatment see Björk [1998]). This technique will be useful when considering forward contracts in the next section. Usually, we think of the bank account as the numeraire, but any asset may be used, for instance a bond. For the arbitrage free price process π_t on the stochastic claim $\mathcal{X}$ paid out at T we therefore have that,

$$\pi_t = E_t^Q\left[e^{-\int_t^T r_s ds} \cdot \mathcal{X}\right] = p(t,T)E_t^T[\mathcal{X}]. \tag{2}$$

Simply put, we account for the correlation between the two objects by switching to a new measure, known as the T-forward neutral measure. The term $p(t,T)$ is the price at t of a standardized zero coupon bond giving one certain unit of account at time T. It is as usual defined as the risk-neutral expectation over the short interest rate r_t:

$$p(t,T) = E_t^Q\left[e^{-\int_t^T r_s ds}\right]. \tag{3}$$

1.1 Forward contracts

When deriving an expression for the term structure of lease rates, it is convenient to have an expression for forward contracts on the spot rent stream or the asset. This is also necessary for a discussion of the expectations hypothesis. We therefore begin by deriving expressions for these objects.

A forward contract is such that the holder of the contract pays a fixed amount determined today and receives a stochastic amount at a future date. The contract is set up such that its initial value is zero. In our case, forward contracts could be made on both the underlying rent stream and the asset.

We define[1] $f(t,T)$ as the forward rate for renting the asset over an infinitesimally short interval at T. Thus,

$$E_t^Q \left[e^{-\int_t^T r_s ds} \left(f(t,T) - X_T \right) \right] = 0. \tag{4}$$

Since $f(t,T)$ is known at time t it can be moved out of the expectation:

$$f(t,T) = \frac{E_t^Q \left[e^{-\int_t^T r_s ds} X_T \right]}{p(t,T)}. \tag{5}$$

It follows directly from (2) and (5) that $f(t,T)$ can be written as,

$$f(t,T) = E_t^T \left[X_T \right]. \tag{6}$$

The asset forward price $F(t,T)$ must similarly be,

$$F(t,T) = E_t^T \left[H_T \right]. \tag{7}$$

By inserting the definition of H_T into (7) and applying iterated expectations we obtain,

$$F(t,T) = \frac{\int_T^\infty p(t,s) f(t,s) ds}{p(t,T)}. \tag{8}$$

In particular,

$$H_t = F(t,t) = \int_t^\infty p(t,s) f(t,s) ds. \tag{9}$$

2 Term structure of lease rates

The underlying approach for determining the equilibrium term structure follows the basic principle used in Miller and Upton (1976), McConnell and Schallenheim (1983), Schallenheim and McConnell (1985), Grenadier (1995, 2002) and Stanton and Wallace (2002). The starting point is that leasing is equivalent to purchasing the service flow from the underlying asset for a specified period of time. In equilibrium the rent on leases of all maturities must adjust in such a way that the present value of the rental payments equal the present value of the acquired service flow.[2] From this equilibrium relationship it is straightforward to derive the term structure.

We denote by $R(t,T)$ the fixed rate at time t for using the asset from time t to time T. The present value of the lease payments must be equal to the present value of the service flow the asset provides during the same period:

$$E_t^Q \left[\int_t^T e^{-\int_t^s r_u du} \left(R(t,T) - X_s \right) ds \right] = 0. \tag{10}$$

[1] Some equivalent approaches to represent the instantaneous forward rate are given in appendix A.1.

[2] This approach abstracts from transaction costs. For an in-depth discussion of transaction costs see Miceli and Sirmans (1999). In their static two-period model transaction costs are pivotal and induce landlords to offer lower rent on longer leases in order to minimize the turnover. Thus, their model implies that the term structure of real estate lease rates should generally be downward-sloping. However, this result is partly an effect of keeping the rent constant over the two periods. Higher rent in the second period would allow for an upward-sloping term structure. Nevertheless the result suggests that incorporating transaction costs in our model would result in less upward-sloping (more downward sloping) term structure for any given set of model parameter values.

That is,

$$R(t,T) = \frac{E_t^Q \left[\int_t^T e^{-\int_t^s r_u du} X_s ds \right]}{\int_t^T p(t,s) ds}. \tag{11}$$

As shown below, expression (11) can be re-expressed in terms of forward rates, either on the short rent or the asset:

$$R(t,T) = \frac{\int_t^T p(t,s) f(t,s) ds}{\int_t^T p(t,s) ds} \tag{12}$$

$$R(t,T) = \frac{H_t - F(t,T) p(t,T)}{\int_t^T p(t,s) ds}. \tag{13}$$

The latter representation is similar to the one used by Grenadier (1995, 2002).[3] We now show the results in (12) and (13). The nominator in (11) can be re-expressed in terms of the instantaneous forward rates or in terms of the forward asset price. In the former case we have,

$$\begin{aligned} E_t^Q \left[\int_t^T e^{-\int_t^s r_u du} X_s ds \right] &= \int_t^T E_t^Q \left[e^{-\int_t^s r_u du} X_s \right] ds = \\ \int_t^T p(t,s) E_t^s \left[X_s \right] ds &= \int_t^T p(t,s) f(t,s) ds. \end{aligned} \tag{14}$$

From this formula (12) follows immediately. In order to prove (13) we have the following calculations:

$$\begin{aligned} & E_t^Q \left[\int_t^T e^{-\int_t^s r_u du} X_s ds \right] = \int_t^T p(t,s) f(t,s) ds \\ = & \int_t^\infty p(t,s) f(t,s) ds - \int_T^\infty p(t,s) f(t,s) ds \\ = & H_t - p(t,T) F(t,T). \end{aligned} \tag{15}$$

This result has the interpretation that we may think of buying the asset at t while simultaneously entering into a forward contract to sell the asset at T.

2.1 Properties of the term structure

By taking the limit of (11) we obtain,

$$\lim_{T \to \infty} R(t,T) = \frac{H_t}{\int_t^\infty p(t,s) ds} = \widetilde{r}_t H_t. \tag{16}$$

In the above, $\widetilde{r}_t$ denotes the yield-to-maturity of a consol bond[4]. Further, when the length of the lease goes to zero,

$$R(t,t) = f(t,t) = X_t = \delta_t H_t. \tag{17}$$

[3] Grenadier (1995, 2002) prefers to work in terms of a call option with zero exercise price, rather than forward contracts. These methods are equivalent as explained in appendix A.1.

[4] A consol bond gives an infinite continuous payment stream of one unit of account. Its yield-to-maturity, or internal rate of return, is the constant discount rate that gives the same present value of the payment stream as the market price:

$$Co_t = \int_t^\infty p(t,s) ds = \int_t^\infty e^{-\widetilde{r}_t (s-t)} ds = \frac{1}{\widetilde{r}_t}$$

By differentiating (12) with respect to T we obtain,

$$\frac{\partial R(t,T)}{\partial T} = \frac{p(t,T)}{\int_t^T p(t,s)ds}\left[f(t,T) - R(t,T)\right]. \tag{18}$$

As seen by expression (18) the term structure of lease rates is locally increasing in T whenever $f(t,T) > R(t,T)$ and vice versa. When $f(t,T) = R(t,T)$ it follows that $R(t,T)$ has a stationary point. The relationship between $f(t,T)$ and $R(t,T)$ is very similar to the relationship between marginal and average costs in basic microeconomics. It can be seen that $R(t,T)$ is a weighted average of instantaneous forward rates. Therefore if $f(t,T)$ is a consistently increasing or decreasing function of T, then so is $R(t,T)$. Also, if $f(t,T)$ is strictly concave or convex, then so is $R(t,T)$. Note also that even if $f(t,T)$ tends to infinity or zero as T increases, $R(t,T)$ still converges to the annuity of the asset price. Very informally, $R(t,T)$ is a smoothed version of $f(t,T)$.

It is tempting to compare the term structure of rents to the term structure of zero coupon bond yield. However, we are swapping a stochastic short rent process for a constant payment. The natural analogy to the term structure of rents is therefore the term structure of swap rates, denoted $R^r(t,T)$, in the bond market. To see why this is so, note that if the short rent process is replaced by the short interest rate in the definition of the swap rate given by (10) we obtain,

$$E_t^Q\left[\int_t^T e^{-\int_t^s r_u du}\left(R^r(t,T) - r_s\right)ds\right] = 0. \tag{19}$$

This is this the same expression as for the term structure for lease rates, but with the short interest rate instead of the short rent. Solving for $R^r(t,T)$ gives after some manipulations,

$$R^r(t,T) = \frac{1 - p(t,T)}{\int_t^T p(t,s)ds}. \tag{20}$$

This is also a version of the result reported by Duffie and Singleton (1997). Note the similarity with (13) above.

In appendix A.1 we further consider forward lease agreements over discrete future intervals. In appendix A.2 we derive the real or indexed term structure and in appendix A.3 the term structure in the presence of credit risk is considered.

2.2 The expectations hypothesis

Grenadier (1995, 2002) argues that the shape of the term structure of lease rates should reveal expectations of future short lease rates. This hypothesis has an obvious parallel to the expectations hypothesis of interest rates, which loosely says that the slope of the yield curve is related to expectations of future short interest rates. A more rigorous form of the expectations hypothesis states that forward interest rates are unbiased estimates of expected future interest rates. As is familiar, this form of the expectations hypothesis only holds if the interest rate is deterministic. With uncertain interest rates, the bias of the hypothesis will depend on the stochastic properties of the short interest rate and the degree of risk aversion against interest rate uncertainty that is prevailing in the market.

In this section we examine the factors that might bias a similar expectations hypothesis of lease rates. Continuing the parallel to interest rates, we formulate the expectations hypothesis as in Grenadier (2002):

$$f(t,T) = E_t^P\left[X_T\right], \tag{21}$$

that is, the forward lease rate is an unbiased estimate of the future short lease rate.

We now study the expectations hypothesis more carefully and we give the instantaneous forward rate again for convenience:

$$f(t,T) = E_t^T [X_T] = \frac{E_t^Q \left[e^{-\int_t^T r_s ds} X_T \right]}{p(t,T)}. \tag{22}$$

The expectations hypothesis is therefore always true under the forward neutral measure. This is different for each maturity however, and has no simple link with objective expectations.

The forward rate in (22) can be rewritten using the definition of covariance:

$$f(t,T) = E_t^Q [X_T] + \frac{Cov_t^Q \left(e^{-\int_t^T r_s ds}, X_T \right)}{p(t,T)}. \tag{23}$$

Therefore, if at least one the two processes r_t and X_t is deterministic or they are uncorrelated under Q, then the result reduces to,

$$f(t,T) = E_t^Q [X_T], \tag{24}$$

that is, the expectations hypothesis holds true under the risk-neutral measure. Note that covariance is in levels rather than stochastic increments. Even if the increments of the stochastic processes for the short rent and short interest rate are independent under the objective probability measure, it may still be that the processes are dependent in levels under the subjective probability measure Q. This is because the drift adjustments when moving to the subjective probability measure may in general be stochastic and correlated.

Further, if the short rent is deterministic or the economy is risk-neutral with respect to at least that process, then the distinction between the objective and the risk-neutral measure disappears. We then obtain,

$$f(t,T) = E_t^P [X_T]. \tag{25}$$

To summarize, we find that if the short rent is deterministic, then the instantaneous forward lease rate is an unbiased estimator of future rents. This is somewhat similar to the expectations hypothesis of interest rates, which holds if and only if the interest rate is deterministic. However, the expectations hypothesis for rents may hold even if the short rent is stochastic. This is the case if the market is risk-neutral towards the rent process while either the short interest rate is deterministic or the short rent and the short interest rate are uncorrelated in levels under Q. Since these requisites are unlikely to hold empirically, we would not expect to find a property market in which the expectations hypothesis of rents holds fully.

The fact that the expectations hypothesis does not hold in the general case implies that the shape of the term structure is not directly related to objective rent expectations. The shape instead depends on (*i*) the risk-neutral drift rate of the rent, (*ii*) the Q-covariance between the short rent and the short interest rate and (*iii*) the term structure of interest rates. Some intuition for this can be given by first examining expression (23). Firstly, all else equal, the forward rent is an increasing function of the risk-neutral drift rate of the short rent. Secondly, the forward rate also depends on the Q-covariance between rent changes and interest rate changes. The more positive (negative) the covariance, the higher (lower) the forward rate. Thirdly, the covariance term is weighted by the inverse of the corresponding zero coupon bond price. Thus, the forward rate also depends on the term structure of interest rates.

The definition of the fixed lease rate $R(t,T)$ implies that it is a weighted average of instantaneous forward rates with declining weights. Thus the level of forward rates feeds into the level of the fixed rate. Further, from expression (18) we see that the term structure curve, given by $R(t,T)$, is locally upward-sloping (downward-sloping) when the instantaneous forward lease rate is higher (lower) than the fixed lease rate of the same maturity. In conclusion, the shape of the term structure is a function of the forward rates and therefore depends on the three factors given above.

2.3 Interpretation of the term structure

Since the analysis above holds for any valid parameterization of the short lease rate and interest rate processes, we can interpret the term structure results in the previous literature within our framework. In Grenadier (1995) and Stanton and Wallace (2002) investors are risk averse but the interest rate is deterministic. These assumptions yield the version of the expectations hypothesis given by expression (24), i.e. the expectations hypothesis holds under the risk-neutral measure. Hence the shape of the term structure is determined by risk-neutral rent expectations. The monotonically upward-sloping (downward-sloping) term structure in the simulations in Grenadier (1995) is obtained when the risk-neutral drift rate is strictly positive (negative). The single humped term structure is obtained when the risk-neutral drift rate of the short rent is decreasing in the time argument and goes from positive to negative. Similarly, the empirically estimated term structures in Stanton and Wallace (2002) should be related to risk-neutral rent expectations. In Grenadier (2002) investors are risk-neutral and the interest rate is deterministic. With this setup the expectations hypothesis, according to expression (25), holds under the objective probability measure. Hence in this model the shape of the term structure has a one to one correspondence to rental expectations under the objective probability measure.

Our results underline that one needs to be careful when interpreting empirical term structures in the property market. Only with the tight restrictions that either the rent is deterministic or alternatively that market participants are risk-neutral and that interest rates are deterministic or uncorrelated with the rent, is it possible to directly infer objective market expectations from the shape of the term structure. As a result, the short rent may easily be expected to decrease under the risk-neutral measure but increase under the objective measure. Hence, an expected increase in the rent level may very well be consistent with a downward-sloping term structure. The opposite scenario, i.e. that the term structure is upward-sloping when the short rent is expected to decrease is, however, arguably more unlikely (ignoring effects of interest rate uncertainty). This would require the risk-neutral drift rate to be higher than the objective drift rate, which is a less likely scenario in the real world. Furthermore, different scenarios of interest rate uncertainty can also lead to different term structures for the same objective rent expectations. In section 3 we parameterize our model to develop an understanding for the degree to which the expectations hypothesis is distorted in different scenarios for risk aversion and interest rate uncertainty.

3 A parameterized model

As demonstrated in section 2, the ability to infer market expectations about the level of future lease rates from the term structure depends upon on how seriously the expectations hypothesis is distorted by risk aversion and interest rate uncertainty. In this section, we parameterize the framework derived in section 2 and perform simulations to study these questions. For tractability we first assume a simple geometric Brownian motion rent process and a Vasicek short interest rate process. In section 3.3 we also consider a mean reverting rent process.

Thus assume the following under Q:

$$
\begin{aligned}
dr_t &= \eta(\alpha - r_t)dt + \sigma_r dW_t^1 && (26)\\
dX_t &= \mu X_t dt + \sigma_x X_t dW_t^2 && (27)\\
dW_t^1 dW_t^2 &= \rho dt. && (28)
\end{aligned}
$$

The risk-neutral drift rate μ is defined as,

$$\mu = \mu^P - \lambda\sigma_x, \tag{29}$$

where μ^P is the objective drift rate and λ is the market price of risk. Note that in the general case, the market price of risk can be stochastic and time dependent; in this parameterization we however assume a constant market price of risk.

Further, by using the definition of $f(t,T)$ it can be shown that (see appendix A.3 for a derivation, where Vasicek bond prices are also given),

$$f(t,T) = X_t e^{\mu(T-t) - \frac{\sigma_z \sigma_r \rho}{\eta}\left(T-t-\frac{1-e^{-\eta(T-t)}}{\eta}\right)}. \tag{30}$$

The term structure of lease rates is easily derived using earlier results. That is,

$$R(t,T) = X_t \frac{\int_t^T p(t,s) e^{\mu(s-t) - \frac{\sigma_x \sigma_r \rho}{\eta}\left((s-t)-\frac{1-e^{-\eta(s-t)}}{\eta}\right)} ds}{\int_t^T p(t,s)ds}. \tag{31}$$

Note that the expression for the instantaneous forward rate is analytical, while expression (31) is easily solved by means of a numerical integration.

3.1 Effect of risk aversion

In the following we analyze how risk aversion affects the term structure of lease rates. To suppress the effect of interest rate uncertainty, we assume a constant interest rate, that is $\eta = 0$ and $\sigma_r = 0$.

For the numerical analysis we will use the following base case parameters:

$$
\begin{array}{ll}
\mu^P = 0.04 & \lambda = 0.15 \\
\sigma_x = 0.20 & r = 0.06 \\
X_t = 1. &
\end{array}
$$

In the base case the risk-neutral drift rate of the rent, $\mu = \mu^P - \lambda\sigma_x = 0.04 - 0.15 \cdot 0.20 = 0.01$, or 1%.

Figure 1a displays the base case. A constant positive risk-neutral drift rate implies, as discussed in section 2, a monotonic upward-sloping term structure of the rent when the interest rate is deterministic.

Figure 1b shows the term structure when the risk aversion parameter is increased to 0.25, i.e. the risk-neutral drift rate is reduced to -1%. As discussed in section 2 and shown in the figure, a negative risk-neutral drift rate implies a downward-sloping term structure.

Note that the objective drift rate of the rent is the same in both scenarios, i.e. the dramatic difference between the shape of the term structure in the two figures is only attributed to a slight change in assumptions regarding risk aversion. It is worth stressing that both choices of risk aversion parameter are compatible with risk-return relationships that typically can be found in the property market. This statement can be motivated in the following way.

Assume for tractability that the interest rate is constant and that the short rent follows a geometric Brownian motion, i.e. the base case scenario. With these assumptions the definition of the property value given by expression (1) simplifies to $H_t = X_t / (\mu - r)$. Hence, since r and μ are constants, the rent process and the property value processes are identical and consequently they have the same risk-neutral drift rate. Since we know from the general discussion in section 1 that the risk-neutral drift rate of the asset price is equal to the risk free rate less the payout ratio (δ), we have that $\mu = r - \delta$. A risk-neutral drift rate of the rent and the property value of $\mu = 1\%$, as in the base case, implies a property market in which the payout ratio equals $\delta = r - \mu = 6\% - 1\% = 5\%$ and the required total rate of return equals $\mu^P + \delta = 4\% + 5\% = 9\%$. In the second scenario, the payout ratio and the required total rate of return equals 7% and 11% respectively. Both these scenarios can realistically be found in the property market.

Figure 1c shows the term structure when the base case is changed by substituting the positive objective drift rate for a negative drift rate of 1%, that is $\mu^P = -0.01$. As expected the term structure is downward-sloping since the risk-neutral drift rate is even more negative, namely -4%. Theoretically it would be possible to obtain a positive term structure when the objective drift rate of the short rent is negative, but this would require an assumption of a negative risk aversion parameter (since we assume a constant interest rate), which we find less likely.

3.2 Effect of interest rate uncertainty

We now go on to study the impact of interest rate uncertainty and the correlation between the short interest rate and the short rent. The following parameters are used:

$$\begin{array}{cc} \mu^P = 0.04 & \lambda = 0.15 \\ \sigma_x = 0.20 & \eta = 0.10 \\ \alpha = 0.06 & \sigma_r = 0.03 \\ X_t = 1. & r_t = 0.06 \end{array}$$

The above parameters again gives a risk-neutral drift rate of the short rent equal to 1%, i.e. $\mu = 0.01$. Figure 2a-d show the simulation results. In addition to the case with $\sigma_r = 0.03$ there is also a thinner line corresponding to $\sigma_r = 0.02$. As seen in the figures, different assumptions regarding correlation changes the term structure significantly. The higher the correlation, the more downward sloping the term structure becomes. Also the volatility of the short interest rate matters. If the correlation between the short rent and the short interest rate is positive, then a higher interest rate volatility will make the term structure more downward sloping (or as an intermediate effect single humped). In case of negative correlation, the reverse relationship holds true.

In Grenadier (1995, 2002), a single humped term structure is associated with expectations of new property supply in the medium term. However, as the above shows, the humped shape can also be the result of interest rate uncertainty.

3.3 Effect of trend reverting rent

In the previous sections rents are assumed to follow a lognormal distribution with constant drift term. The literature on rental adjustment processes suggest, however, that the rent process is better described as mean-reverting or trend-reverting. Recent examples are Hendershott *et al.* (1999) and Hendershott *et al.* (2002). In these adjustment models, rents are constrained to return to their long run average. The gap between actual and trend level rent is found to have explanatory power for rent changes and suggests that rents revert towards the average rent level. Hendershott *et al.* (2002) also estimates error correction models, in which they find significant error correction coefficients, which also indicates reverting rents.

In this section we present a parameterization of our basic model that allow for trend reversion in the rent process. As in Lo and Wang (1995), we implement trend reversion in a continuous time setting by modelling the logarithm of the stochastic process as a trending Ornstein-Uhlenbeck process. Thus assume under Q a Vasicek short interest rate and a log spot rent process, $Z_t = \ln X_t$, that follows a trending Ornstein-Uhlenbeck process:

$$\begin{aligned} dr_t &= \eta(\alpha - r_t)dt + \sigma_r dW_t^1 && (32) \\ dZ_t &= \left[\gamma(\mu t - Z_t) + \mu\right] dt + \sigma_z dW_t^2 && (33) \\ dW_t^1 dW_t^2 &= \rho dt. && (34) \end{aligned}$$

Further, by using the definition of $f(t,T)$ it can be shown that (see appendix A.5),

$$f(t,T) = \left(\frac{X_t}{e^{\mu t}}\right)^{e^{-\gamma(T-t)}} e^{\mu T - \frac{\sigma_z \sigma_r \rho}{\eta}\left(\frac{1-e^{-\gamma(T-t)}}{\gamma} - \frac{1-e^{-(\eta+\gamma)(T-t)}}{\eta+\gamma}\right) + \frac{\sigma_z^2}{4\gamma}\left(1-e^{-2\gamma(T-t)}\right)}. \quad (35)$$

This is a simple way of modelling rents that follow a long term trend but due to occasional shocks or business cycles, are pushed away from the trend level. When this happens, rents tend to be pulled back to the trend level. When the economy is off the trend, the best forecast is that it will eventually converge back to trend, but nothing beyond that, i.e. when new major shocks might occur, can be predicted. In appendix A.4 it is further shown that the specification of Z_t is also consistent with a trend-reverting rent under the objective probability measure.

To study the effect of trend-reversion on the term structure we use a constant interest rate and further,

$$\begin{aligned} \mu &= 0.01 \quad & \gamma &= 0.2 \\ \sigma_z &= 0.20 \quad & r &= 0.04 \end{aligned}$$

Figure 3a-c displays the term structure in the case when the initial rent is at its trend value ($X_t = 1$) as well as 50% below and 50% above. Since the rent in Figure 3b initially is below its trend value, the risk-neutral drift rate is initially higher than when rent is at its equilibrium. Hence, for shorter terms, the slope of the term structure curve is steeper compared to Figure 3a. When the rent follows a geometric Brownian motion the term structure can only be single humped as an effect of interest rate uncertainty. When rents are trend reverting, a single humped term structure can also occur due to the trend reversion. The humped term structure in Figure 3c occurs because the risk-neutral drift rate of the rent goes from negative (the short term reversion effect) to positive.

Finally, Figure 3d plots the case when the drift rate is negative ($\mu = -0.01$), but the rent is currently 50% below trend. This leads to a term structure with humped shape that is initially upward sloping. For shorter lease, the positive mean reversion effect is larger than the drift effect.

3.4 Discussion of simulation results

The simulations show that risk-aversion, interest rate uncertainty and trend reversion significantly affect the shape of the term structure in our model. These results once again underline that great care is needed when attempting to infer expectations of future rent from the term structure. A specific shape can be attributed to a number of scenarios for the economy and the specific property market. Hence, in order not to draw erroneous conclusions regarding objective rent expectations from the term structure, careful analysis is required.

Gunnelin and Söderberg (2002) and Englund *et al.* (2002) provide empirical evidence that can be interpreted in the light of the above simulations. Gunnelin and Söderberg found mainly positive term structures during the pronounced upswing in the office market of the Stockholm CBD during the 1980's. Since risk-neutral rent expectations, as shown in section 2.2, are pivotal for the slope of the term structure, this indicates that the expectations were mainly positive during this time period. Although it is difficult to measure risk-neutral expectations correctly, the fact that the risk free interest rate was higher than the payout ratio in the Stockholm CBD during the whole time period supports that this was the case (see discussion in section 3.1).

Since the risk-neutral drift rate typically is lower than the objective, we would, however, expect the term structure to be less steeply upward-sloping than would be the case if objective rental expectations was the main determinant of the term structure. This is also consistent with the findings in Gunnelin and Söderberg. During the second half of the 1980's, rental expectations were extremely high in the Stockholm CBD and rents doubled during this time period. Although positive term structures were found, they were not as steeply upward-sloping as one would expect if objective rental expectations were the main factor determining the shape of the term structure.

Englund *et al.* (2002) study the term structure in the same office market during the time period 1998-2002. The rent increase during this period was of similar magnitude as that during the peak of the boom in the late 1980's. As was the case in the study of Gunnelin and Söderberg, the very high rental increase does not translate into steeply upward-sloping term structures. Instead, for most of the years the estimated term structures are trendless or slightly positive. If we once again look at risk-neutral rental expectations, this result seems plausible. The risk-free interest rate was lower than the payout ratio during the whole period under study, indicating that risk-neutral expectations were low or even negative. It should, however, be pointed out that the use of the difference between the interest rate and the payout ratio as an indicator of risk-neutral expectations is based on the assumption that the rent is at its trend value. The fact that rents doubled from 1998 to 2002 indicates that this was not the case. Considering the severe down-turn of the Stockholm property market in the first half of the 1990's, it is more likely that the rental market exhibited a positive mean- or trend-reversion. As shown in Figure 3b and 3d, when rents are below trend, the term structure is more upward-sloping compared to when rents are at the trend level. Hence, taking trend reversion into consideration, the negative difference between interest rates and payout ratio may very well be consistent with a slightly positive term structure. Another possible explanation could be that correlation between interest rates and rents in the Stockholm office market was negative, which as shown in Figure 2, tends to increase the slope of the term structure. However, since no study has attempted to model the covariance under the risk-neutral measure between interest rate levels and rent levels in the Stockholm CBD, we cannot confirm or reject this hypothesis.

4 Conclusion

In this paper we have extended previous work on the term structure of lease rates by deriving equilibrium relationships in a general continuous time setting where both the short rent and the short interest rate is uncertain. Since the framework is non-parametric our results hold under very general conditions.

We show that risk aversion and interest rate uncertainty can significantly bias an expectations hypothesis of lease rate similar to that of interest rates. It is the risk-neutral expectation of future rents, not the objective, that in combination with the characteristics of the interest rate process determine the relationship between expected future lease rates and forward lease rates. As a result, objective expectations about future rent levels can not be directly inferred from an inspection of the lease term structure. The effect of risk-aversion and interest rate uncertainty on equilibrium rents in the local property market must first be taken into consideration when interpreting the term structure. To directly infer market expectations from an inspection of the term structure, without considering these aspects, can lead to erroneous conclusions.

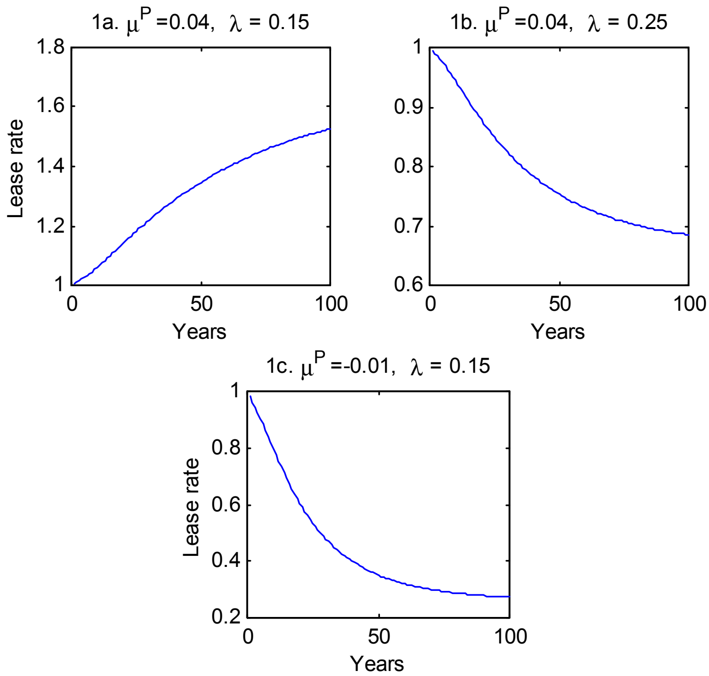

Figure 1a-1c. Effect of risk aversion on the term structure of lease rates.

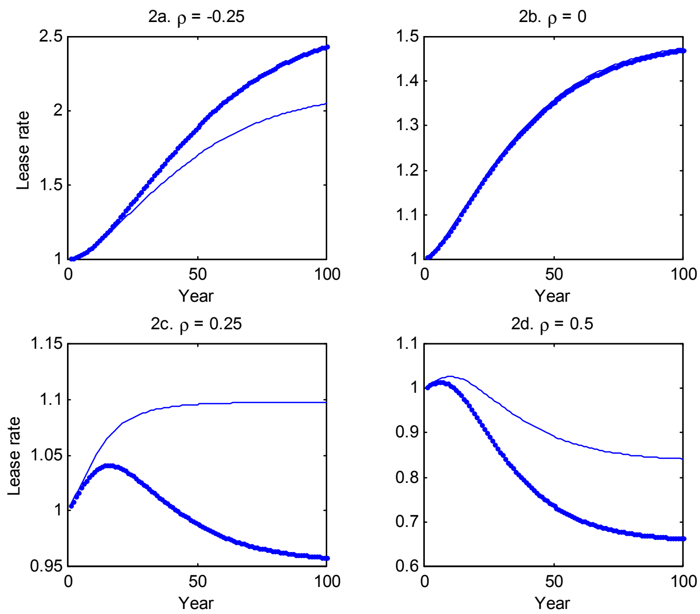

Figure 2a-2d. Effect of interest rate uncertainty on the term structure of lease rates. (Thicker line is $\sigma_r = 0.03$, thinner line $\sigma_r = 0.02$.)

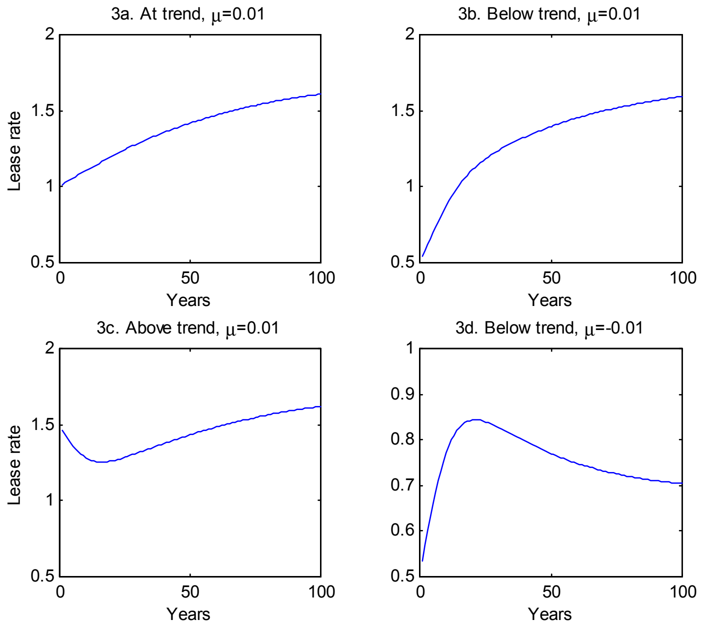

Figure 3a-3d. Effect of mean reversion on the term structure of lease rates.

References

[1] Björk, T. 1998. *Arbitrage Theory in Continuous Time*. Oxford University Press.

[2] Cox, J., J. Ingersoll and S. Ross. 1981. A Re-examination of Traditional Hypotheses about the Term Structure of Interest Rates. *Journal of Finance* 35(4): 769-799.

[3] Duffie, D. and K.J. Singleton. 1997. An Econometric Model of the Term Structure of Interest-Rate Swap Yields. *Journal of Finance* 52(4): 1287-1321.

[4] Englund, P., Å. Gunnelin., M. Hoesli and B. Söderberg 2002. Implicit Forward Rents as Predictors of Future Rents. Forthcoming, *Real Estate Economics*.

[5] Frachot, A. 1996. A Reexamination of the Uncovered Interest Rate Hypothesis. *Journal of International Money and Finance* 15(3): 419-437.

[6] Grenadier, S. 1995. Valuing Lease Contracts: A Real-Options Approach. *Journal of Financial Economics* 38(3): 297-331.

[7] Grenadier, S. 1996. Leasing and Credit Risk. *Journal of Financial Economics* 42(3): 333-364.

[8] Grenadier, S. 2002. An Equilibrium Analysis of Real Estate Leases. NBER Working Paper 9475.

[9] Gunnelin, Å. and B. Söderberg 2002. Term Structures in the Office Rental Markets in Stockholm. *Journal of Real Estate Finance and Economics* 26(2&3).

[10] Hendershott, P., C. Lizieri and G. Matysiak. 1999. The Workings of the London Office Market. *Real Estate Economics* 27(2): 365-387.

[11] Hendershott, P., B. MacGregor and R. Tse. 2002. Estimation of the Rental Adjustment Process. *Real Estate Economics* 30(2): 165-183.

[12] Lo, A. and J. Wang. 1995. Implementing Option Pricing Models when Asset Returns are Predictable. *Journal of Finance* 50(1): 87-129.

[13] McConnell, J. and J. Schallheim. 1983. Valuation of Asset Leasing Contracts. *Journal of Financial Economics* 12(2): 237-261.

[14] Miller, M. and C. Upton. 1976. Leasing, Buying and the Cost of Capital Services. *Journal of Finance* 31(3): 761-786.

[15] Miceli, T. J. and C.F. Sirmans. 1999. Tenant Tunover, Rental Contracts and Self-Selection. *Journal of Housing Economics 8(4): 301-311.*

[16] Rubinstein, M. 1976. The Valuation of Uncertain Income Streams and the Pricing of Options. *Bell Journal of Economics* 7(2): 407-425.

[17] Schallheim, J. and J. McConnell. 1985. A Model for the Determination of "Fair" Premiums on Lease Cancellation Insurance Policies. *Journal of Finance* 40: 1439-1457.

[18] Stanton, R. and N. Wallace. 2002. An Empirical Test of a Contingent Claims Lease Valuation Model. Working paper. U.C. Berkeley.

[19] Vasicek, O. 1977. An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics* 5: 177-188.

A Appendix

A.1 Term structure relationships

The fixed forward rate at t for renting over the future period $[T_1, T_2]$, to be paid continuously during the contract period, is denoted $R^F(t, T_1, T_2)$. Based on the principle that equivalent contracts have the same price, the following obtains:

$$R(t, T_2)\int_t^{T_2} p(t,s)ds = R(t, T_1)\int_t^{T_1} p(t,s)ds + R^F(t, T_1, T_2)\int_{T_1}^{T_2} p(t,s)ds. \quad \text{(A.1)}$$

By solving for the forward rate and using the definitions for the term structure of lease rates in (12) and (13) in section 2, we can express the forward contract as follows:

$$R^F(t, T_1, T_2) = \frac{\int_{T_1}^{T_2} p(t,s)f(t,s)ds}{\int_{T_1}^{T_2} p(t,s)ds}. \quad \text{(A.2)}$$

$$R^F(t, T_1, T_2) = \frac{F(t,T_1)p(t,T_1) - F(t,T_2)p(t,T_2)}{\int_{T_1}^{T_2} p(t,s)ds}. \quad \text{(A.3)}$$

Grenadier (1995, 2002) prefers to work in terms of call options with zero exercise price rather than forward contracts. This contract differs from a forward on the house only in that payment is made at origin rather than at maturity. Thus, to avoid arbitrage:

$$C(H_t, 0, T) = p(t,T)F(t,T) = \int_T^{\infty} p(t,s)f(t,s)ds. \quad \text{(A.4)}$$

Section 2 expresses several results in terms of the instantaneous forward rate, but this can also also be reversed:

$$f(t,T) = -\frac{\partial C(H_t, 0, T)/\partial T}{p(t,T)} \quad \text{(A.5)}$$

$$f(t,T) = R(t,T) + \frac{\partial R(t,T)}{\partial T} \cdot \frac{\int_t^T p(t,s)ds}{p(t,T)} \quad \text{(A.6)}$$

$$f(t,T) = \lim_{\delta \to 0} R^F(t, T, T+\delta). \quad \text{(A.7)}$$

The result (A.5) is obtained by differentiating (A.4) with respect to T. Result (A.6) follows by differentiating the expression for the term structure of lease rates in (12) with respect to T. Finally, (A.7) follows from (A.2). The first and last relationships above are given by Grenadier (2002) for the case with constant interest rate. In principle the above relationships could be used to compute an empirically observed term structure, and then calibrate it to some model.

Although we have considered continuous rent payments, one could also think of payments in discrete installments. If payments are made at n time points $t_1, ..., t_n$ then the fixed rate becomes, $R^d(t,T) = R(t,T)\int_t^T p(t,s)ds / \sum_{k=1}^n p(t, t_k)$.

A.2 Term structure with indexing

A real contract denoted $R^R(t,T)$ is adjusted according to the change in some stochastic index, denoted I_t, which can be thought of for instance as the consumer price index. Grenadier (1995) considers this case with a constant interest rate. With a stochastic interest rate, it is useful to introduce notation $p^R(t,T)$ for a real bond that gives one inflation adjusted unit of account:

$$p^R(t,T) = E_t^Q \left[e^{-\int_t^T r_s ds} \cdot \frac{I_T}{I_t} \right] = p(t,T) E_t^T \left[I_T \right] / I_t. \tag{A.8}$$

The payment streams must have the same present value:

$$R^R(t,T) = \frac{\int_t^T p(t,s)ds}{\int_t^T p^R(t,s)ds} R(t,T) = \frac{\int_t^T f(t,s)p(t,s)ds}{\int_t^T p^R(t,s)ds}. \tag{A.9}$$

From (A.8) we can see that the difference in expected nominal yield between the nominal and real bond depends on three factors: expected increase in the price index (inflation), risk aversion against inflation, and correlation between nominal interest rates and the price index. A negative risk-neutral drift rate in the price index, or correlation between the index and the short rent may lead to a higher real rent than nominal rent. The correlation between the rent process and the index has no affect on the term structure of real lease rates.

Rather than paying a constant rate, it could be specified to adjust according to some deterministic scheme. We denote that by φ_s. That is, the initial rent becomes,

$$R^{adj}(t,T) = \frac{\int_t^T p(t,s)f(t,s)ds}{\int_t^T \varphi_s p(t,s)ds} = \frac{\int_t^T p(t,s)ds}{\int_t^T \varphi_s p(t,s)ds} R(t,T). \tag{A.10}$$

Arguably the simplest implementation is the case when the rent is adjusted at a constant rate ϕ,

$$\varphi_s = e^{\phi(s-t)}. \tag{A.11}$$

A.3 Term structure and credit risk

Now assume that the lessee may go bankrupt, as in Grenadier (1996). We model bankruptcy using the indicator function Ω_s (indicator for non-bankruptcy is $\overline{\Omega}_s$). The indicator function may in turn depend on fundamental firm characteristics, which are left unspecified here. If the lessee goes bankrupt, then the lessor can release at the going market rate, but can only recoup a fraction $1-\omega$ due to transaction costs (this follows Grenadier, 1996). The lessee with credit risk pays rate $R^c(t,T)$, as opposed to $R(t,T)$. Equilibrium pricing satisfies,

$$E_t^Q \left[\int_t^T e^{-\int_t^s r_u du} \cdot \left(\overline{\Omega}_s R^c(t,T) + \Omega_s (1-\omega) X_s - X_s \right) ds \right] = 0. \tag{A.12}$$

We introduce notation, $p^c(t,s) = E_t^Q \left[\exp\left(-\int_t^s r_s du\right) \cdot \overline{\Omega}_s \right]$, as this has the interpretation of a risky bond. If rent is not correlated with default risk a particularly simple expression obtains,

$$R^c(t,T) = \frac{\int_t^T f(t,s) \left[\omega p(t,s) + (1-\omega) p^c(t,s) \right] ds}{\int_t^T p^c(t,s)\, ds}.$$

One possible implementation of default risk would be a constant hazard rate of κ, independent of other processes, which would imply that $p^c(t,T) = p(t,T)e^{-\kappa(T-t)}$. Note that if the lessor cannot recover any of the revenue lost from bankruptcy, i.e. $\omega = 1$, then $R^c(t,T) \geq R(t,T)$ because $p^c(t,s) \leq p(t,s)$. Consider the case when ω is low, i.e. there are low or no transaction costs associated with default. Even in this case the default risk is not innocuous to the pricing problem: For instance, if the default occurs in a market with increasing rent the lessor can release at a higher rate. In the latter case, default risk is actually beneficial to the lessor (see also Grenadier, 1996, for a discussion based on an economic model of bankruptcy).

A.4 The parameterized model

We have the following two processes for the short rent and the short interest rate:

$$\begin{aligned} dr_t &= \eta(\alpha - r_t)dt + \sigma_r dW_t^1 \\ dX_t &= \mu X_t dt + \sigma_x X_t dW_t^2 && (A.13) \\ dW_t^1 dW_t^2 &= \rho dt. && (A.14) \end{aligned}$$

Thus analytic bond prices follow Vasicek (1977). The price of a zero coupon bond is,

$$p(t,T) = E_t^Q \left[e^{-\int_t^T r_s ds} \right] = e^{A(t,T) - B(t,T) r_t}. \tag{A.15}$$

Here the following definitions apply,

$$\begin{aligned} B(t,T) &= \frac{1}{\eta}\left[1 - e^{-\eta(T-t)}\right]. && (A.16) \\ A(t,T) &= \frac{\left[B(t,T) - (T-t)\right]\left[\alpha - \sigma_r^2/2\right]}{a^2} - \frac{\sigma_r^2 B^2(t,T)}{4a}. && (A.17) \end{aligned}$$

Risk-neutral bond dynamics may be expressed as,

$$\begin{aligned} dp(t,T) &= r_t p(t,T)dt + v(t,T)p(t,T)dW_t^1, && (A.18) \\ v(t,T) &= -\frac{\sigma_r}{\eta}\left[1 - e^{-\eta(T-t)}\right]. && (A.19) \end{aligned}$$

The rent process thus has the following Q^s dynamics:

$$dX_t = \left(\mu + v(t,s)\sigma_x\rho\right) X_t dt + \sigma_x X_t dW_t^{2,s}. \tag{A.20}$$

Further,

$$\begin{aligned} &\int_t^s v(u,s)du = \\ &= -\frac{\sigma_r}{\eta}\int_t^s \left[1 - e^{-\eta(s-u)}\right] du \\ &= -\frac{\sigma_r}{\eta}\left| u - \frac{e^{-\eta(s-u)}}{\eta}\right|_t^s \\ &= -\frac{\sigma_r}{\eta}\left((s-t) - \frac{1 - e^{-\eta(s-t)}}{\eta}\right). && (A.21) \end{aligned}$$

Hence,

$$\begin{aligned} f(t,s) &= E_t^s\left[X_s\right] = X_t e^{\mu(s-t) + \sigma_x\rho\int_t^s v(u,s)du} && (A.22) \\ \int_t^T p(t,s)f(t,s)ds &= X_t \int_t^T p(t,s) e^{\mu(s-t) - \frac{\sigma_x\sigma_r\rho}{\eta}\left((s-t) - \frac{1-e^{-\eta(s-t)}}{\eta}\right)} ds. && (A.23) \end{aligned}$$

The expression for $R(t,T)$ becomes,

$$R(t,T) = X_t \frac{\int_t^T p(t,s) e^{\mu(s-t) - \frac{\sigma_x\sigma_r\rho}{\eta}\left((s-t) - \frac{1-e^{-\eta(s-t)}}{\eta}\right)} ds}{\int_t^T p(t,s)ds}. \tag{A.24}$$

A.5 The parameterized model with trend reversion

When the short interest rate follows a Vasicek process and the log rent process follows a trending Ornstein-Uhlenbeck process, the latter has Q^s dynamics as follows:

$$dZ_t = \left[\gamma\left(\mu t - Z_t\right) + \mu + v(t,s)\sigma_Z\rho\right]dt + \sigma_Z dW_t^{2,s}. \tag{A.25}$$

Further,

$$\begin{aligned}
&\int_t^s e^{-\gamma(s-u)}v(u,s)du = \\
&= -\frac{\sigma_r}{\eta}\int_t^s \left[e^{-\gamma(s-u)} - e^{-(\eta+\gamma)(s-u)}\right]du \\
&= -\frac{\sigma_r}{\eta}\left|\frac{e^{-\gamma(s-u)}}{\gamma} - \frac{e^{-(\eta+\gamma)(s-u)}}{\eta+\gamma}\right|_t^s \\
&= -\frac{\sigma_r}{\eta}\left(\frac{1-e^{-\gamma(s-t)}}{\gamma} - \frac{1-e^{-(\eta+\gamma)(s-t)}}{\eta+\gamma}\right)
\end{aligned} \tag{A.26}$$

Hence,

$$\begin{aligned}
f(t,s) &= E_t^s\left[X_s\right] \\
&= \left(\frac{X_t}{e^{\mu t}}\right)^{e^{-\gamma(s-t)}} e^{\mu s - \frac{\sigma_z\sigma_r\rho}{\eta}\left(\frac{1-e^{-\gamma(s-t)}}{\gamma} - \frac{1-e^{-(\eta+\gamma)(s-t)}}{\eta+\gamma}\right) + \frac{\sigma_z^2}{4\gamma}\left(1-e^{-2\gamma(s-t)}\right)}.
\end{aligned}$$

Note that,

$$\lim_{\gamma\to 0}\left[\frac{1-e^{-\gamma(s-t)}}{\gamma}\right] = s-t.$$

This gives the special case when $\gamma = 0$.

The rent process may follow a trending Urnstein-Ohlenbeck process both under P and Q, given specific assumptions of risk aversion. We assume that risk aversion against the log rent process $\lambda^z = q(\gamma t+1)$. If we define $\mu^P = \mu + q\sigma_z$ then the objective dynamics are,

$$dZ_t = \left[\gamma(\mu^P t - Z_t) + \mu^P\right]dt + \sigma_z dW_t^2.$$

That is, the rent is trend reverting under the objective measure.

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