Fractional Graphs

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Abstract. Edge-colorings are used to extend the notion of the graph Cartesian product to a quotient operation that allows for the formation of graph fractions. Fractional graphs form a group that is isomorphic to the positive rational numbers.

1 Introduction

The *Cartesian product* of two simple graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph $G \Box H$ with $V(G \Box H) = V(G) \times V(H)$, and $(u, x)(v, y) \in E(G \Box H)$ if either $u = v$ and $xy \in E(H)$, or $uv \in E(G)$ and $x = y$. Figure 1 shows an example. Notice that if x_0 is a fixed vertex of H, the edges $\{(u, x_0)(v, x_0) \mid uv \in E(G)\}$ of $G \Box H$ form a subgraph that is isomorphic to G. Call this the *fiber* over x_0. Likewise, if $u_0 \in V(G)$, the fiber over u_0 is the subgraph with edges $\{(u_0, x)(u_0, y) \mid xy \in E(H)\}$, and it is isomorphic to H.

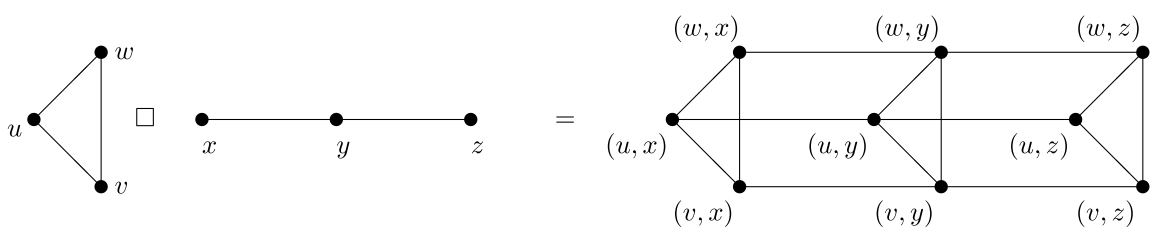

Figure 1

This paper describess how to extend the idea of the Cartesian product to an operation that allows for the formation of quotients of graphs. Doing this involves enriching graphs by coloring their edges with two colors that encode information about numerator and denominator.

2 Colorings

An *edge two-coloring* of a graph G is a function $\kappa : E(G) \to \{1, -1\}$. We think of 1 and -1 as two colors, so κ just assigns colors to the edges of G. (Notice this definition allows incident edges to have the same color.) Henceforth we will simply call such a function κ a *coloring* of G. In drawing a colored graph, we draw the edges of color 1 solid and those of color -1 dotted.

If $\kappa : E(G) \to \{1, -1\}$ is identically 1, it is called the *trivial coloring* of G. Henceforth, every graph under discussion is assumed to have a coloring. If a coloring is not stated, it is trivial by default. Trivially colored graphs can be identified with ordinary (i.e. uncolored) ones.

There is a natural way to color a Cartesian product. In this article, the product of two colored graphs is always assumed to have the coloring given by the following definition.

Definition 1. *Suppose graphs G and H have colorings κ_G and κ_H. Then $G\Box H$ has coloring $\kappa_{G\Box H}$ defined as $\kappa_{G\Box H}((u,x)(v,x)) = \kappa_G(uv)$ and $\kappa((u,x)(u,y)) = \kappa_H(xy)$.*

Figure 2 gives an example. Notice that fibers over $x_0 \in V(H)$ have the same coloring as G, and fibers over $u_0 \in V(G)$ have the same coloring as H.

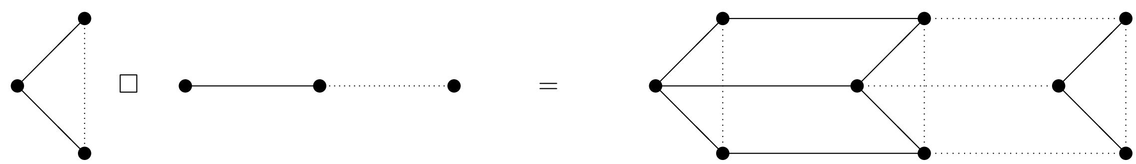

Figure 2

Definition 2. *Colored graphs G and H are equal, written $G = H$, if there is a color-preserving isomorphism between them. Specifically, $G = H$ means there is a bijective function $\alpha : V(G) \to V(H)$ with $uv \in E(G)$ if and only if $\alpha(u)\alpha(v) \in E(H)$, and $\kappa_G(uv) = \kappa_H(\alpha(u)\alpha(v))$ for all $uv \in E(G)$.*

Lemma 1. *The Cartesian product of colored graphs is commutative and associative, that is $G\Box H = H\Box G$ and $G\Box(H\Box K) = (G\Box H)\Box K$ for all (colored) graphs G, H and K.*

Proof. Let G, H and K have colorings κ_G, κ_H and κ_K, respectively.

Consider commutativity. The function $\alpha : V(G\Box H) \to V(H\Box G)$ defined as $\alpha(u,x) = (x,u)$ is a bijection. We examine the two types of edges in $G\Box H$. In the first case, an edge of form $(u,x)(v,x)$ has color $\kappa_G(uv)$ by Definition 1. This edge maps to $\alpha(u,x)\alpha(v,x) = (x,u)(x,v)$ which an edge of $H\Box G$, again with color $\kappa_G(uv)$ by Definition 1. In the second case, an edge of form $(u,x)(u,y)$ has color $\kappa_H(xy)$ and maps to $\alpha(u,x)\alpha(u,y) = (x,u)(y,u)$, an edge of $H\Box G$ with color $\kappa_H(xy)$. In the same way, we check that α^{-1} preserves adjacencies. Then $G\Box H = H\Box G$ by Definition 2.

For associativity, consider $\alpha : V(G\Box(H\Box K)) \to V((G\Box H)\Box K)$ defined as $\alpha(u,(x,z)) = ((u,x),z)$. We consider the three types of edges in $G\Box(H\Box K)$ separately. In the first case, consider an edge of form $(u,(x,z))(v,(x,z))$, which has color $\kappa_G(uv)$. The map α sends this edge to the edge $((u,x),z)((v,x),z)$ of $(G\Box H)\Box K$, which according to Definition 1 has color $\kappa_{G\Box H}((u,x)(v,x)) = \kappa_G(uv)$. Second, an edge of form $(u,(x,z))(u,(y,z))$, of color $\kappa_{H\Box K}((x,z)(y,z)) = \kappa_H(xy)$, is mapped to edge $((u,x),z)((u,y),z)$ of color $\kappa_{G\Box H}((u,x)(u,y)) = \kappa_H(xy)$. Third, an edge of form $(u,(x,z))(u,(x,w))$, of color $\kappa_{H\Box K}((x,z)(x,w)) = \kappa_K(zw)$, is mapped to edge $((u,x),z)((u,x),w)$ of color $\kappa_K(zw)$. These three cases show α preserves adjacencies and colors. Similarly, α^{-1} also preserves adjacencies, so $G\Box(H\Box K) = (G\Box H)\Box K$ follows. ■

3 Graph Fractions

If G has coloring κ, let G^{-1} designate the graph G with coloring $-\kappa$, that is G^{-1} is G with the colors interchanged. Call G^{-1} the *inverse* of G. Obviously $(G^{-1})^{-1} = G$, and it is easy to check that $(G\Box H)^{-1} = G^{-1}\Box H^{-1}$. Now comes our main definition.

Definition 3. *Let G and H be colored graphs. The quotient of G by H is the graph* $\dfrac{G}{H} = G\,\Box H^{-1}$.

For typographical reasons, $\frac{G}{H}$ may be written as G/H. Figure 3 shows a few examples. In each case the numerator and denominator are either trivially colored graphs or their inverses, though that is not required in general.

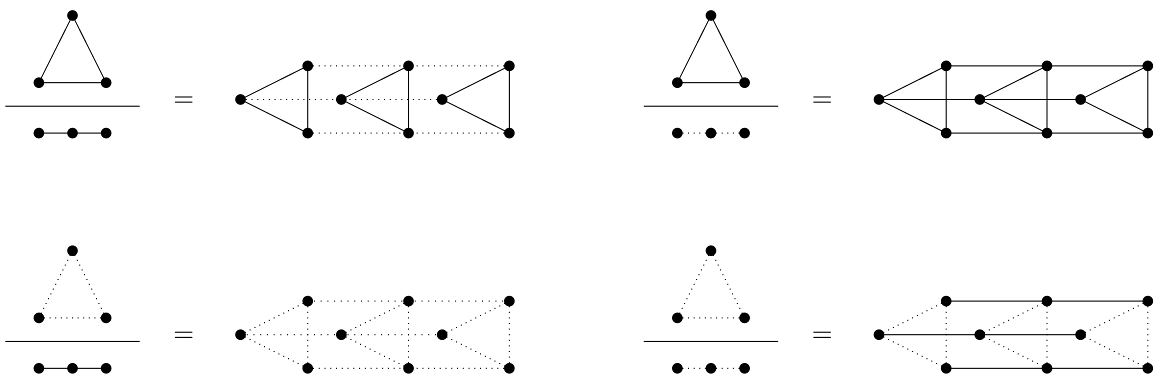

Figure 3

The next four propositions show that graph fractions behave much as fractions of integers do, but with $\Box$ playing the role of multiplication.

Proposition 1. *If F, G, H and K are colored graphs, then* $\dfrac{F}{G}\Box\dfrac{H}{K} = \dfrac{F\Box H}{G\Box K}$.

Proof. This follows from Definition 3, Lemma 1, and the fact $(G\Box H)^{-1} = G^{-1}\Box H^{-1}$. Indeed, $\dfrac{F}{G}\,\Box\,\dfrac{H}{K} = (F\Box G^{-1})\Box(H\Box K^{-1}) = (F\Box H)\Box(G^{-1}\Box K^{-1}) = (F\Box H)\Box(G\Box K)^{-1} = \dfrac{F\Box H}{G\Box K}$. ■

Proposition 2. *If F, G, H, K are colored graphs,* $\dfrac{F/G}{H/K} = \dfrac{F}{G}\Box\dfrac{K}{H} = \dfrac{F\Box K}{G\Box H}$.

Proof. By the definitions, $\dfrac{F/G}{H/K} = \dfrac{F\Box G^{-1}}{H\Box K^{-1}} = (F\Box G^{-1})\Box(H\Box K^{-1})^{-1} = (F\Box G^{-1})\Box(H^{-1}\Box(K^{-1})^{-1})$ $= (F\Box G^{-1})\Box(H^{-1}\Box K) = (F\Box G^{-1})\Box(K\Box H^{-1}) = \dfrac{F}{G}\Box\dfrac{K}{H}$. This latter expression equals $\dfrac{F\Box K}{G\Box H}$ by Proposition 1. ■

Proposition 3. *If G and H are colored graphs, then* $\left(\dfrac{G}{H}\right)^{-1} = \dfrac{H}{G}$.

Proof. Observe $\left(\dfrac{G}{H}\right)^{-1} = (G\Box H^{-1})^{-1} = G^{-1}\Box H = H\Box G^{-1} = \dfrac{H}{G}$. ■

Define the *trivial graph* I to be the graph with one vertex and no edges. Its coloring can thus only be the empty function $\kappa_I : \emptyset \to \emptyset$. The definitions show $I^{-1} = I$ and $G\Box I = G$ for all graphs G.

Proposition 4. *If G is any colored graph, then* $\dfrac{G}{I} = G$, *and* $\dfrac{I}{G} = G^{-1}$.

Proof. Observe $\dfrac{G}{I} = G\Box I^{-1} = G\Box I = G$ and $\dfrac{I}{G} = I\Box G^{-1} = G^{-1}$. ∎

The analogy between fractions of integers and fractions of graphs goes further. Just as the nonzero rational numbers form a group, so do fractional graphs.

4 A Group of Graphs

One way to define the multiplicative group of positive rational numbers is to declare fractions a/b and c/d of positive integers to be equivalent if $ad = bc$, and confirm that this is an equivalence relation. The positive rational numbers are then the equivalence classes of this relation, and these classes form a group. The same approach works with graphs.

Let Γ be the set of all *connected* graphs, all with trivial colorings. Set $\mathbb{G} = \{G/H \mid G, H \in \Gamma\}$, which is closed under $\Box$, by Proposition 1 and the fact that Cartesian products of connected graphs are connected. Define a relation $\sim$ on $\mathbb{G}$ as $F/G \sim H/K$ if $F\Box K = G\Box H$. One immediately checks that this relation is reflexive and symmetric, but transitivity is not so obvious.

Suppose $F/G \sim H/J$ and $H/J \sim K/L$, so $F\Box J = G\Box H$ and $H\Box L = J\Box K$, whence $F\Box J\Box H\Box L$ $= G\Box H\Box J\Box K$. We want to "cancel" the $J\Box H$ to get $F\Box L = G\Box K$ which would make $F/G \sim K/L$. The fact that the cancellation is justified follows from a theorem—proved by Sabidussi and Vizing in the early 1960's—stating that every connected graph has a unique prime factorization relative to $\Box$. (A graph G is *prime* if $G \neq I$, and $G = H\Box K$ implies $H = I$ or $K = I$. Sabidussi and Vizings' theorem states that if a (uncolored) connected graph can be factored into prime graphs as $P_1\Box P_2\Box\cdots\Box P_m$ and $Q_1\Box Q_2\Box\cdots\Box Q_n$, then $m = n$ and the indices can be relabeled so that $P_i \cong Q_i$ for $1 \leq i \leq m$. See Chapter 4 of [1] or [2, 3] for proofs of this theorem.) Sabidussi and Vizings' theorem implies that $\Box$ has a cancellation property: If $A\Box B = A\Box C$ for connected graphs A, B and C, then $B = C$, for otherwise B and C have different prime factorizations, giving $A\Box B$ $= A\Box C$ two distinct prime factorizations. Owing to this cancellation property, $\sim$ is transitive, and therefore an equivalence relation. Let $\mathbb{G}^* = \mathbb{G}/\sim$ be the set of equivalence classes. Denote the equivalence class containing a graph G/H as $[G/H]$.

From the definition of $\sim$, the class $[I] = [I/I]$ consists of all graphs of form G/G. In such a graph, the fibers of color 1 and the fibers of color -1 are all isomorphic to G. Figure 4 shows an example.

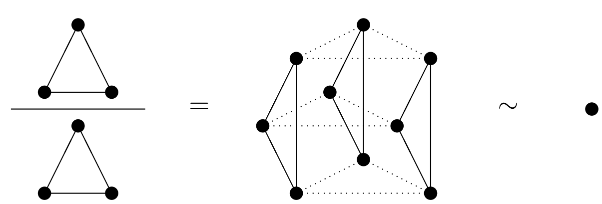

Figure 4

Using Proposition 1, it is easy to check that there is a well-defined binary operation $\Box$ on $\mathbb{G}^*$ defined as $[F/G]\Box[K/H] = [(F/G)\Box(K/H)]$. Associativity and commutativity of $\Box$ on $\mathbb{G}^*$ are inherited from associativity and commutativity of $\Box$ on $\mathbb{G}$ (Lemma 1). Observe $[I]\Box[G/H] = [I\Box(G/H)] = [G/H]$, so $[I]$ is an identity. Lastly, $[G/H]\Box[H/G] = [(G/H)\Box(H/G)] = [(G\Box H)/(H\Box G)] = [(G\Box H)/(G\Box H)] = [I]$, so any element $[G/H]$ has an inverse $[H/G]$. (In fact, by Proposition 3, $[K]^{-1} = [K^{-1}]$.) Therefore $\mathbb{G}^*$ is an abelian group. As with the rational numbers, we drop the brackets and let any fraction represent the equivalence class it belongs to. For example, $G\Box G^{-1} = I$, though this is expressing equality of equivalence classes, not graphs.

Proposition 5. *The group $\mathbb{G}^*$ of fractional graphs is isomorphic to the group $\mathbb{Q}^+$ of positive rational numbers.*

Proof. Let $\mathscr{P} = \{p_1, p_2, p_3, \ldots\}$ be the set of all prime numbers. The fundamental theorem of arithmetic implies any rational number unequal to 1 has a unique expression $p_{i_1}^{n_1} p_{i_2}^{n_2} \cdots p_{i_k}^{n_k}$ with $n_1, n_2, \ldots, n_k$ nonzero integers, whence $\mathbb{Q}^+$ is the free abelian group generated on $\mathscr{P}$. Analogously, if $\mathscr{G} = \{P_1, P_2, P_3, \ldots\}$ is the set of all connected prime graphs (with trivial colorings), then Sabidussi and Vizing's theorem implies any $G/H \neq I$ has unique expression $P_{i_1}^{n_1}\Box P_{i_2}^{n_2}\Box\cdots\Box P_{i_k}^{n_k}$, so $\mathbb{G}^*$ is the free abelian group generated on $\mathscr{G}$. Then $\mathbb{G}^* \cong \mathbb{Q}^+$, since $|\mathscr{G}| = |\mathscr{P}| = \aleph_0$. ∎

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