COLORING LINK DIAGRAMS WITH A CONTINUOUS PALETTE

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ABSTRACT. The well-known technique of n-coloring a diagram of an oriented link l is generalized using elements of the circle $\mathbf{T}$ for colors. For any positive integer r, the more general notion of a $(\mathbf{T}, r)$-coloring is defined by labeling the arcs of a diagram D with elements of the torus $\mathbf{T}^{r-1}$. The set of $(\mathbf{T}, r)$-colorings of D is an abelian group, and its quotient by the connected component of the idenitity is isomorphic to the torsion subgroup of $H_1(M_r(l); \mathbf{Z})$. Here $M_r(l)$ denotes the r-fold cyclic cover of S^3 branched over the link l. Results about braid entropy are obtained using techniques of symbolic dynamical systems.

1. INTRODUCTION. The well-known technique of tricoloring offers an elementary method of distinguishing a trefoil from the trivial knot [**CrFo**], [**Fo1**], [**Fo2**]. A *tricoloring* of a link diagram is an assignment of three colors to the arcs of the diagram such that at any crossing either all three colors appear or only one color appears. Obviously any diagram has three trivial, monochromatic diagrams. After checking that the number of tricolorings of a diagram is unaffected by the three Reidemeister moves, one can deduce that the trefoil knot is different from the trivial knot simply by observing that the standard diagram for the trefoil knot can be tricolored nontrivially.

Expanding our palette to n colors, identified with the elements of the finite cyclic group $\mathbf{Z}/n$, we can consider the more general notion of *n-coloring*. An n-coloring of a link diagram is an assignment of colors to the arcs such that at any crossing the sum of the colors of the undercrossings is equal to twice the color of the overcrossing modulo n. Since again Reidemeister moves do not affect the number of n-colorings of the diagram, the number is an invariant of l. The idea but not the terminology can be found in [**Fo1**]. The requisite mathematical ideas were certainly known to Reidemeister [**Re**].

There are important relationships between the set of n-colorings of a diagram for a link l and the topology of the link space $S^3 - l$. It has long been known, for example, that the set of n-colorings is an abelian group that is isomorphic to $H_1(M_2(l); \mathbf{Z}/n) \oplus \mathbf{Z}/n$, where $M_r(l)$ denotes the r-fold cyclic cover of S^3 branched over l. More recently, the

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notion of n-coloring has inspired useful exotic constructions in knot theory such as those found in [**Kau**].

Here we trade our finite palette for a continuous one, coloring diagrams of oriented links with elements of the continuous group $\mathbf{T}$ ($= \mathbf{R}/\mathbf{Z}$). Of course, one no longer expects the set of $\mathbf{T}$-colorings to be finite. However, the set is a compact abelian topological group with attractive properties. We will see that its quotient by the connected component of the identity element is isomorphic to the torsion subgroup $TH_1(M_2(l);\mathbf{Z})$. Using a slightly more general scheme, coloring arcs with elements of the torus $\mathbf{T}^{r-1}$, we will recover $TH_1(M_r(l);\mathbf{Z})$, for any $r \geq 1$, in a similar way.

The groups of colorings that we define all have natural "shift automorphisms" that can be studied from the point of view of dynamics. What can be learned from such a study? For example, if l_s is the closure of an iterated braid α^s, we determine the exponential growth rates $\lim_{s\to\infty}(1/s)\log|TH_1(M_r(l_s);\mathbf{Z})|$, for each r, using results of [**LiScWa**]. We then prove that if the growth rate is positive for some r, then α has positive braid entropy.

Our approach can be generalized with $\mathbf{T}$ replaced by an arbitrary topological group Σ. We do this in the last section, and indicate directions for further research.

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2. $\mathbf{T}^{\mathbf{Z}}$-COLORINGS. We consider the compact abelian group $\mathbf{T}^{\mathbf{Z}}$ consisting of all bi-infinite sequences (α_j) of elements $\alpha_j \in \mathbf{T}$. The *shift map* $\sigma : \mathbf{T}^{\mathbf{Z}} \to \mathbf{T}^{\mathbf{Z}}$ which sends (α_j) to (α'_j) where $\alpha'_j = \alpha_{j+1}$, is an automorphism.

Definition 2.1. Assume that D is a diagram of an oriented link. A $\mathbf{T}^{\mathbf{Z}}$**-coloring** of D is an assignment of elements $C \in \mathbf{T}^{\mathbf{Z}}$ to the arcs of D such that the condition

$$\sigma(C_i - C_k) = C_j - C_k. \tag{2.1}$$

is satisfied at any crossing. Here C_k corresponds to the overcrossing, while C_i, C_j correspond to the undercrossings, and we encounter C_j as we travel in the preferred direction along the arc labeled by C_k, turing left at the crossing.

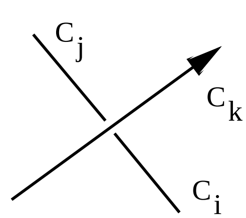


Figure 1

We denote the set of $\mathbf{T}^{\mathbf{Z}}$-colorings of D by $\mathrm{Col}_{\mathbf{T},\infty}(D)$. At first glance it appears to be hopelessly large. The following observation is intended to encourage the reader and also motivate the direction we intend to follow. For the present, identity an element $\alpha \in \mathbf{T}$ with the periodic sequence $(\ldots, \alpha, -\alpha, \underline{\alpha}, -\alpha, \alpha, \ldots)$. (The bar indicates the 0th coordinate.) It is easy to check that when using only elements of this type, condition (2.1) reduces to the familiar coloring condition that the sum of the colors of the undercrossings is equal to twice the color of the overcrossing. Since any finite cyclic group embeds in $\mathbf{T}$ in a natural way, the subgroup of $\mathrm{Col}_{\mathbf{T},\infty}(D)$ consisting of periodic $\mathbf{T}^{\mathbf{Z}}$-colorings of D with period 2 contains every n-coloring of D. We will describe the information contained in the subgroup of periodic $\mathbf{T}^{\mathbf{Z}}$-colorings with arbitrary period.

Definition 2.2. A $\mathbf{T}^{\mathbf{Z}}$-coloring of an oriented link diagram D is **periodic** if there exists a positive integer r such that $\sigma^r(C) = C$, for every assigned label C. In such a case we say that $\mathbf{T}^{\mathbf{Z}}$-coloring has **period r**.

We can color D trivially by assigning C to every arc. Such a coloring is **monochromatic**. Clearly, a monochromatic $\mathbf{T}^{\mathbf{Z}}$-coloring of D has period r, for every positive integer r, and the subgroup of monochromatic colorings is isomorphic to $\mathbf{T}^{\mathbf{Z}}$.

One can show that if a diagram D' is obtained from D by a finite sequence of Reidemeister moves (see [**BuZi**], for example), then the group $Col_{\mathbf{T},\infty}(D')$ is isomorphic to $\mathrm{Col}_{\mathbf{T},\infty}(D)$, and hence is an oriented link invariant. Rather than do this, we will prove invariance using algebraic topology.

Let l be an oriented link with group $G_l = \pi_1(S^3 - l)$. The **augmentation homomorphism** (or total linking number homomorphism) is the epimorphism $\chi : G_l \to \mathbf{Z}$ that sends each oriented meridianal generator to 1. Adopting the terminology of [**BrCr**], we refer to the kernel K as the **augmentation subgroup** of G_l. (If l is a knot, then K is the commutator subgroup of G_l.) We will call the covering space $X_\infty(l)$ associated to χ the **total linking number cover** of the link. The fundamental group $\pi_1 X_\infty(l)$ of the cover is isomorphic to K.

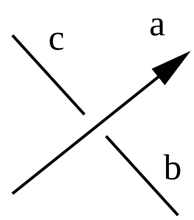
Theorem 2.3. Let D be a diagram of an oriented link l. Then

$$\mathrm{Col}_{\mathbf{T},\infty}(D) \cong \mathrm{Hom}(K, \mathbf{T}) \oplus \mathbf{T}^{\mathbf{Z}}.$$

Corollary 2.4. The group $\mathrm{Col}_{\mathbf{T},\infty}(D)$ is an invariant $\mathrm{Col}_{\mathbf{T},\infty}(l)$ of the oriented link l.

Proof of Theorem 2.3. We can find a presentation for K using the following procedure which combines the Wirtinger and Reidemeister-Schreier methods [**SiWi2**]. Distinguish some arc of the diagram, and dentote the corresponding meridianal generator by x.

The generators corresponding to the remaining arcs of the diagram can be expressed as $ax, bx, cx, \ldots$, where $a, b, c, \ldots$ are elements of K. Now label the distinguished arc with the identity element of K, and label each of the remaining arcs by the corresponding element $a, b, c, \ldots$ The augmentation subgroup K has generators $x^{-j}ax, x^{-j}bx^j, s^{-j}cx^j, \ldots$ $(j \in \mathbf{Z})$. We abbreviate them by $a_j, b_j, c_j, \ldots$. The relations correspond to the crossings of the diagram: $a_{j+1}c_j = b_{j+1}a_j$ for the crossing in Figure 2. As is true for Wirtinger presentations of link groups, any single relator is a consequence of the remaining relators and can be omitted.



$a_{j+1}c_j = b_{j+1}a_j$
FIGURE 2

For any homomorphism $\rho : K \to \mathbf{T}$, we obtain a $T^{\mathbf{Z}}$-coloring of D by sending the arcs labeled $a, b, \ldots$ to $(\rho(a_j)), (\rho(b_j)), \ldots$. This gives a one-to-one correspondence between homomorphisms $\rho : K \to \mathbf{T}$ and the subgroup of $\mathbf{T}^{\mathbf{Z}}$-colorings of D such that $(\ldots, 0, 0, 0, \ldots)$ is assigned to the distinguished arc.

Consider the following short exact sequence of abelian groups

$$0 \to \mathrm{Hom}(K, \mathbf{T}) \to \mathrm{Col}_{\mathbf{T},\infty}(D) \xrightarrow{\pi} \mathbf{T}^{\mathbf{Z}} \to 0,$$

where π is the map that sends any $\mathbf{T}^{\mathbf{Z}}$-coloring to the element of $\mathbf{T}^{\mathbf{Z}}$ assigned to the distinguished arc of D. If we define $s : \mathbf{T}^{\mathbf{Z}} \to \mathrm{Col}_{\mathbf{T},\infty}(D)$ by sending any $C \in \mathbf{T}^{\mathbf{Z}}$ to the monochromatic coloring of D in which C is assigned to every arc, then $\pi s = 1_{\mathbf{T}^{\mathbf{Z}}}$. Hence the sequence splits, and the theorem is proved. ■

Definition 2.5. Let D be a diagram of an oriented link l with some arc distinguished. A $\mathbf{T}^{\mathbf{Z}}$-coloring of D is **based** if it assigns the identity element to the distinguished arc.

The group of based $\mathbf{T}^{\mathbf{Z}}$ colorings of an oriented link diagram D clearly is independent of the choice of distinguished arc. We will denote the group by $\mathrm{Col}^0_{\mathbf{T},\infty}(l)$. The proof of Theorem 2.3 shows that $\mathrm{Col}^0_{\mathbf{T},\infty}(l) \cong \mathrm{Hom}(\pi_1 X_\infty(l), \mathbf{T})$, which is isomorphic to $\mathrm{Hom}(H_1(X_\infty(l); \mathbf{Z}), \mathbf{T})$, since $\mathbf{T}$ is abelian.

If A is any locally compact abelian group, then $\mathrm{Hom}(A, \mathbf{T})$ with the compact-open topology is the **dual group** of A, and it is usually denoted by $A^\wedge$ [**Kat**]. Endowed with the discrete topology, $H_1(X_\infty(l);\mathbf{Z})$ is a locally compact abelian group. It is easy to check that the (topological) group of based $\mathbf{T}^{\mathbf{Z}}$-colorings is isomorphic to the dual group $H_1(X_\infty(l);\mathbf{Z})^\wedge$.

Let $l = l_1 \cup \cdots \cup l_\mu$ be an oriented link, and let r be a positive integer. The cyclic cover $M_r(l)$ of S^3 branched over l is constructed as follows. Begin with the exterior $X(l)$ of l, obtained from S^3 by removing a tubular neighborhood $N_1 \cup \cdots \cup N_\mu$ of l, and consider its r-fold cyclic cover $X_r(l)$, corresponding to the homomorphism $G_l \xrightarrow{\chi} \mathbf{Z} \to \mathbf{Z}/r$. The boundary of $X(l)$ consists of disjoint tori $\partial N_1 \cup \cdots \cup \partial N_\mu$, and the preimage of each is again a torus. The preferred longitude on each $\partial N_i,\ i = 1, \ldots, \mu$, is covered by r disjoint loops, while the meridian is covered r times by a single loop. We obtain $M_r(l)$ by attatching a solid torus $S^1 \times D^2$ to each lift ∂N_i, matching a meridian $* \times S^1$ with a preimage of a meridian of ∂N_i. Additional details can be found in [**BuZi**].

Consider now a diagram D for the link with a distinguished arc. By the proof of Theorem 1.3 the augmentation subgroup K has a presentation of the form

$$\langle a_j, b_j, \ldots \mid r_j, s_j, \ldots \quad (j \in \mathbf{Z})\rangle, \tag{2.2}$$

where $a_j, b_j, \ldots$ denote the elements $x^{-j}ax^j, x^{-j}bx^j, \ldots$. The relators $r_{j+k}, s_{j+k}, \ldots$ can be obtained from $r_j, s_j, \ldots$, respectively, by "shifting subscripts," adding k to the subscript of every generator that occurs. For any $r \geq 1$, the following is a presentation of $\pi_1(X_r(l))$.

$$\langle y, a_j, b_j, \ldots \mid r_j, s_j, \ldots, y^{-1}a_jy = a_{j+r},\ y^{-1}b_jy = b_{j+r}, \ldots \quad (j \in \mathbf{Z})\rangle \tag{2.3}$$

The generator y is represented by a lift of the meridian of the distinguished arc of D; it is equal to x^r in the link group G_l. We can build a presentation for $\pi_1(M_r(l))$ by adding μ relators to (2.3), one for each solid torus added to $X_r(l)$. The relators that we add are $x^r, (ax)^r, (bx)^r, \ldots$; they are represented up to conjugacy by the lifts of the rth power of the various meridianal generators. Equivalently, we can introduce $x^r, x^{-r}(ax)^r, x^{-r}(bx)^r, \ldots$. Of course, the first relator merely eliminates the generator y. The second relator can be written as $(x^{-r}ax^r)(x^{-r+1}ax^{r-1})\cdots(x^{-1}ax)$ which is the same as $a_ra_{r-1}\cdots a_1$. Moreover, for any integer j, a loop in $M_r(l)$ representing $x^{-j+1}(ax)^rx^{j-1}$ is freely homotopic to one that represents $(ax)^r$, and hence $x^{-r}x^{-j+1}(ax)^rx^{j-1} = a_{j+r-1}a_{k+r-2}\cdots a_j$ is trivial in $\pi_1(M_r(l))$. Similarly, the third relator $x^{-r}(bx)^r$ implies that $b_rb_{r-1}\cdots b_1$ is trivial. Hence $b_{j+r-1}b_{k+r-2}\cdots b_j$ is trivial for all j. We conclude that $\pi_1(M_r(l))$ has a presentation consisting of generators $a_j, b_j, \ldots$ and relators $r_j, s_j, \ldots$, together with

$$a_j = a_{j+r}, \quad b_j = b_{j+r},\ \ldots, \tag{2.4 i}$$

$$a_{j+r-1}a_{k+r-2}\cdots a_j, \quad b_{j+r-1}b_{k+r-2}\cdots b_j,\ \ldots \tag{2.4 ii}$$

where j ranges over $\mathbf{Z}$.

When l has only one component the relators (2.4 ii) follow immediately from the single relator x^r, since in that case every meridianal generator $ax, bx, \ldots$ is conjugate to x. On the other hand, for any link l the relators (2.4 i) clearly are a consequence of (2.4 ii). Hence we have shown

Lemma 2.7. Let l be an oriented link, and r a positive integer. Consider a presentation for the augmentation subgroup K of the form (2.2). Then $\pi_1(M_r(l))$ is the quotient of K by the normal subgroup generated by the elements (2.4 ii).

Abelianization yields a more familiar result. Regard $H_1(X_\infty(l);\mathbf{Z})$ as a $\mathbf{Z}[t,t^{-1}]$-module in the usual way, with the action of t corresponding to conjugation by x in the group of the link. It follows from Lemma 1.7 that the quotient of the $\mathbf{Z}[t,t^{-1}]$-module $H_1(X_\infty(l);\mathbf{Z})$ by the submodule $(t^{r-1}+\cdots+t+1)H_1(X_\infty(l);\mathbf{Z})$ is isomorphic to $H_1(M_r(l);\mathbf{Z})$, for any positive integer r.

Definition 2.8. Assume that l is an oriented link with diagram D, and r is a positive integer. A $(\mathbf{T},r)$-**coloring** of D is a periodic $T^{\mathbf{Z}}$-coloring with period r such that the sum of any r consecutive coordinates of any assigned label vanishes.

We will denote the subgroup of all $(\mathbf{T},r)$-colorings of D by $\mathrm{Col}_{\mathbf{T},r}(D)$. (The group $\mathrm{Col}_{\mathbf{T},1}(D)$ is obviously trivial.) The subgroup of based $(\mathbf{T},r)$-colorings will be denoted by $\mathrm{Col}^0_{\mathbf{T},r}(D)$.

We give $H_1(M_r(l));\mathbf{Z})$ the discrete topology. From what has been said it follows that the dual group $[H_1(M_r(l);\mathbf{Z})]^\wedge$ is isomorphic to $\mathrm{Col}^0_{\mathbf{T},r}(D)$. Also, $[H_1(M_r(l);\mathbf{Z})\oplus\mathbf{Z}^{r-1}]^\wedge$ is isomorphic to $\mathrm{Col}_{\mathbf{T},r}(D)$.

For a fixed positive integer r, we can determine $\mathrm{Col}^0_{\mathbf{T},r}(D)$ by the following device (cf. [**SiWi1**]). Label arcs of D by vectors $C\in\mathbf{T}^{r-1}$ such that at any crossing

$$(C_i - C_j)S_r^\epsilon = C_j - C_k, \tag{2.5}$$

where S_r is the $(r-1)\times(r-1)$-matrix

$$S_r = \begin{pmatrix} 0 & 0 & \ldots & 0 & -1 \\ 1 & 0 & \ldots & 0 & -1 \\ 0 & 1 & \ldots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -1 \end{pmatrix}.$$

As in Definition 2.1, the vector C_k corresponds to the overcrossing, C_i, C_j correspond to the undercrossings, $\epsilon = \pm 1$ is the algebraic sign of the crossing. Any vector $C = (\alpha_1, \ldots, \alpha_{r-1}) \in \mathbf{T}^{r-1}$ extends to an element $(\alpha_j) \in \mathbf{T}^{\mathbf{Z}}$ by defining α_{j+r-1} to be the matrix product $C \cdot S^j$, for $j \in \mathbf{Z}$. Note that the sum of any r consecutive coordinates of (α_j) vanishes; conversely, any element $(\alpha_j) \in \mathbf{T}^{\mathbf{Z}}$ for which the sum of every r consecutive coordinates vanishes arises uniquely in this way. The correspondence establishes an isomorphism between $\mathrm{Col}_{\mathbf{T},r}(D)$ and the subgroup of $\mathbf{T}^{\mathbf{Z}}$-colorings which satisfy condition (2.4).

Example 2.9. Consider any diagram D of the trefoil knot 3_1 with three arcs. A $(\mathbf{T}, 2)$-coloring of D is a triple $(\alpha, \beta, \gamma) \in \mathbf{T}^3$ such that $2\alpha = \beta + \gamma$, $2\beta = \gamma + \alpha$, and $2\gamma = \alpha + \beta$. The first condition allows us express γ as $2\alpha - \beta$. The second condition then becomes $3\alpha = 3\beta$. Given any value $\alpha \in \mathbf{T}$, there are exactly three values of β (and one of γ) such that (α, β, γ) determines a $(\mathbf{T}, 2)$-coloring of D.

It is apparent that $\mathrm{Col}_{\mathbf{T},2}(D)$ consists of three disjoint circles in $\mathbf{T}^3$. One of the circles is the connected component N of the identity. Notice that $\mathrm{Col}_{\mathbf{T},2}(D)/N \cong H_1(M_2; \mathbf{Z})$, a cyclic group of order 3. In Theorem 2.10 we establish a general result.

Any $(\mathbf{T}, r)$-coloring of D can be regarded as a triple $(A, B, C) \in (\mathbf{T}^{r-1})^3$ such that

$$(A, B, C) \cdot M(S_r) = (0, 0, 0) \pmod 1,$$

where $M(S_r)$ is the block matrix

$$\begin{pmatrix} I - S_r & S_r & -I \\ -I & I - S_r & S_r \\ S_r & -I & I - S_r \end{pmatrix}.$$

The closely related matrix

$$M(t) = \begin{pmatrix} 1-t & t & -1 \\ -1 & 1-t & t \\ t & -1 & 1-t \end{pmatrix},$$

is almost an Alexander matrix of 3_1. In fact, it is a reduced Alexander matrix of the link $3_1 \sqcup \bigcirc$, the union of l with an unknotted, unlinked circle [**CrFo**], [**Kau**, pp. 201–203]. Note that M presents $H_1(X_\infty(3_1); \mathbf{Z}) \oplus \mathbf{Z}$.

The entries of M are polynomials $m_{i,j}(t)$. It follows from Lemma 2.7 that the block matrix $M(S_r) = (m_{i,j}(S_r))$ that we use to find $(\mathbf{T}, r)$-colorings of D is a presentation matrix for the abelian group $H_1(M_r(3_1); \mathbf{Z}) \oplus \mathbf{Z}^{r-1}$.

Theorem 2.10. Assume that l is an oriented link with diagram D, and r is a positive integer. Let N be the connected component of the identity in $\mathrm{Col}_{\mathbf{T},r}(D)$. Then

$$\mathrm{Col}_{\mathbf{T},r}(D)/N \cong TH_1(M_r(l); \mathbf{Z}).$$

Proof. An exact sequence similar to that in the proof of Theorem 1.3 shows that $\mathrm{Col}_{\mathbf{T},r}(D) \cong [H_1(M_r(l);\mathbf{Z})]^\wedge \oplus \mathbf{T}^{r-1}$. We can decompose $H_1(M_r(l);\mathbf{Z})$ as the direct sum of its torsion subgroup $TH_1(M_r(l);\mathbf{Z})$ and a free abelian group $\mathbf{Z}^k$ of finite rank. Then $\mathrm{Col}_{\mathbf{T},r}(D) \cong [TH_1(M_r(l);\mathbf{Z})]^\wedge \oplus \mathbf{T}^k \oplus \mathbf{T}^{r-1}$. Since the first factor is finite, N is isomorphic to the torus $\mathbf{T}^{k+r-1}$. Hence $\mathrm{Col}_{\mathbf{T},r}(D)/N \cong [TH_1(M_r(l);\mathbf{Z})]^\wedge$, which is isomorphic by Pontryagin duality to $TH_1(M_r(l);\mathbf{Z})$. ∎

3. BRAID ENTROPY AND BRANCHED COVERS. Assume that α is an n-braid with its strands coherently oriented. Let l_s be the oriented link obtained as the closure of the iterated product α^s, for any positive integer s. We combine the idea of $(\mathbf{T}, r)$-colorings with results of symbolic dynamics in order to prove asymptotic results about the homology of $M_r(l_s)$. It is convenient to regard the n arcs at the bottom of α as "input arcs." Those at the top are "output arcs." The braid product $\alpha_1\alpha_2$ is visualized as usual by joining the output strands of α_2 to the input strands of α_1.

Let r be a postive integer. Any labeling of the n input arcs with vectors in $\mathbf{T}^{r-1}$ uniquely determines vectors for the output arcs, using (2.5). The correspondence defines an automorphism f_r of the $n(r-1)$-torus.

Let B_n denote the group of n-braids, with standard generators $\sigma_1, \ldots, \sigma_{n-1}$ chosen so that when strands are coherently oriented each crossing is positve. We recall that the Burau representation $B : B_n \to GL(n, \mathbf{Z}[t, t^{-1}])$ maps σ_i to the block diagonal matrix

$$B_{\sigma_i}(t) = \begin{pmatrix} I_{i-1} & & & \\ & 1-t & 1 & \\ & t & 0 & \\ & & & I_{n-i-1} \end{pmatrix},$$

where I_k denotes the $k \times k$ identity submatrix. The entries of $B_\alpha(t)$ are Laurent polynomials $b_{i,j}(t)$. (See [**Bi**], for example.) It is easy to see that f_r is induced by the block matrix $B_\alpha(S_r)$ obtained from $(b_{i,j}(t))$ by substituting the transpose S_r' of the $(r-1)\times(r-1)$-matrix S_r everywhere for t.

Topological entropy $h(f)$ is defined for any continuous map f of a compact space X [**AdKoMc**]. This measure of the dynamical complexity is generally difficult to compute. However, the topological entropy of a toral automorphism is known [**Yu**]. If f is a torus automorphism induced by an integer matrix M, then $h(f)$ is equal to the log of the Mahler measure of the characteristic polynomial $\mathrm{Det}(xI - M)$.

Definition 3.1. [**Ma**] (See also [**Sc**].) The **Mahler measure** of a polynomial $p(t) = c_d t^d + \cdots + c_1 t + c_0$ $(c_d \neq 0)$ is

$$\mathcal{M}(p(t)) = |c_d| \cdot \prod_{j=1}^{d} \max(|r_j|, 1),$$

where $r_1, \ldots, r_d$ are the roots of $p(t)$.

Theorem 3.2. Let α be an n-braid, and l_s the closure of α^s. Let $\Delta_\alpha(t,x)$ be the characterstic polynomial of the Burau matrix $B_\alpha(t)$. Then for any positive integer r,

$$\lim_{s\to\infty} \frac{1}{s} \log |TH_1(M_r(l_s); \mathbf{Z})| \tag{3.1}$$

exists and it is equal to

$$\Big|\prod_{j=1}^{r-1} \mathcal{M}(\Delta_\alpha(\zeta^j, x))\Big|, \tag{3.2}$$

where ζ is a primitive rth root of unity.

Proof. Let $\mathrm{Fix}(f_r^s)$ denote the subgroup of period s points of f_r. Clearly, $\mathrm{Col}_{\mathbf{T},r}(l_s) \cong \mathrm{Fix}(f_r^s)$. By Theorem 2.10 we have $TH_1(M_r(l_s); \mathbf{Z}) \cong \mathrm{Fix}(f_r^s)/\mathrm{Fix}(f_r^s)^\circ$, where $\mathrm{Fix}(f_r^s)^\circ$ denotes the connected component of the identity in $\mathrm{Fix}(f_r^s)$.

It is shown in [**LiScWa**] that $\lim_{s\to\infty}\big|\mathrm{Fix}(f_r^s)/\mathrm{Fix}(f_r^s)^\circ\big|$ exists and is equal to the topological entropy $h(f_r)$. In view of the comments above, $h(f_r)$ is equal to $\log \mathcal{M}[\mathrm{Det}(xI - B_\alpha(S_r'))]$. The $(r-1)\times(r-1)$ blocks that comprise $xI - B_\alpha(S_r')$ commute, and hence $\mathrm{Det}(xI - B_\alpha(S_r'))$ is equal to the determinant of the $(r-1)\times(r-1)$-matrix $\Delta_\alpha(S_r, x)$, by Lemma A.1 with $R = \mathbf{Z}[t^{\pm 1}, x^{\pm 1}]$ and $\mathcal{R} = \mathbf{Z}[x^{\pm 1}]$ (see Appendix). The matrix S_r' is similar to the diagonal matrix with rth roots of unity $\zeta^j,\ j \neq 1$, along the diagonal. Hence $\mathrm{Det}(\Delta_\alpha(S_r', x)) = \Delta_\alpha(\zeta, x)\cdots\Delta_\alpha(\zeta^{r-1}, x)$, and the theorem follows. ∎

The quantity (3.1) can be regarded as the exponential growth rate of $|TH_1(M_r(l_s); \mathbf{Z})|$. Clearly, conjugate n-braids produce the same links l_s, for each s, and hence the same growth rates.

It is well known that B_n is isomorphic to the mapping class group of $D - P_n$, the 2-disk minus n interior points. Hence any braid can be represented by a homeomorphism of $D - P_n$; two homeomorphisms represent the same braid if and only if they are isotopic rel ∂. The **braid entropy** of $\alpha \in B_n$ is the infimum of $h(\phi)$, taken over all homeomorphisms ϕ representing α.

We recall the description of a useful lower bound for braid entropy. Assume that ϕ is a homeomorphism of $D - P_n$ that representing an n-braid α. Fixing a finite set of generators $g_1, \cdots, g_n$ for the (free) fundamental group of the puntured disk, one iterates the induced automorphism $\phi_\sharp$ on each generator. Then $\gamma(\phi_\sharp)$ is defined to be

$$\max_i \overline{\lim_{k\to\infty}} \frac{1}{k} \log |\phi_\sharp^k(g_i)|,$$

where $|g|$ denotes the length of a shortest word in $g_1^{\pm 1}, \ldots, g_n^{\pm 1}$ that represents g. The quantity is finite and independent of the generator set. It follows from results of R. Bowen [**Bo**] that $\log \gamma(\phi_\sharp) \le h(\phi)$. It is clear that $g(\phi_\sharp)$ is the same for all homeomorphisms ϕ representing α. Hence if it positive, then α has positive braid entropy.

In [**Ko**] B. Kolev proved that $\gamma(\phi_\sharp)$ is bounded below by the spectral radius of the Burau matrix $B_\alpha(t)$ (this result is implicit in [**Fr**]).

Corollary 3.3. Let α be an n-braid, and l_s the closure of α^s. If for some positive integer r, the exponential growth rate of $|TH_1(M_r(l_s); \mathbf{Z})|$ is positive, then the braid α has positive entropy.

Proof. By the hypothesis and Theorem 4.2 there exists an integer j, $1 \le j < r$, such that $\mathcal{M}(\Delta_\alpha(\zeta^j, x)) > 1$. Consequently, some root of $\Delta_\alpha(\zeta^j, x)$ has modulus greater than 1, and so the spectral radius of the Burau matrix $B_\alpha(t)$ is greater than 1. For an arbitrary homeomorphism ϕ representing α, Kolev's result implies that $\log \gamma(\phi_\sharp)$ is positive, and so by Bowen's result $h(\phi) > 0$. Hence α has positve braid entropy. ∎

When the hypothesis of Corollary 3.3 holds, the exponential growth rates increase without bound as r approaches infinity.

Corollary 3.4. Let α be an n-braid, and l_s the closure of α^s. If for some positive integer r, the exponential growth rate of $|TH_1(M_r(l_s); \mathbf{Z})|$ is positive, then

$$\lim_{r \to \infty} \lim_{s \to \infty} \frac{1}{s} \log |TH_1(M_r(l_s); \mathbf{Z})| = \infty.$$

Proof. By the proof of Corollary 3.3 there exists an integer j, $1 \le j < r$, such that some root of $\Delta_\alpha(\zeta^j, x)$ has modulus greater than 1. Let I be an interval about the modulus that does not include zero. For any natural number N, we can choose r' sufficiently large so that there exist at least N r'th roots of unity ζ' such that $\Delta_\alpha(\zeta', x)$ has a root with modulus contained in I. Now Theorem 3.2 implies that the exponential growth rate of $|TH_1(M_{r'}(l_s); \mathbf{Z})|$ is at least N times that of $|TH_1(M_r(l_s); \mathbf{Z})|$. ∎

4. LIE GROUP PALETTES. Colorings can be defined for any topological group Σ. We consider the topological group $\Sigma^{\mathbf{Z}}$ consisting of all bi-infinite sequences (α_j) of elements $\alpha_j \in \Sigma$. As before the shift map $\sigma : \Sigma^{\mathbf{Z}} \to \Sigma^{\mathbf{Z}}$ which sends (α_j) to (α'_j), where $\alpha'_j = \alpha_{j+1}$, is an automorphism.

Definition 4.1. Let Σ be any topological group. Assume that D is a diagram of an oriented link. A $\Sigma^{\mathbf{Z}}$-**coloring** of D is an assignment of elements $C \in \Sigma^{\mathbf{Z}}$ to the arcs of D such that at any crossing

$$(4.1) \qquad (\sigma C_i)C_k = (\sigma C_k)C_j$$

As in Definition 2.1, C_k corresponds to the overcrossing, while C_i, C_j correspond to the undercrossings. The same orientation convention should be applied.

Periodic $\Sigma^{\mathbf{Z}}$-colorings are defined as in Definition 2.2.

The set $\mathrm{Col}_{\Sigma,\infty}(D)$ of $\Sigma^{\mathbf{Z}}$-colorings of a diagram D for an oriented link l is a closed σ-invariant subspace of $\Sigma^{\mathbf{Z}}$. In general it is not a subgroup. However, as a dynamical system it is an invariant $\mathrm{Col}_{\Sigma,\infty}(l)$ of l. In fact, if Y denotes the infinite cyclic cover of the link $l \cup \bigcirc$, then $\mathrm{Col}_{\Sigma,\infty}(D)$ is homeomorphic to $\mathrm{Hom}(\pi_1 Y, \Sigma)$. Here $\pi_1 Y$ is given the discrete topology, and $\mathrm{Hom}(\pi_1 Y, \Sigma)$ receives the compact-open topology.

When Σ is finite, $\mathrm{Col}_{\Sigma,\infty}(l)$ coincides with the representation shift $\Phi_\Sigma(l)$ studied in [**SiWi3**]. In that case we obtain a shift of finite type, a special type of dynamical system that can be completely described by a finite directed graph Γ. The elements ($\Sigma^{\mathbf{Z}}$-colorings) correspond to bi-infinite paths in Γ, while the action of σ on an element is realized by shifting the vertices of the corresponding path.

When Σ is infinite no such simple model is available. Special subspaces such as those consisting of period r points are more amenable.

Definition 4.2. Assume that l is an oriented link with diagram D, and r is a postive integer. A (Σ, r)-**coloring** of D is a periodic $\Sigma^{\mathbf{Z}}$-coloring with period r such that the product in reverse order of any r consecutive coordinates of any assigned label is the identity.

The set $\mathrm{Col}^0_{\Sigma,\infty}(D)$ of based $\Sigma^{\mathbf{Z}}$-colorings and the sets $\mathrm{Col}^0_{\Sigma,r}(D)$ of based (Σ, r)-colorings, for each positive integer r, are defined as section 2. Give $\pi_1 M_r(l)$ the discrete topology, and $\mathrm{Hom}(\pi_1 M_r(l); \Sigma)$ the compact-open topology. The following is immediate from Lemma 2.7.

Proposition 4.3. Assume that l is an oriented link with diagram D, and r is a positive integer. Then $\mathrm{Col}^0_{\Sigma,\infty}(D)$ is homeomorphic to $\mathrm{Hom}(\pi_1 M_r(l), \Sigma)$.

For any topological group, the set of connected components of $\mathrm{Col}_{\Sigma,r}(D)$ is a natural generalization of Fox's n-coloring. By a theorem of W. Goldman [**Go**] the set is finite whenever Σ is a semisimple Lie group with finite center.

The construction for braids also generalizes. If α is an n-braid as in Section 3, then any labeling of the input arcs with vectors in Σ^{r-1} uniquely determines vectors for the output arcs using (4.1). We obtain a homeomorphism f_r of $\Sigma^{n(r-1)}$. The connected components of $\mathrm{Fix}(f_r^s)$ correspond bijectively to those of $\mathrm{Hom}(\pi_1 M_r(l); \Sigma)$.

Conjecture 4.4. Assume that Σ is a semisimple Lie group with finite center. If the topological entropy of $h(f_r)$ is positive, for some positive integer r, then the braid entropy $h(\alpha)$ is positive.

APPENDIX: On determinants of block matrices

Assume that R is a commutative ring, and $\mathcal{R}$ is a commutative ring without zero divisors. Assume further that $\rho : R \to \mathrm{Mat}_n(\mathcal{R})$ is a representation of R in the ring of $n \times n$ matrices over $\mathcal{R}$. For any positive integer k, we extend ρ to a homomorphism $\bar{\rho} : \mathrm{Mat}_k(R) \to \mathrm{Mat}_{nk}(\mathcal{R})$; the image of $M \in \mathrm{Mat}_k(R)$ is a matrix of $n \times n$ blocks in which any two blocks commute.

The following result might be well known to some group of experts. We have been unable to find it in the literature.

Theorem A.1. $\det \bar{\rho}(M) = \det\, [\rho(\det M)]$, for every $M \in \mathrm{Mat}_k(R)$.

Proof. We will prove the result by induction on k. If $k = 1$, the conclusion is obvious. We assume that it is true for $k-1$, and show it for k.

Assume that $M = (m_{i,j})$. Consider the matrix equation:

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ -m_{2,1} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{k,1} & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & m_{1,1} & & \\ & & \ddots & \\ & & & m_{1,1} \end{pmatrix} M = \begin{pmatrix} m_{1,1} & * & * & * \\ 0 & & & \\ \vdots & & N & \\ 0 & & & \end{pmatrix}$$

Taking determinants, we see at once that

$$m_{1,1}^{k-1} (\det M) = m_{1,1}\ (\det N). \tag{A.1}$$

Applying $\bar{\rho}$ to both sides of the matrix equation above, and taking determinants:

$$\begin{aligned} [\det \rho(m_{1,1})]^{k-1} \cdot \det \rho(M) &= \det \rho(m_{1,1}) \cdot \det \rho(N) \\ &= \det \rho(m_{1,1}) \cdot \det\, [\rho(\det N)] \\ &\quad \text{(by the induction hypothesis)} \\ &= \det\, [\rho(m_{1,1}) \cdot \rho(\det N)] \end{aligned}$$

$$= \det\,[\rho(m_{1,1} \cdot \det\, N)]$$

$$= \det\,[\rho(m_{1,1}^{k-1} \cdot \det\, M)]$$

(using (A.1))

$$= [\det\,\rho(m_{1,1})]^{k-1} \cdot \det\,[\rho(\det\, M)]].$$

If $\det\,[\rho(m_{1,1})^{k-1}] \neq 0$, then the proof is complete, since $\mathcal{R}$ is assumed to have no zero divisors. For the general case, embed $\mathcal{R}$ in the polynomial ring $\mathcal{R}[z]$, and replace $\rho(m_{1,1})$ by $\rho(m_{1,1})+zI_n$ in the computation above. The determinant of $\rho(m_{1,1})+zI_n$ is a polynomial of degree n, and hence does not vanish. Examination of the constant terms of the polynomials obtained completes the proof. ∎

The following elementary consequence of Theorem A.1 is worth stating.

Corollary A.2. Assume that $A = (A_{i,j})$ is a $kn \times kn$-matrix consisting of $n \times n$-block matrices $A_{i,j}$ over a commutative ring $\mathcal{R}$ without zero divisors. If the block matrices are pairwise commutative, then

$$\det\, A = \det[\prod_{\pi \in S_k} (\mathrm{sgn}\ \pi) A_{1,\pi(1)} \cdots A_{k,\pi(k)}].$$

The product is the usual one that appears in the definition of determinant. The corollary states that we can first treat the block matrices as matrix coefficients.

Proof. The block matrices generate a subring R of $\mathrm{Mat}_n(\mathcal{R})$. ∎

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