Scaled trace forms of central simple algebras

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Introduction

Any central simple algebra over a field of characteristic not two has a well-defined non-singular quadratic form called the trace form attached to it. This quadratic form was studied in [6] from the viewpoint of the algebraic theory of quadratic forms. In this article we examine a generalization of trace forms to "scaled trace forms" which are defined via the reduced trace map together with scaling by a non-zero element of the algebra.

In section 1 of the paper we give our basic definitions and obtain necessary and sufficient conditions for the scaled trace forms to be non-singular. In section 2 we deal with the "split" case, i.e. the case when the algebra is a full matrix algebra over the field. In section 3 we investigate the algebraic invariants of scaled trace forms. We show that the determinant of a scaled trace form is, up to sign, equal to the reduced norm of the scaling element. Also we give formulae for the signature at each ordering when the underlying field is formally real. We cannot calculate the Hasse-Witt invariant in general but we determine it in two special cases only. In sections 4 and 5 we relate scaled trace forms to the transfer homomorphism of quadratic form theory and use them to obtain information about the kernel and image of the extension of scalars homomorphism of Witt rings.

We use the terminology and notation of the book of Scharlau [9] and refer the reader to this book for any further background information on quadratic form theory.

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1 Scaled trace forms

Let A be a central simple algebra over a field F of characteristic not two. Let A have degree n, i.e. A has dimension n^2 as an F-vector space. The *reduced trace* map $\mathrm{tr}\colon A \to F$ is defined by taking a splitting field L of A, an isomorphism $\rho : A \otimes_F L \to M_n L$, and then defining $\mathrm{tr}\ z$ to be the usual trace of the matrix $\rho(z \otimes 1)$ for each $z \in A$. It is well-known that the value of $\mathrm{tr}\, z$ lies in F and is independent of the choice of L and ρ. See [9, p. 296].

(The *characteristic polynomial* of z over F is defined to be the characteristic polynomial of $\rho(z \otimes 1)$ over L which turns out to have coefficients in F and to be independent of the choice of L and ρ! We could also define $\mathrm{tr}\, z$ as minus the coefficient of the second highest degree term in this characteristic polynomial.)

Let z be a non-zero element of A. The mapping $q_z : A \to F$ given by $q_z(x) = \mathrm{tr}\,(zx^2)$ for each $x \in A$ will be called a *scaled trace form*. Note that for $z = 1$ this reduces to the ordinary trace form which we studied in [6]. Also in [7] we examined scaled trace forms in the split case, i.e. the case $A = M_n F$, the ring of all $n \times n$ matrices with entries in F.

Lemma 1.1. *The map q_z is a quadratic form over F and is non-singular if and only if the equation $zx + xz = 0$ has only the trivial solution $x = 0$ for $x \in A$.*

Proof It is easy to check that q_z is a quadratic form and that its associated symmetric bilinear form is $\phi_z : A \times A \to F$, $\phi_z(x, y) = \frac{1}{2}\,\mathrm{tr}\,\Big(z(xy + yx)\Big)$ for each $x, y \in A$.

The definition of non-singularity yields that q_z is non-singular if and only if the equation $\phi_z(x, y) = 0$ for all $y \in A$ implies $x = 0$. Using the properties of the trace map we can also write $\phi_z(x, y) = \frac{1}{2}\mathrm{tr}\,\Big((zx + xz)y\Big)$. It is well-known [6,lemma 1.1] that the trace form of A is non-singular. The condition for non-singularity of ϕ_z given in the lemma now follows easily.

Corollary 1.2. (Some special cases of note).
(a) If A is a quaternion division algebra and $z \in A$ then q_z is non-singular if and only if $\mathrm{tr}\, z \neq 0$. (Here $\mathrm{tr}\, z = z + \overline{z}$, where $\overline{z}$ is the conjugate of z in A.)
(b) If A is a division algebra of odd degree then q_z is non-singular for all $z \neq 0$.
(c) If $A = M_n F$, (the split case), then q_z is non-singular if and only if $\sigma(z) \cap \sigma(-z)$ is the empty set, $\sigma(z)$ being the set of all eigenvalues of the matrix z in an algebraic closure of F.

Proof
(a) Let A be generated in the usual way by elements i, j satisfying $ij = -ji$, $i^2 = a, j^2 = b$ for some $a, b \in F$. If the equation $zx + xz = 0$ has a non-zero solution $x \in A$ then $z = -xzx^{-1}$ so that $\mathrm{tr}\, z = -\mathrm{tr}\, z$ by properties of trace. Hence $\mathrm{tr}\, z = 0$. Conversely if $\mathrm{tr}\, z = 0$ then $z = \alpha_1 i + \alpha_2 j + \alpha_3 ij$ for some α, β, γ in F. Taking $x = \delta_1 i + \delta_2 j + \delta_3 ij$ with $\delta_1\alpha_1 a + \delta_2\alpha_2 b - \delta_3\alpha_3 ab = 0$ we see that $zx + xz = 0$.
(b) If q_z fails to be non-singular then $zx + xz = 0$ for some $x \in A$, $x \neq 0$, so that $z = -xzx^{-1}$. Let $nr : A \to F$ be the reduced norm map. Recall that, for any $z \in A, nr(z)$ is the determinant of $\rho(z \otimes 1)$ where ρ is the isomorphism used above in defining the reduced trace. (Also $\mathrm{nr}\, z$ could be defined as $(-1)^n$ times the constant term in the characteristic polynomial of z over F.) Then $\mathrm{nr}\, z = -\mathrm{nr}\, z$ using properties of the reduced norm and the fact that the degree of A is odd. Hence $\mathrm{nr}\, z = 0$ so that $z = 0$ in the division algebra A.

(c) The equation $zx + xz = 0$ can be viewed as a set of n^2 linear equations in n^2 unknowns, the entries of the matrix x. The dimension of the solution set of this system does not change on passing to the algebraic closure of F. In this algebraic closure the matrix z is similar to a matrix j in Jordan form, i.e. there exists an invertible matrix p such that $p^{-1}zp = j$. Then the equation $zx+xz = 0$ is equivalent to the equation $jy + yj = 0$ after putting $y = p^{-1}xp$. The eigenvalues of z are the diagonal entries of j and it is an easy exercise to check that $\sigma(z) \cap \sigma(-z)$ being empty is the necessary and sufficient condition for $jy + yj = 0$ to have the unique solution $y = 0$.
See also [7].

Lemma 1.3.
(a) If the elements z_1 and z_2 are conjugate in the central simple algebra A, i.e. $z_2 = tz_1t^{-1}$ for some invertible $t \in A$, then q_{z_1} and q_{z_2} are isometric.
(b) If z is invertible in the central simple algebra A then q_z and $q_{z^{-1}}$ are isometric.

Proof:
(a) The map $A \to A$, $x \to txt^{-1}$ is an isometry.
(b) The map $A \to A$, $x \to z^{-1}x$ is an isometry.

We now look at some examples of scaled trace forms. In each case we will assume that the scaling factor z satisfies the conditions, as in 1.1 and 1.2, which ensure non-singularity of the scaled trace form q_z.

Example 1. *Let A be a quaternion algebra, and write*

$$A = \left(\frac{a, b}{F}\right),$$

i.e. A is generated in the usual way by elements i, j satisfying $ij = -ji$, $i^2 = a$, $j^2 = b$ for some $a, b \in F$. Let $z \in A$ satisfy $\operatorname{tr} z \neq 0$. From (a) and (c) of 1.2 this is the necessary and sufficient condition for q_z to be non-singular.

Writing $t = \operatorname{tr} z$ and $n = \operatorname{nr} z$ we obtain the following diagonalization of q_z.

$$q_z \simeq \langle tn, ta, tb, -tab\rangle$$

This comes from the basis $\{\overline{z}, i, j, ij\}$ of A which is orthogonal with respect to q_z, $\overline{z}$ being the conjugate of z.

Note that the split case, i.e. $A \cong M_2F$, is included here by putting $a = 1$ so that

$$q_z \simeq \langle tn, t\rangle \perp \langle 1, -1\rangle.$$

Example 2. *Let A be a biquaternion algebra, i.e. a tensor product of two quaternion algebras A_1 and A_2. We write*

$$A_r = \left(\frac{a_r, b_r}{F}\right)$$

for $r = 1, 2$ and let i_r, j_r be the generators of A_r so that $i_r^2 = a_r$, $j_r^2 = b_r$, $i_rj_r = -j_ri_r$ etc.

Let $z \in A$ be a basic tensor of the form $z_1 \otimes z_2$ where $z_r \in A_r$ for $r = 1, 2$.

The formula for the reduced trace of a tensor product;

$$\operatorname{tr}_{A_1\otimes A_2}(x \otimes y) = \operatorname{tr}_{A_1}(x)\operatorname{tr}_{A_2}(y) \text{ for all } x \in A_1, y \in A_2$$

shows that $q_z : A \to F$ is the product of the forms q_{z_1} and q_{z_2}.

(This last statement does not require A_1 and A_2 to be quaternion algebras and remains true for any central simple algebras A_1 and A_2).

The description in example 1 now yields the diagonalization

$$q_z \simeq \langle t_1n_1, t_1a_1, t_1b_1, -t_1a_1b_1\rangle\langle t_2n_2, t_2a_2, t_2b_2, -t_2a_2b_2\rangle$$

t_r, n_r denoting the reduced trace and reduced norm of z_r, $r = 1, 2$.

It seems a lot more difficult to obtain a description of q_z for a general element z of a biquaternion algebra.

2 The Split Case

Proposition 2.1.

Let $A = M_nF$ and let $z \in A$ satisfy the condition of 1.2 (c) ensuring non-singularity of q_z. If z is an upper (or lower) triangular matrix then

$$q_z \simeq \langle z_{11}, z_{22}, \ldots, z_{nn}\rangle \perp h$$

where h is a sum of $n(n-1)/2$ hyperbolic planes and z_{ii} are the diagonal entries of z. (These diagonal entries are of course the eigenvalues of z which belong to F since z is triangular.)

Proof: The above description of q_z arises as follows;

The subspace of diagonal matrices in A yields a subform $\langle z_{11}, z_{22}, \ldots, z_{nn}\rangle$ of q_z by using the standard basis matrices e_{ii}, $i = 1, 2, \ldots, n$, which have entry one in the (i, i)-place and zero elsewhere. Its orthogonal complement with respect to q_z is hyperbolic because it contains a totally isotropic subspace of half its dimension, namely the subspace of strictly upper (or strictly lower) triangular matrices.

Proposition 2.2. *Let $A = M_nF$ and let $z \in A$ satisfy the condition of 1.2 (c) ensuring non-singularity of q_z. The Witt index of q_z is at least $n(n-1)/2$.*

Write $z_1 \oplus z_2$ for the block diagonal matrix

$$\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$

It is easy to see that $q_{z_1\oplus z_2}$ is Witt equivalent to the orthogonal sum $q_{z_1} \perp q_{z_2}$. (The subspace of matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ yields the sum $q_{z_1} \perp q_{z_2}$ while its orthogonal complement is hyperbolic because it contains a totally isotropic subspace, the matrices of the form $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$, of half the dimension.) Now any matrix in M_nF is similar to a direct sum of companion matrices, the rational canonical form, so it suffices to prove the result for a companion matrix z, i.e.

$$z = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \end{pmatrix}$$

for some elements $a_1, a_2, \ldots, a_n$ of F.

For such a matrix z one may check that q_z contains a hyperbolic subspace of dimension $n(n-1)$, namely the subspace W of matrices in M_nF which have zero as each entry in the bottom row. The restriction of q_z to W can be shown to be non-singular. (Let $x \in W$ and one can show that $\phi_z(x, y) = 0$ for all $y \in W$ implies $x = 0$.) The strictly upper triangular matrices form a totally isotropic subspace of W which has half the dimension of W. This shows that q_z has a hyperbolic subspace of dimension $n(n-1)$.

Proposition 2.3. *Let $A = M_nF$ and let $z \in A$ satisfy the condition of 1.2 (c) ensuring non-singularity of q_z. Let L be a splitting field of the characteristic polynomial of z over F. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be a complete set of roots of z in L, (i.e. a complete set of eigenvalues of z), including repetitions. Then $q_z^L \simeq \langle\lambda_1, \lambda_2, \ldots, \lambda_n\rangle \perp h$ where h is hyperbolic of dimension $n(n-1)$ and q_z^L denotes the form q_z extended to L.*

Proof. By 1.3 (a) and 2.1 it suffices to show that over L the matrix z becomes similar to an upper triangular matrix. This is shown by induction on n. For $n = 1$ the result is trivially true. Assume the result true for $n-1$. Let λ be an eigenvalue in L of the $n \times n$ matrix z. Let $v \in L^n$ be an eigenvector so that $zv = \lambda v$. Letting v be the first element of some basis of L^n we see that z is similar to a matrix of the form

$$\begin{pmatrix} \lambda & * & * & * & * \\ 0 & & & & \\ & & C & & \\ 0 & & & & \end{pmatrix}$$

where C is an $(n-1) \times (n-1)$ matrix with entries in L. (The above $n \times n$ matrix represents the linear map $f : L^n \to L^n$, $f(v) = zv$, with respect to the chosen basis). The result follows by applying the inductive assumption to C.

Corollary 2.4. *Let $A = M_nF$ and let z_1, z_2 be two elements of A which have the same characteristic polynomial over F. Let L be the splitting field of this polynomial over F and assume $[L : F]$ is odd, i.e. L is an odd degree extension of F. Then q_{z_1} and q_{z_2} are isometric.*

Proof. By 2.3 we see that $q_{z_1}^L$ and $q_{z_2}^L$ are isometric. By Springer's theorem, [9, p. 46–7], it follows that q_{z_1} and q_{z_2} are isometric.

Proof. For an alternative approach to scaled trace forms in the split case see [7] where it is shown that q_z is represented by the matrix $\frac{1}{2}(z^t \otimes i + i \otimes z)w$, where i is the $n \times n$ identity matrix, z^t is the transpose of z, and w is the "shuffle matrix" defined by $w = \sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij} \otimes e_{ji}$. (Here e_{ij} is the standard basis matrix with entry one in the (i, j)-place and zero elsewhere.)

The matrix w is a symmetric matrix which represents the usual trace form of M_nF. The above results on q_z can also be obtained from this viewpoint [7].

Remark 2. *It remains open to find a diagonalization of q_z for matrices z in general. One would hope that there is a description involving the coefficients of the characteristic polynomial of z. Also one may ask whether corollary 2.4 is true for even degree extensions.*

3 Isometry Class Invariants

We now discuss some of the fundamental isometry class invariants of scaled trace forms.

The Determinant

Recall that the determinant of any non-singular quadratic form is the element of the group of square classes F/F^2 given by the determinant of any matrix representing the form.

Proposition 3.1. *Let $q_z : A \to F$ be a scaled trace form on a central simple algebra of degree n over a field F and assume q_z is non-singular. The determinant of q_z is equal to $(-1)^{n(n-1)/2}nrz$ in F/F^2.*

Proof. By the argument in [6, theorem 1.3] it suffices to calculate the determinant in the split case. (It is shown there that the determinant does not change under the extension of F to a generic splitting field of A.)

But this calculation has been done in the split case [7] by using the matrix representation $\frac{1}{2}(z^t \otimes i + i \otimes z)w$ for q_z as described in Remark 1 above. It is shown there that the determinant of this matrix is equal to $(-1)^{n(n-1)/2} \det z$. In fact $\det w = (-1)^{n(n-1)/2}$ while properties of Kronecker sums show that $\det \frac{1}{2}(z^t \otimes i + i \otimes z) = \det z$ (modulo F^2).

By the definition of reduced norm the result follows.

The Signatures

Let F be a formally real field, let $\mathcal{P}$ be an ordering of F, and let q be a quadratic form over F. We write $\mathrm{sig}_{\mathcal{P}} q$ for the signature of the quadratic form q at the ordering P of F. First we can obtain a signature formula for scaled trace forms over quaternion algebras.

3.2. *Let A be a quaternion algebra, and write $A = \left(\frac{a,b}{F}\right)$, i.e. A is generated in the usual way by elements i, j satisfying $ij = -ji$, $i^2 = a$, $j^2 = b$ for some $a, b \in F$. Let $z \in A$ satisfy $\mathrm{tr}\, z \neq 0$ so that q_z is non-singular. Let $t = \mathrm{tr}\, z$ and $n = nrz$. Let F be a formally real field, and let $\mathcal{P}$ be an ordering of F.*

We have the following signature formula;

$$\mathrm{sig}_{\mathcal{P}} q_z = \{(\mathrm{sig}_{\mathcal{P}} q_1) + (\mathrm{sig}_{\mathcal{P}} \langle n \rangle) - 1\} \mathrm{sig}_{\mathcal{P}} \langle t \rangle$$

(Note that q_1 is the usual trace form of A, labelled T_A in $[L]$ where we showed that $\mathrm{sig}_{\mathcal{P}} q_1 = \pm 2$, the minus sign occurring precisely when both a and b are negative in $\mathcal{P}$.)

Proof. We saw earlier that $q_z \simeq \langle tn, ta, tb, -tab \rangle$ and $q_1 = \langle 2, 2a, 2b, -2ab \rangle$. Hence $\mathrm{sig}_{\mathcal{P}} q_z = \mathrm{sig}_{\mathcal{P}} \langle t \rangle \{\mathrm{sig}_{\mathcal{P}} q_1 - \mathrm{sig}_{\mathcal{P}} \langle 1, -n \rangle\}$ because in the Witt ring of F we have $\langle n, a, b, -ab \rangle = \langle 1, a, b, -ab \rangle - \langle 1, -n \rangle$. This gives the result.

Remark We may rewrite this signature formula as follows;
Let $A = \left(\frac{a,b}{F}\right)$ with both a and b being negative in the ordering $\mathcal{P}$ of F. (This includes the case of the real quaternions). Then we have

$$\mathrm{sig}_{\mathcal{P}} q_z = -2 \mathrm{sig}_{\mathcal{P}} \langle t \rangle$$

since n is positive in $\mathcal{P}$ when a and b are both negative. ($z = \alpha + \beta i + \gamma j + \delta ij$ implies $n = \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2$).

Let $A = \left(\frac{a,b}{F}\right)$ with at least one of a and b positive in $\mathcal{P}$. Then we have

$$\mathrm{sig}_{\mathcal{P}} q_z = 2\mathrm{sig}_{\mathcal{P}}\langle t\rangle$$

when n is positive in $\mathcal{P}$,

$$\mathrm{sig}_{\mathcal{P}} q_z = 0$$

when n is negative in $\mathcal{P}$.

Next we will obtain a signature formula in the split case.

Proposition 3..3 *Let $A = M_nF$ and let $z \in A$ satisfy the condition of 1.2 (c) ensuring non-singularity of q_z. Let F be formally real and let $\mathcal{P}$ be an ordering on F.*

Let $F_{\mathcal{P}}$ be a real closure of F so that $F_{\mathcal{P}}(\sqrt{-1})$ is algebraically closed. Let the eigenvalues of the matrix z be $\mu_1, \mu_2, \dots, \mu_r, \lambda_1, \overline{\lambda}_1, \lambda_2, \overline{\lambda}_2, \dots, \lambda_s, \overline{\lambda}_s$ where $\mu_i \in F_{\mathcal{P}}$ for each $i, \lambda_i \in F_{\mathcal{P}}(\sqrt{-1})$ for each $i, \overline{\lambda}$ denotes the conjugate of λ in $F_{\mathcal{P}}(\sqrt{-1})$, and $r + 2s = n$. (We do not exclude the possibility of repetitions in the μ_i or λ_i). We have the following signature formula;

$$\mathrm{sig}_{\mathcal{P}} q_z = \sum_{i=1}^{r} \mathrm{sig}_{\overline{\mathcal{P}}}\langle \mu_i + 2\sum_{j=1}^{s} \mathrm{sig}_{\overline{\mathcal{P}}}\langle \lambda_j + \overline{\lambda}_j\rangle$$

where $\overline{\mathcal{P}}$ is the unique ordering on $F_{\mathcal{P}}$.
(Note that $\mu_i \neq 0$ for all i and $\lambda_j + \overline{\lambda}_j \neq 0$ for all j by the assumption of non-singularity of q_z.)

Proof. Using the real Jordan form [3] we have that over $F_{\mathcal{P}}$ the matrix z is similar to a sum blocks of the following kind. For each eigenvalue $\mu \in F_{\mathcal{P}}$ we have the usual kind of Jordan blocks which are upper triangular. For each unrepeated eigenvalue $\lambda = \alpha + \beta\sqrt{-1}$ we get a 2×2 block $B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ in the real Jordan form. (This corresponds to the conjugate pair λ, $\overline{\lambda}$.) For each repeated conjugate pair $\lambda, \overline{\lambda}$ we have blocks like

$$\begin{pmatrix} B & I & 0 & \cdots & 0 \\ 0 & B & I & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & B & I \\ 0 & 0 & 0 & 0 & B \end{pmatrix}$$

where B is as above and I is the identity 2×2 matrix. This is a "block upper triangular matrix" and the scaled trace form it yields is Witt equivalent to that given by the block diagonal matrix

$$\begin{pmatrix} B & 0 & 0 & \cdots & 0 \\ 0 & B & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & B & 0 \\ 0 & 0 & 0 & 0 & B \end{pmatrix}$$

(In the same way as in 2.1 the subspace of diagonal 2×2 blocks yields the form given by this latter matrix while its orthogonal complement is hyperbolic because the set of strictly upper triangular blocks give a totally isotropic subspace of half the dimension.)

Using the split case of example 1 earlier we see that q_B is Witt equivalent over $F_{\mathcal{P}}$ to the two-dimensional form $\langle \lambda + \overline{\lambda},\ \lambda\overline{\lambda}(\lambda + \overline{\lambda})\rangle$. (Note that if $B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ we have $t = 2\alpha = \lambda + \overline{\lambda}$ and $n = \alpha^2 + \beta^2 = \lambda\overline{\lambda}$.) Now, as in the proof of 2.2, the scaled trace form q_z will decompose into a sum

$$\langle \mu_1, \mu_2, \ldots, \mu_r \rangle \perp \langle \lambda_1 + \overline{\lambda}_1, \lambda_1\overline{\lambda}_1(\lambda_1 + \overline{\lambda}_1), \ldots, \lambda_s + \overline{\lambda}_s, \lambda_s\overline{\lambda}_s(\lambda_s + \overline{\lambda}_s)\rangle \perp h$$

with h hyperbolic. The signature formula follows since $\lambda\overline{\lambda}$ is positive in $F_{\mathcal{P}}$.

Proposition 3.4. *Let A be any central simple algebra over F and let $z \in A$ satisfy the condition of 1.1 ensuring non-singularity of q_z. Let F be formally real and let $\mathcal{P}$ be an ordering on F. Let $F_{\mathcal{P}}$ be a real closure of F so that $F_{\mathcal{P}}(\sqrt{-1})$ is algebraically closed. Let the roots of the characteristic polynomial of z over F be $\mu_1, \mu_2, \ldots, \mu_r, \lambda_1, \overline{\lambda}_1, \lambda_2, \overline{\lambda}_2, \ldots, \lambda_s, \overline{\lambda}_s$ where $\mu_i \in F_{\mathcal{P}}$ for each i, $\lambda_i \in F_{\mathcal{P}}(\sqrt{-1})$ for each i, $\overline{\lambda}$ denotes the conjugate of λ in $F_{\mathcal{P}}(\sqrt{-1})$, and where $r + 2s = n$. We do not exclude the possibility of repetitions in the μ_i or λ_i.*
We have the following signature formulae;

$$sig_{\mathcal{P}} q_z = (-1)^{sgn_{\mathcal{P}} A} \left(\sum_{i=1}^{r} sig_{\overline{\mathcal{P}}} \langle \mu_i \rangle + 2 \sum_{j=1}^{s} sig_{\overline{\mathcal{P}}} \langle \lambda_j + \overline{\lambda}_j \rangle \right)$$

where $\overline{\mathcal{P}}$ is the unique ordering on $F_{\mathcal{P}}$, and $sgn_{\mathcal{P}} A$ is the "sign function" defined as in [10, p 74], i.e. $sgn_{\mathcal{P}} A = 0$ if $F_{\mathcal{P}}$ splits A, $sgn_{\mathcal{P}} A = 1$ otherwise.

Proof. Either $F_{\mathcal{P}}$ splits A completely, i.e. $A \otimes F_{\mathcal{P}} \cong M_n F_{\mathcal{P}}$, or else $A \otimes F_{\mathcal{P}} \cong M_m D$ where $m = n/2$ and $D = \left(\dfrac{-1, -1}{F_{\mathcal{P}}} \right)$, the "quaternions" over $F_{\mathcal{P}}$. In the first case the result follows from 3.3 since any isomorphism $\rho : A \otimes F_{\mathcal{P}} \to M_n F_{\mathcal{P}}$ yields $\rho(z \otimes 1)$ as a matrix whose eigenvalues are the roots of the characteristic polynomial of z over F.

In the second case $\rho(z \otimes 1)$ will be similar in $M_m D$ to an upper triangular matrix of the form

$$\begin{pmatrix} d_1 & * & * & \cdots & * \\ 0 & d_2 & * & \cdots & * \\ \vdots & & & \ddots & * \\ 0 & 0 & \cdots & 0 & d_m \end{pmatrix}$$

where each $d_i \in D$ and the characteristic polynomial of z over F decomposes over $F_{\mathcal{P}}$ into a product $\prod_{i=1}^{m} (x - d_i)(x - \overline{d}_i)$. In the same way as we dealt with triangular matrices earlier we find that q_z becomes Witt equivalent to an orthogonal sum $\sum_{i=1}^{m} q_{d_i}$ of scaled trace forms over D. Writing $t_i = d_i + \overline{d}_i$ and $n_i = d_i\overline{d}_i$ we have, by example 1 earlier, that $q_{d_i} \simeq \langle t_i n_i, t_i a, t_i b, -t_i ab \rangle$ and hence, by 3.2, $\mathrm{sig}_{\mathcal{P}} q_{d_i} = -2\ \mathrm{sig}_{\mathcal{P}} \langle t_i \rangle$. But corresponding to the quadratic $(x - d_i)(x - \overline{d}_i)$ we have a pair of roots $\lambda = \frac{1}{2}\left(t_i \pm \sqrt{(t_i^2 - 4n_i)}\right)$ of the characteristic polynomial of z over F. If $t_i^2 - 4n_i \geq 0$ in $\mathcal{P}$ then the two roots are in $F_{\mathcal{P}}$ and they both have the same sign as t_i. (Note

that $n_i > 0$ in $F_{\mathcal{P}}0$. If $t_i^2 - 4n_i < 0$ in $\mathcal{P}$ then we have a pair of conjugate roots $\lambda_i, \overline{\lambda}_i$ in $F_{\mathcal{P}}(\sqrt{-1})$ and $t_i = \lambda_i + \overline{\lambda}_i$.
This completes the proof.

Remark. *In the special case when $z = 1$ the signature formula in 3.4 reduces to that of [6, theorem 2.3(i)].*

We now give another signature formula involving extension fields of F in the case when A is a division algebra.

Proposition 3.5. *Let A be a central division algebra over the field F and let $\mathcal{P}$ be an ordering of F. Let $z \in A$ satisfy the condition of 1.1 ensuring the non-singularity of q_z. Let L be a finite separable extension field of F and suppose that the ordering $\mathcal{P}$ extends to at least one ordering of L.*
We have the following signature formula;

$$\mathrm{sig}_{\mathcal{P}} q_z = \frac{1}{m}\left(\sum_{\mathcal{R}\supseteq\mathcal{P}} \mathrm{sig}_R q_z^L\right)$$

where m is the number of extensions of the ordering $\mathcal{P}$ to an ordering $\mathcal{R}$ of L, and the sum is over all such orderings R.

Proof. Consider the transfer map $\mathrm{tr}_* : W(L) \to W(F)$ of Witt rings induced by the reduced trace map $L \to F$. We apply the Frobenius reciprocity theorem in quadratic form theory [9,p 48] to the forms q_z over F and $\langle 1\rangle$ over L. This yields that $\mathrm{tr}_*(q_z^L) \simeq q_z \cdot \mathrm{tr}_*\langle 1\rangle$. Now $m = \mathrm{sig}_{\mathcal{P}}\mathrm{tr}_*\langle 1\rangle$ by $[9, p\ 116]$ so that taking signatures yields $\mathrm{sig}_{\mathcal{P}}\mathrm{tr}_*(q_z^L) = m\ \mathrm{sig}_{\mathcal{P}} q_z$. By a theorem of Knebusch, $[9, p\ 124]$,

$$\mathrm{sig}_{\mathcal{P}}\mathrm{tr}_*(q_z^L) = \sum_{R\supseteq\mathcal{P}} \mathrm{sig}_R q_z^L$$

and this completes the proof.

Corollary 3.6. *Let A be a central division algebra over the field F and let $\mathcal{P}$ be an ordering of F. Let L be a finite separable extension field of F and suppose the ordering $\mathcal{P}$ extends to at least one ordering of L. We have the following formula for the "sign function";*

$$(-1)^{\mathrm{sgn}_{\mathcal{P}} A} = \sum_{\mathcal{R}\supseteq\mathcal{P}} (-1)^{\mathrm{sgn}_{\mathcal{R}}(A\otimes_F L)}$$

Proof. Let $z = 1$ in 3.5, use the formula $\mathrm{sig}_{\mathcal{P}} q_1 = (-1)^{\mathrm{sgn}_{\mathcal{P}} A}(\deg A)$ obtained in [6], and the fact that the degree is unchanged on extending to L.

Proposition 3.7. *Let A be a central division algebra over the field F and let $\mathcal{P}$ be an ordering of F. Let $z \in A$ satisfy the condition of 1.1 ensuring the non-singularity of q_z. Let L be a maximal subfield of A which contains the subfield $F(z)$ of A. Assume that L is a Galois extension of F and that the ordering $\mathcal{P}$ extends to at least one ordering of L. We have the following signature formula;*

$$\mathrm{sig}_{\mathcal{P}} q_z = \frac{1}{m}(\deg A)\left(\sum_{\mathcal{R}\supseteq\mathcal{P}} \mathrm{sig}_R\langle z\rangle\right)$$

where m is the number of extensions of the ordering $\mathcal{P}$ to an ordering $\mathcal{R}$ of L, and the sum is over all such orderings R.

Proof. L is a splitting field of A so that $A \otimes_F L \cong M_n L$, where $n = \deg A$, and under any isomorphism $z \otimes 1$ will map to a matrix similar to a diagonal matrix with entries z^σ where σ runs through the elements of the Galois group G of L over F. (We can use the canonical isomorphism $A \otimes_F L \to \mathrm{End}_L A, x \otimes y \to f_{x\otimes y}$ where $f_{x\otimes y}(w) = xwy$ for $w \in L$, A being viewed as a right L-vector space of dimension n. Since L is Galois, A is a crossed product algebra with an L-basis of elements $\{t_\sigma\}, \sigma \in G$, σ extending to the inner automorphism of A via t_σ. Then $f_{z\otimes 1}$ has the required diagonal matrix representation with respect to the above L-basis.)

Hence, using 2.1 and 1.3 (a), q_z^L is Witt equivalent to $\sum_{\sigma\in G}\langle z^\sigma\rangle$. Since $\mathrm{tr}_*\langle z^\sigma\rangle = \mathrm{tr}_*\langle z\rangle$ for all $\sigma \in G$ we see that $tr_* q_z^L$ is the sum of n copies of $\mathrm{tr}_*\langle z\rangle$ up to Witt equivalence. As in the proof of 3.5 we use the Frobenius reciprocity theorem and the Knebusch theorem to obtain the result.

The Hasse-Witt invariant.

In general it does not seem so easy to calculate the Hasse-Witt invariant of the forms q_z. In the split case it seems that almost any value may occur as the Hasse-Witt invariant of q_z depending on the choice of z.

In the case of $A = \left(\dfrac{a,b}{F}\right)$, a quaternion algebra, we can easily see from our description in example 1 earlier that the Hasse-Witt invariant [9,p 80] is given by $s(q_z) = (a,b)(t,-n)(n,-1)$ in the Brauer group $B(F)$ where t and n are the reduced trace and reduced norm of z respectively.

In the case of a biquaternion algebra and scaled trace form q_z for $z = z_1 \otimes z_2$ as in example 2 earlier we saw that

$$q_z \simeq \langle t_1n_1, t_1a_1, t_1b_1, -t_1a_1b_1\rangle\langle t_2n_2, t_2a_2, t_2b_2, -t_2a_2b_2\rangle$$

Using the formula in [5] for the Hasse-Witt invariant of a product of quadratic forms it is easy to check that $s(q_z) = (-n_1, -n_2)$ in $B(F)$ for this example.

An adaption of the Galois cohomology technique used in [8] to calculate the Hasse-Witt invariant of the usual trace form, i.e. q_1, may well be the best approach to try in general. In [8] we showed that

$$s(q_1) = (-1,-1)^{n(n-2)/8} A^{n/2} \text{ in } B(F)$$

for a central simple algebra A of even degree n over F. (For n odd the trace form q_1 is isometric to the trace form in the split case and so is not so interesting). We leave it as an open problem to calculate $s(q_z)$ in general.

4 Witt kernels

We now describe a relationship between scaled trace forms and the usual transfer map [9,p 48] and show how this viewpoint may give new information about Witt kernels.

Let L be a finite separable extension of the field F. The trace map $tr : L \to F$ yields a transfer map $\mathrm{tr}^* : W(L) \to W(F)$, an additive homomorphism of Witt rings, and extension of scalars gives a ring homomorphism $r_* : W(F) \to W(L)$.

The kernel of r_* is an ideal in $W(F)$ often known as a *Witt kernel*. Let $y \in L$ and consider the map $m_y : L \to L$ given by left multiplication by y. Now m_y is represented by a matrix in M_nF where $n = [L : F]$, the matrix being unique up to similarity, and hence we may define a scaled trace form $q_{m_y} : M_nF \to F$ as in section 1. (Note that q_{m_y} is not necessarily non-singular. Take $F = \mathbf{R}$, $L = \mathbf{C}$, $y = i$ and q_{m_y} is singular).

Proposition 4.1. *Let L be a Galois extension of F and let $y \in L$ be such that the scaled trace form q_{m_y} is non-singular. Let $tr^* : W(L) \to W(F)$ be the transfer map induced by the trace $tr\, L \to F$. Then the element $q_{m_y} - tr^*\langle y\rangle$ of $W(F)$ is in the kernel of the ring homomorphism $r_* : W(F) \to W(L)$. Moreover if $[L : F]$ is odd then $q_{m_y} = tr^*\langle y\rangle$ in $W(F)$.*

Proof. Let G be the Galois group of L over F and let $n = [L : F]$. Then there is a canonical isometry $(tr^*\langle y\rangle)^L \simeq_{\sigma\in G} \langle y^\sigma\rangle$. See [9,p6 1]. The characteristic polynomial over F of the matrix m_y similar in M_nL to a diagonal matrix with the diagonal entries $y^\sigma, \sigma \in G$. (The characteristic polynomial is a power of the minimal polynomial of y over F, the minimal polynomial is irreducible, and m_y is similar to a sum of copies of the companion matrix of the minimal polynomial. The companion matrix is diagonalizable over L since it has distinct roots in L. If H is the subgroup of G corresponding to the subfield $F(y)$ of L then $y^\sigma = y^{\sigma'}$ if and only if the cosets σH and $\sigma' H$ are equal.) Now 2.1 shows that $q^L_{m_y}$ is Witt equivalent to $\sum_{\sigma\in G}\langle y^\sigma\rangle$. Hence $q_{m_y} - \mathrm{tr}^*\langle y\rangle$ is in the Witt kernel. When $[L : F]$ is odd this Witt kernel is well-known to be zero by Springer's theorem [9,p 46]. Thus $q_{m_y} = \mathrm{tr}^*\langle y\rangle$ in $W(F)$ for $[L : F]$ odd.

Remark 1.

For $[L : F]$ even it is generally not the case that $q_{m_y} = \mathrm{tr}^*\langle y\rangle$ in $W(F)$. For example, taking $y = 1$ yields $q_{m_y} = \langle 1, 1, \dots, 1\rangle$, the sum of n copies of $\langle 1\rangle$, in $W(F)$ while $\mathrm{tr}^*\langle 1\rangle$ is the trace form of L over F.

Remark 2.

Witt kernels are not so well understood in general. See [1], [2], [4], [11] for some results on Witt kernels, mainly in the cases of quadratic, biquadratic and quartic extensions.

We have a method of generating elements of the Witt kernel by taking all elements in $W(F)$ of the kind $q_{m_y} - \mathrm{tr}^*\langle y\rangle$ where y runs through all the elements of L for which q_{m_y} is non-singular. As suggested by Jan van Geel one may ask whether the full Witt kernel is equal to the ideal in $W(F)$ generated by all elements of the above type. We cannot answer this question in general, but for quadratic extensions it is easy to see that the answer is yes. (Taking $y = 1$ we see that $q_{m_1} - \mathrm{tr}^*\langle 1\rangle$ generates the Witt kernel.)

5 Witt images

Let L be an extension field of F and $r_* : W(F) \to W(L)$ be the usual extension of scalars homomorphism. The image of r_* will be called a *Witt image*. We will see now that scaled trace forms can yield information about Witt images.

Proposition 5.1. *Let L be an extension field of F and $r_* : W(F) \to W(L)$ be the usual extension of scalars homomorphism. Let ϕ be an anisotropic quadratic form over L. Suppose that ϕ has a diagonal representation $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ for some elements $\alpha_i \in L$, $i = 1, 2, \ldots, n$, and for each $i = 1, 2, \ldots, n$ the value of the elementary symmetric function $p_i(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is in F. Then ϕ is in the Witt image of $r_* : W(F) \to W(L)$.*

Proof. Recall that the functions p_j are defined as follows ;

$$\begin{aligned}
p_1(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \sum_{i=1}^{n} \alpha_i, \\
p_2(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \sum_{i<j}^{n} \alpha_i \alpha_j, \\
p_3(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \sum_{i<j<k}^{n} \alpha_i, \alpha_j, \alpha_k \\
&\vdots \quad \vdots \\
p_n(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \prod_{i=1}^{n} \alpha_i
\end{aligned}$$

Then the polynomial $\prod_{i=1}^{n}(x - \alpha_i)$ is in the polynomial ring $F[x]$ since its coefficients are, up to sign, given by the elementary symmetric functions of the roots α_i. Let c denote the companion matrix of this polynomial so that $c \in M_n F$ and consider the scaled trace form q_c over F. The characteristic polynomial of c is $\prod_{i=1}^{n}(x - \alpha_i)$ which splits completely in L. Thus $r_*(q_c) = \phi$ in $W(L)$ by 2.1.

Corollary 5.2. *Let K be a field of characteristic not two. Let $L = K(x_1, x_2, \ldots, x_n)$, the field of rational functions over K in the indeterminates $x_1, x_2, \ldots, x_n$. Let $F = K(p_1, p_2, \ldots, p_n)$, the subfield of symmetric rational functions in $x_1, x_2, \ldots, x_n$. (We have written p_i as short for $p_i(x_1, x_2, \ldots, x_n)$.) Then the quadratic form $\langle x_1, x_2, \ldots, x_n \rangle$ over L is in the Witt image of $r_* : W(F) \to W(L)$.*

Proof. Take $\phi = \langle x_1, x_2, \ldots, x_n \rangle$ in 5.1.

Remark 1.
It is easy to obtain the result of 5.2 directly for $n = 2$ by using the Witt relation $\langle \alpha, \beta \rangle = \langle \alpha + \beta, \alpha\beta(\alpha + \beta) \rangle$. Specifically we have

$$\langle x_1, x_2 \rangle = \langle x_1 + x_2, x_1 x_2 (x_1 + x_2) \rangle = \langle p_1, p_1 p_2 \rangle$$

With a little more work we can do a similar thing for $n = 3$. We find that

$$\langle x_1, x_2, x_3 \rangle = \langle p_1, p_1 p_2 - p_3, p_1 p_3 (p_1 p_2 - p_3) \rangle$$

where $p_1 = x_1 + x_2 + x_3$, and $p_1 p_2 - p_3 = (x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$.

Maybe it is possible to obtain the result of 5.2 for general n by performing some more complicated manipulations involving the Witt relation but this does not seem obvious. Also the Witt kernel of $r_* : W(F) \to W(L)$ is most certainly not trivial so there can be many different elements of $W(F)$ which map to the element $\langle x_1, x_2, \ldots, x_n\rangle$ of $W(L)$.

Remark 2.
Note that $K(x_1, x_2, \ldots, x_n)$ is a Galois extension of $K(p_1, p_2, \ldots, p_n)$ with Galois group S_n acting in the obvious way on $K(x_1, x_2, \ldots, x_n)$. I am indebted to David Leep for pointing out the following alternative proof of 5.2. The result of Knebusch-Scharlau [9,p 61], as mentioned at the start of the proof of 4.1, is easily generalized to the following;

Let K be a Galois extension of a field F with Galois group G. Let E be an intermediate field corresponding to the subgroup H of G, i.e. E is the subfield of K fixed by H. Let C be a complete set of coset representatives of H in G so that G is the disjoint union of the cosets σH where σ runs through C. Let ϕ be a nonsingular quadratic form over E and let $tr^*_{E/F}$ be the transfer map given by the trace map from E to F. Then the generalized version of the theorem of [9,p 61] says that $(tr^*_{E/F}\phi)^K = \sum_{\sigma\in C}\langle\phi^\sigma\rangle$.

We now apply this to the situation of 5.2. Take K and F as in the statement of 5.2, take $E = F(x_1)$, $G = S_n, H$ the subgroup of S_n fixing x_1, so $H \cong S_{n-1}$, and take $\phi = \langle x_1\rangle$ viewed as a form over E. For coset representatives we may take $C = \{\tau_1, \tau_2, \tau_3, \ldots, \tau_n\}$ where τ_1 is the identity of S_n and τ_j is the transposition switching x_1 and x_j for each $j = 2, 3, \ldots, n$. The conclusion of 5.2 follows at once.

Remark 3.
Another proof of 5.2, via matrix theory, can be seen as follows.

The diagonal matrix D with entries $x_1, x_2, \ldots, x_n$ is congruent over L to the symmetric matrix

$$S = \begin{pmatrix} s_1 & s_2 & s_3 & \cdots & s_n \\ s_2 & s_3 & s_4 & \cdots & s_{n+1} \\ \vdots & & & \ddots & \vdots \\ s_n & s_{n+1} & \cdots & \cdots & s_{2n-1} \end{pmatrix}$$

whose entry in the (i,j)-place is the power sum s_{i+j-1}, where $s_k = \sum_{i=1}^{n} x_i^k$, each s_k being in F since the power sums are symmetric functions of $x_1, x_2, \ldots, x_n$.

Specifically it is easy to check that $VDV^t = S$ where V is the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & & & \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}$$

which is an invertible matrix in L since the x_i are distinct.

References

[1] R. Elman, T. Y. Lam and A. Wadsworth, *Quadratic forms under multiquadratic extensions*, Indagationes Math. **42**, (1980), 131–145.

[2] R. Elman, T. Y. Lam, J-P. Tignol, and A. Wadsworth, *Witt rings and Brauer groups under multiquadratic extensions*, Amer. J. Math. **105**, (1983), 1119–1170.

[3] R. A. Horn and C. R. Johnson, Matrix analysis, Cambridge University Press, Cambridge 1985.

[4] T. Y. Lam, D. B. Leep and J-P.Tignol, *Biquaternion algebras and quartic extensions*, I.H.E.S. Pub.Math. no.77 (1993), 63–102.

[5] D. W. Lewis, *Some canonical pairings of Witt groups of forms*, J. Algebra, **81**, (1983), 108–120.

[6] D. W. Lewis, *Trace forms of central simple algebras*, Math. Zeitschrift, **215**, (1994), 367–375.

[7] D. W. Lewis, *Trace forms, Kronecker sums, and the shuffle matrix*, Linear and Multilinear Algebra, **40**, (1996), 221–227.

[8] D. W. Lewis and J. Morales, *The Hasse invariant of the trace form of a central simple algebra*, Pub. Math. de Besançon, Théorie des nombres, Année 1993-94, 1–6.

[9] W. Scharlau, *Quadratic and hermitian forms*, Springer-Verlag, Berlin Heidelberg New York (1985).

[10] D. B. Shapiro, J-P.Tignol and A.Wadsworth, *Witt rings and Brauer groups under multiquadratic extensions II*, J. Algebra **78**, (1982), 58–90.

[11] R.Ware, *Some remarks on the map between Witt rings of an algebraic extension*, Conf. on quadratic forms, ed.g.Orzech, Queens papers in pure and applied math. no. 46 (1976), 634–649.

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