Drinfeld cusp forms and their combinatorics

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Outline

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Basic notation

$\mathbb{F}_q$ the field of $q = p^n$ elements
K_∞ a local field with residue field $\mathbb{F}_q$
$\mathcal{O}_\infty$ the ring of integers of K_∞
π a uniformizer of K_∞

The Bruhat-Tits tree

K_∞, $\mathcal{O}_\infty$, π, $\mathbb{F}_q$

Definition (Bruhat-Tits tree)

$\mathcal{T}$:= the simplicial complex of dimension 1 with

set of vertices $\mathrm{Vert}(\mathcal{T}) :=$
homothety classes $[\mathrm{L}]$ of rank 2 $\mathcal{O}_\infty$-lattices $\mathrm{L} \subset \mathrm{K}_\infty^2$

set of edges $\mathrm{Edge}(\mathcal{T}) :=$
pairs $([\mathrm{L}], [\mathrm{L}'])$ such that $\pi\mathrm{L} \subsetneq \mathrm{L}' \subsetneq \mathrm{L}$.

$|\mathcal{T}|$ the geometric realization of $\mathcal{T}$

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Lemma

$\mathcal{T}$ is a $q + 1$-regular tree.

Definition

Vertices $\Lambda_i := [\mathcal{O}_\infty \oplus \pi^i \mathcal{O}_\infty]$, $i \in \mathbb{Z}$.
standard vertex Λ_0, standard edge $e_0 := (\Lambda_0, \Lambda_1)$

Group action

K_∞, $\mathcal{O}_\infty$, π, $\mathbb{F}_\mathrm{q}$

Consider elements of K_∞^2 as **column vectors**
$\Rightarrow$ have natural left action of $\mathrm{GL}_2(\mathrm{K}_\infty)$ on K_∞^2.

Group action

K_∞, $\mathcal{O}_\infty$, π, $\mathbb{F}_q$

Consider elements of K_∞^2 as **column vectors**
$\Rightarrow$ have natural left action of $GL_2(K_\infty)$ on K_∞^2.

Definition ($GL_2(K_\infty)$-action on $\mathcal{T}$)

$$GL_2(K_\infty) \times \mathcal{T} \to \mathcal{T} : (\gamma, [L]) \mapsto [\gamma L]$$

Set $\Gamma_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_\infty) \mid c \in \pi\mathcal{O}_\infty \right\}$.

Lemma

$GL_2(K_\infty)$ acts transitively on $Vert(\mathcal{T})$ and $Edge(\mathcal{T})$.

$Vert(\mathcal{T}) = GL_2(K_\infty)/GL_2(\mathcal{O}_\infty)K_\infty^*$,
$Edge(\mathcal{T}) = GL_2(K_\infty)/\Gamma_\infty K_\infty^*$.

Drinfeld's upper half plane Ω

$\mathbb{C}_\infty := \widehat{K_\infty^{alg}}$

Definition (Drinfeld's upper half plane)

$\Omega := \mathbb{P}^1(\mathbb{C}_\infty) \smallsetminus \mathbb{P}^1(K_\infty)$

Definition ($GL_2(K_\infty)$-action on Ω)

$GL_2(K_\infty) \times \Omega \to \Omega : (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z) \mapsto \gamma z = \frac{az+b}{cz+d}$

(rigid) analysis on Ω

$$K_\infty,\ \mathcal{O}_\infty,\ \pi,\ \mathbb{F}_q,\ \mathbb{C}_\infty,\ \Omega$$

Let $b_1, \ldots, b_q \in \mathcal{O}_\infty$ be representatives of $\mathcal{O}_\infty/\pi$.

Proposition (reduction map)

$\exists$ *a (natural)* $GL_2(K_\infty)$*-equivariant map*

$$\rho\colon \Omega \to |\mathcal{T}| \text{ such that}$$

$$\rho^{-1}(|e_0| \smallsetminus \{\Lambda_0, \Lambda_1\}) = \{z \in \mathbb{C}_\infty \mid 1 < |z| < q\}$$

$$\rho^{-1}(\Lambda_0) = \{z \in \mathbb{C}_\infty \mid \underset{i=1,\ldots,q}{\forall} |z - b_i| = 1\}$$

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On these charts use Laurent series type expansions to define **(rigid) analytic functions on** Ω.

Drinfeld modular forms

From now on: $K_\infty := \mathbb{F}_q((\frac{1}{T}))$, $\pi := \frac{1}{T}$.

For $A := \mathbb{F}_q[T]$ and $K = \mathrm{Frac}(A)$ have

$$GL_2(A) \hookrightarrow GL_2(K) \hookrightarrow GL_2(K_\infty).$$

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For $\Gamma \subset GL_2(A)$ a congruence subgroup:

Definition (Drinfeld modular form (Goss))

A **Drinfeld modular form of weight k (and trivial type) for** Γ is a rigid analytic function

$$f : \Omega \to \mathbb{C}_\infty$$

such that
(a) $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
(b) f has a Laurent series expansion at all cusps with vanishing principal part.

Results

One defines cusp forms (in the obvious way).
Have Hecke operators for prime ideals $0 \neq \mathfrak{p} \subset \mathbb{F}_q[T]$
No known analog of a Petersson inner product.

Let f be a Hecke eigenform with eigenvalues $a_{\mathfrak{p}}(f)$.

Theorem (Goss)

The $a_{\mathfrak{p}}(f)$ are integral
$K_f := K(\{a_{\mathfrak{p}}(f)\}_{\mathfrak{p}})$ *is finite over K.*

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Theorem (B.)

There is a strictly compatible system

$$\left(\rho_{f,\lambda}\colon Gal(\overline{K}/K) \to GL_1(\widehat{K_f}^{\lambda})\right)_{\lambda \text{ finite}}$$

such that $\rho_{f,\lambda}(Frob_{\mathfrak{p}}) = a_{\mathfrak{p}}(f)$ for almost all p.

The sequence $(a_{\mathfrak{p}}(f))_{\mathfrak{p}}$ is given by a Hecke character.

Questions

There is no multiplicity one result!

Does multiplicity one hold for fixed weight?

Does it hold in weight 2

Does it hold in weight 2 and for $\Gamma_0(\mathfrak{p})$ with $\mathfrak{p}$ prime?

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There may be $\mathfrak{p}$ not dividing the level N of f such that

$$\rho_{f,\lambda}(Frob_{\mathfrak{p}}) \neq a_{\mathfrak{p}}(f)$$

(because of non–ordinariness of modular curves of level Nq)

What happens at these $\mathfrak{p}$?

Harmonic cocycles

$\mathcal{T}$, $Edge(\mathcal{T})$, Γ

How to compute Drinfeld modular forms?.

Let M be a $K[GL_2(A)]$-module with $\dim_K(M)$ finite.

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Definition

The K-vector space $C_{har}(\Gamma, M)$ of M-**valued** Γ-**invariant harmonic cocycles** is the set of maps

$$c\colon Edge(\mathcal{T}) \to M : e \mapsto c(e),$$

such that:

1. For all edges e one has $c(-e) = -c(e)$.
2. For all vertices v one has $\sum_{e\to v} c(e) = 0$,
   where the sum is over all edges e ending at v.
3. For all $\gamma \in \Gamma$ and $e \in Edge(\mathcal{T})$ one has $c(\gamma e) = \gamma c(e)$.

Basic properties

$\mathcal{T}$, $Edge(\mathcal{T})$, Γ

Proposition (automatic cuspidality; Teitelbaum)

Given M there exists a finite subset Z of $\Gamma\backslash\mathcal{T}$ such that any $c \in C_{har}(\Gamma, M)$ vanishes on all edges e not in a class of Z.

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Definition

A simplex $t \in Vert(\mathcal{T}) \cup Edge(\mathcal{T})$ is Γ-**stable** iff

$$Stab_{\Gamma}(t) = \{1\}.$$

Proposition

There are only finitely many Γ-stable orbits of simplices.

Theorem (Teitelbaum)

Suppose Γ is p'-torsion free. Then:

- *Any Γ-invariant harmonic cocycle is determined by its values on the Γ-stable orbits of edges.*
- *The only relations are those coming from Γ-stable vertices.*

Remark:
The space $C_{har}(\Gamma, M)$ has an interpretation in terms of relative group homology. Let $\Gamma_v \subset \Gamma$ be the stabilizers of a set of representatives for the cusps. Then:

$$C_{har}(\Gamma, M) \cong H_1(\Gamma, \Gamma_v, M).$$

The residue map

$$\mathcal{T},\ Edge(\mathcal{T}),\ \Gamma,\ C_{har}(\Gamma, M)$$

Recall:

A Drinfeld modular form f is a rigid analytic function on Ω.

Ω is a tubular neighborhood of $\mathcal{T}$ via ρ.

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ρ^{-1} of the inner part of an edge e is an annulus $A(e)$.

For f of weight 2 define

$$\mathrm{Res}_2 : Edge(\mathcal{T}) \to \mathbb{C}_\infty : e \mapsto \mathrm{Res}_{A(e)}(fdz).$$

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Theorem (Teitelbaum)

Res_2 defines an isomorphism from the $\mathbb{C}_\infty$ vector space of Drinfeld cusp forms of weight 2 and level Γ to $C_{har}(\Gamma, K) \otimes_K \mathbb{C}_\infty$.

An analogous theorem holds in weight k with $M \approx Sym^{k-2}$.

On the proof of Teitelbaum's theorem:
It suffices to prove it for $\Gamma = \Gamma(N)$ with $N \in \mathbb{F}_q[T] \smallsetminus \mathbb{F}_q$.
Injectivity: Using a π-adic measure theory, Teitelbaum constructs an explicit section for

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Surjectivity: Compute $\dim S_2(\Gamma(N))$ via Riemann-Roch and a canonical line bundle on $\Gamma(N)\backslash\Omega$.
Express $\dim C_{har}(\Gamma(N), \mathbb{C}_\infty)$ as the number of stable orbits of edges minus the number of stable orbits of vertices..
Show that the dimensions are equal.

How to understand the quotient tree?

Proposition

The quotient tree $GL_2(\mathbb{F}_q[T])\backslash\mathcal{T}$ is represented by the half line with vertices $\{\Lambda_i\}_{i\geq 0}$.

There are no $GL_2(\mathbb{F}_q[T])$-stable simplices of $\mathcal{T}$. For $i \geq 1$, the stabilizer of Λ_{i+1} is strictly larger than that of Λ_i.

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For general Γ:

Consider $\Gamma\backslash\mathcal{T}$ as a finite 'covering' of the above half line.

The stabilizers of simplices of the 'cover' have a similar monotonicity property as those of $GL_2(\mathbb{F}_q[T])\backslash\mathcal{T}$.

Stable simplices can only be found above Λ_i for small i (depending on Γ).

Using the above idea, one can show all 'basic properties' on harmonic cocycles we quoted.

References

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