

Estimating default correlations using a reduced-form model

Robert Jarrow and Donald van Deventer show how to estimate default event correlations using a reduced-form model with historical default data. Default event correlations between two firms can be calculated from time-series observations of the firms' default probabilities using a simple formula. The firms' default probabilities can be estimated with a standard hazard rate model, and the procedure is illustrated using market default data. Default event correlations are key inputs to the pricing of collateralised default correlations

To price basket default swaps and collateralised default obligations (CDOs), estimating firm default correlations is essential. Different default correlations can dramatically change the relevant valuations (Lehman Brothers, 2004). Because default is a zero/one event, it is well known that sample default correlations are uninformative. Consequently, to estimate these default correlations, one must use a model. The most discussed approach for estimating these default correlations is to use a structural model¹ in conjunction with a copula function (see Bluhm, Overbeck & Wagner, 2003). The parameters of the copula function are often estimated using stock price correlations instead of asset price correlations, which are unobservable.

Unfortunately, this approach inherits the well-known problem associated with structural models in matching short maturity credit spreads, plus the problems associated with assuming an arbitrary copula function to capture the correlation structure (see Yu, 2003). Alternatively, a reduced-form model can be utilised. By assuming a particular hazard rate process, implying that default follows a jump or point process, reduced-form models better match short maturity credit spreads. The purpose of this article is to discuss default correlation estimation for reduced-form models and to illustrate how easy it is to obtain these estimates using standard hazard rate estimation procedures. We illustrate this approach using the Jarrow & Chava (2002) default model.

An outline of this article is as follows. The next section introduces the notation necessary to discuss the issues. We then review the fundamental problem associated with default correlation estimation using historical data. The hazard rate estimation procedure is then discussed and the correlation estimates presented. The final section concludes.

Notation

This section sets up the notation for a reduced-form credit risk model. We are given a finite time horizon $[0, \bar{T}]$ and a filtered probability space $(\Omega, \mathcal{F}, (F_t)_{t=0}^{\bar{T}}, P)$ where $F = F_{\bar{T}}$ and P is the statistical probability (as contrasted with the risk-neutral or martingale probability often used for valuation). For the purposes of this article, we need only concern ourselves with the statistical probability P .

Let us consider two risky firms A and B with default times τ_A and τ_B , respectively. The default times are random variables on this filtered probability space. Let the relevant state variables in the economy be represented by the vector X_t . This vector represents the micro (balance-sheet data) and macroeconomic variables that determine the likelihood of firms A and B defaulting. We assume that X_t is measurable with respect to the given filtration. The point processes associated with the event of default are denoted by the indicator variables $N_A(t) = 1_{(\tau_A \leq t)}$ and $N_B(t) = 1_{(\tau_B \leq t)}$, which become unity in the event of default. We assume that these point processes follow a Cox process with intensities $\lambda_A(t)$ and $\lambda_B(t)$, respec-

tively.² Let $\mathbf{X} \equiv \sigma\{X_t : 0 \leq t \leq \bar{T}\}$ denote the information set generated by the state variable vector over the model's horizon. Recall that for Cox processes, given \mathbf{X} , the default event variables $1_{(\tau_A \leq t)}$ and $1_{(\tau_B \leq t)}$ are independent.

For subsequent usage, we need to recall some facts. The default probability for firm A over the time period $[0, T]$ is given by:

$$\Pr(\tau_A \leq T) = E(1_{(\tau_A \leq T)}) = E(E(1_{(\tau_A \leq T)} | \mathbf{X})) = E\left(1 - e^{-\int_0^T \lambda_A(s) ds}\right) \quad (1)$$

The joint default probability for firm A and firm B is given by:

$$\begin{aligned} \Pr\{(\tau_A \leq T) \cap (\tau_B \leq T)\} &= E\left(E(1_{(\tau_A \leq T)} 1_{(\tau_B \leq T)} | \mathbf{X})\right) \\ &= E\left(E(1_{(\tau_A \leq T)} | \mathbf{X}) E(1_{(\tau_B \leq T)} | \mathbf{X})\right) \\ &= E\left(1 - e^{-\int_0^T \lambda_A(s) ds}\right) E\left(1 - e^{-\int_0^T \lambda_B(s) ds}\right) \end{aligned} \quad (2)$$

These observations are used below.

The problem

This section discusses the problem associated with estimating default correlations using historical data. First, let us be precise. When we talk about the correlation between the defaults of firms A and B we usually mean the following: standing at time zero, fix a future time period T , say one year ahead. This interval $[0, T]$ is the time horizon over which we are interested in measuring default correlations. Formally, we want the correlation of the joint default events $1_{(\tau_A \leq T)}$ and $1_{(\tau_B \leq T)}$ over this horizon, denoted $Corr(1_{(\tau_A \leq T)}, 1_{(\tau_B \leq T)})$.

Given historical data on firm defaults, it is well known that the sample correlation coefficient provides a poor estimate. Consider why. To calculate this sample correlation, we need to partition the past time horizon into one-year intervals and observe the realisations of these indicator default variables. From these realisations, the sample correlation coefficient is calculated. But, for all non-defaulting firms, these realisations are uniformly zero, and thus the sample default correlation is also zero. This correlation statistic is not very informative. It is the fact that this is a zero/one event that precludes the direct estimation using historical data.

Consequently, to obtain an estimate, one needs to: use a credit risk model; estimate the parameters of such a model; and then use the model

¹ In structured approaches default is modelled as an asset valuation process hitting a predetermined barrier. Reduced-form models employ a hazard rate model for the determination of the default event. For a review of these different approaches, see Jarrow & Protter (2004)

² See Lando (1998) for more details

A. Hazard rate estimation using 1962–1999 data for the NYSE, Amex and Nasdaq

	Private firm model	Public firm model
Intercept	-6.4215 5,603.5273 < 0.0001	-9.4398 1,362.0116 < 0.0001
NITA	-1.6408 297.0374 < 0.0001	-0.1680 2.1674 0.1410
TLTA	2.2816 375.3822 < 0.0001	2.5447 457.6793 < 0.0001
EXRET		-1.6406 282.4544 < 0.0001
RSIZ		-0.1667 55.5170 < 0.0001
Sigma		0.6803 155.9063 < 0.0001
Model fit χ^2	899.53 < 0.0001	2,282.22 < 0.0001
Number of bankruptcies	1,090	1,066
Firm-year obs	147,698	146,102
Number of firms	17,460	17,460

Note: from Jarrow & Chava (2002). Parameter estimates are given in the first row of each cell, followed by χ^2 and *P*-values in the second and third row respectively. The χ^2 and *P*-values for the likelihood ratio test for the model are reported in the model fit column. The hazard model is estimated for 1962–1999 data with yearly observation intervals

and the estimates to obtain the implied default correlations. A frequently used approach in this regard is the structural model for credit risk, in conjunction with assuming a particular copula function. The parameters of the copula function are obtained using stock price correlation data (see Bluhm, Overbeck & Wagner, 2003). Unfortunately, this approach to default correlation estimation inherits the well-known problem associated with the univariate structural model not being able to match short maturity credit spreads, plus problems associated with specifying an *ad hoc* copula function (see Yu, 2003). Alternatively, a reduced-form model can be utilised. By assuming a particular hazard rate process, implying that default follows a jump or point process, reduced-form models better match short maturity credit spreads. The following section discusses the correlation estimation procedure using a reduced-form model.

Hazard rate estimation

This section briefly reviews hazard rate estimation (a standard reference is Kiefer, 1988). We use time-series observations of a collection of firms' market and balance sheet data, represented by the vector X_t . Also observed is the default status of each firm, and if it defaults, the date of default.

The time series starts at some date t_0 , is observed monthly, and terminates at some ending date T_0 . At termination, for each firm, it has either defaulted, remains in the database or has left the database earlier due to a merger/acquisition. For the last two possibilities (healthy or merged), we say that the firm is censored at the ending or merger date. For firm *A*, we denote this censoring time as T_A . Next, we define the random time $Y_A = \min\{\tau_A, T_A\}$, which corresponds to the last date we observe the firm in the time period $[t_0, T_0]$.

Define the discrete time conditioned (on the state variables) hazard rate process as $p_A(t) = \lambda_A(t)\Delta$, where $\Delta = 1/12$, corresponding to a monthly observation period. We assume that $p_A(t)$ follows the logistic model:

$$p_A(t) = \frac{1}{1 + e^{-\alpha - \beta X_t}} \tag{3}$$

where α is a constant, and β is a vector of constants, to be estimated.

Now, for each firm standing at time Y_A , we are interested in the point process $N_A(t) = 1_{(\tau_A \leq t)}$ evaluated at time Y_A , that is, $N_A(Y_A) = \{1 \text{ if } Y_A = \tau_A, 0 \text{ if } Y_A < \tau_A\}$. For estimation, we will use a maximum-likelihood procedure. To obtain the likelihood function $L(\cdot)$, note first that:

$$\begin{aligned} L\{N_A(Y_A) | X_s : s = t_0, \dots, Y_A\} &= \\ & \left\{ \Pr(\tau_A = Y_A | X_s : s = t_0, \dots, Y_A) \text{ if } N_A(Y_A) = 1, \right. \\ & \left. \Pr(\tau_A > Y_A | X_s : s = t_0, \dots, Y_A) \text{ if } N_A(Y_A) = 0 \right\} \\ &= \Pr(\tau_A = Y_A | X_s : s = t_0, \dots, Y_A)^{N_A(Y_A)} \\ & \Pr(\tau_A > Y_A | X_s : s = t_0, \dots, Y_A)^{1 - N_A(Y_A)} \end{aligned}$$

But:

$$\begin{aligned} \Pr(\tau_A = Y_A | X_s : s = t_0, \dots, Y_A) &= p_A(Y_A) \prod_{t=t_0}^{Y_A-1} (1 - p_A(t)) \\ \Pr(\tau_A > Y_A | X_s : s = t_0, \dots, Y_A) &= \prod_{t=t_0}^{Y_A} (1 - p_A(t)) \end{aligned}$$

Thus, the log-likelihood function can be rewritten as:

$$\begin{aligned} \log L\{N_A(Y_A) | X_s : s = t_0, \dots, Y_A\} \\ = N_A(Y_A) \log\left(\frac{p_A(Y_A)}{1 - p_A(Y_A)}\right) + \sum_{t=t_0}^{Y_A} \log(1 - p_A(t)) \end{aligned}$$

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Using the independence across firms, we get the joint log-likelihood function as³:

$$\sum_A \log L\{N_A(Y_A)|X_s : s = t_0, \dots, Y_A\} = \sum_A \sum_{t=t_0}^{Y_A} [N_A(t) - N_A(t-1)] \log\left(\frac{p_A(t)}{1-p_A(t)}\right) + \sum_A \sum_{t=t_0}^{Y_A} \log(1-p_A(t))$$

Given this likelihood function, one can use a standard statistical package such as SAS to do the maximum likelihood estimation.

Jarrow & Chava (2002) estimate the preceding model over the time period 1962–1999 for all NYSE-Amex-Nasdaq listed companies. The state variables used in the estimation were net income to total assets (NITA), total liabilities to total assets (TLTA), monthly return on the firm less the value-weighted CRSP NYSE/Amex return (EXRET), logarithm of firm equity value divided by the total NYSE and Amex market value (RSIZ), and the firm's equity volatility (SIGMA). The estimates for the parameters (α , β) are contained in table A. Using the parameter estimates in table A, in conjunction with expression (3), one can obtain time-series estimates of the default probabilities for firm A.

To illustrate this, we select two financial firms, Citigroup and FleetBoston Financial. The calculations were obtained using an extension of the Jarrow & Chava (2002) model. Figure 1 (top) shows the one-month and five-year default probabilities for Citigroup from 1990 to the present.⁴ One can clearly see the adverse effects of the recessions in 1990–1991, 1994–1995 and 1999–2003 on the default probabilities of Citigroup across the maturity spectrum of defaults. The second financial institution considered is Boston-based FleetBoston Financial, which has recently announced a merger with Bank of America. Not surprisingly, a similar graph for FleetBoston Financial (middle of figure 1) shows the same cyclicality in the one-month and five-year default probabilities.

The high correlation between Citigroup's and FleetBoston's default probabilities can be made clearer by graphing the default probabilities (one month horizon) of both institutions on the same graph (bottom of figure 1). This graph shows a high degree of correlation between the default probabilities of the two companies, with the exception of a brief period in 1999–2000 when FleetBoston suffered earlier from the impact of retail and small business defaults during the 1999–2003 credit events.

The next section shows how to obtain the default event correlation from these joint time series of default probabilities.

Default correlation estimation

This section shows how to estimate joint default probabilities and default correlations in a reduced-form model. The estimates are based on expressions (1) and (2), in conjunction with a standard approximation to an integral and the exponential function. It is important to emphasise that the correlations obtained depend crucially on the assumed form of the hazard rate process in expression (3). Different specifications will provide different implied correlations.

Let $p_A(t)$, $p_B(t)$ correspond to the maximum likelihood estimates at time t of the probability of default on firm A and B, respectively, using monthly observation intervals ($\Delta = 1/12$). As noted before, the intensity process for firm A can be estimated as:

$$\lambda_A(t) = p_A(t) \frac{1}{\Delta} \quad (4)$$

This hazard rate intensity can be thought of as the annualised default probability of firm A. It can be shown that over any relevant time interval $[0, T]$:

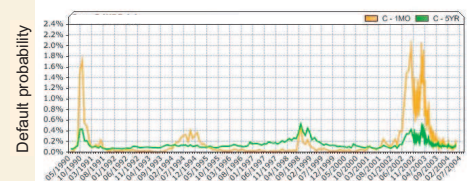
$$\Pr\{\tau_A \leq T\} \approx \lambda_A(t^*)T \quad \text{for some } t^* \in [0, T] \quad (5)$$

Then, we obtain for T small (say less than a year)⁵:

1. Citigroup (top)/FleetBoston (middle)/joint graph (bottom)

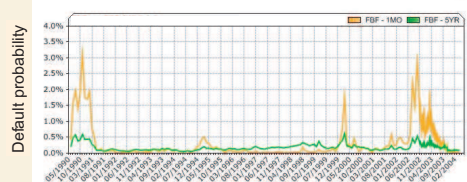
Default probability (DP) profile for Citigroup

Model	DP (%)	1-day chg	1-mo chg	3-mo chg	6-mo chg	1-yr chg
JC	0.18	+0.01	+0.10	+0.09	0.00	-0.55
MS	0.79	+0.01	+0.05	+0.04	-0.04	-0.25
P-JM	0.74	0.00	+0.01	+0.01	-0.01	-0.07



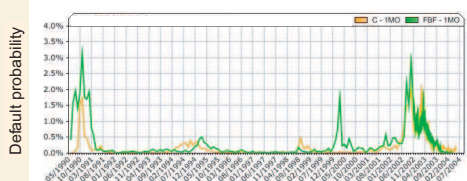
Default probability (DP) profile for FleetBoston

Model	DP (%)	1-day chg	1-mo chg	3-mo chg	6-mo chg	1-yr chg
JC	0.03	0.00	0.00	0.00	-0.22	-1.15
MS	0.50	0.00	0.00	-0.01	-0.16	-0.31
P-JM	0.69	0.00	0.00	0.00	-0.03	-0.09



Default probability (DP) profile for Citigroup/FleetBoston

Model	DP (%)	1-day chg	1-mo chg	3-mo chg	6-mo chg	1-yr chg
JC	0.18	+0.01	+0.10	+0.09	0.00	-0.55
MS	0.79	+0.01	+0.05	+0.04	-0.04	-0.25
P-JM	0.74	0.00	+0.01	+0.01	-0.01	-0.07



$$\Pr\{(\tau_A \leq T) \cap (\tau_B \leq T)\} \approx \left[\text{Cov}(\lambda_A(t^*), \lambda_B(t^*)) + E(\lambda_A(t^*))E(\lambda_B(t^*)) \right] T^2 \quad (6)$$

and:

$$\text{Corr}(1_{(\tau_A \leq T)}, 1_{(\tau_B \leq T)}) \approx \text{Corr}(\lambda_A(t^*), \lambda_B(t^*)) \Theta(\lambda_A(t^*), \lambda_B(t^*)) \quad (7)$$

where:

$$\Theta(\lambda_A(t^*), \lambda_B(t^*)) = \frac{\text{Std}(\lambda_A(t^*))\text{Std}(\lambda_B(t^*))T}{\sqrt{\left\{E(\lambda_A(t^*)) - [E(\lambda_A(t^*))]^2\right\}T \left\{E(\lambda_B(t^*)) - [E(\lambda_B(t^*))]^2\right\}T}} < 1$$

$\text{Cov}(\cdot, \cdot)$ corresponds to the covariance and $\text{Std}(\cdot)$ corresponds to the standard deviation. The proof is in the appendix.

Expression (7) states that the correlation of two firms' default events over the time period $[0, T]$ is equal to the correlation of the two firms' conditional default probabilities over the period $[0, T]$, after an adjustment.

³ This uses the identity $N_A(Y_A) = \sum_{t=t_0}^{Y_A} [N_A(t) - N_A(t-1)]$

⁴ These charts were obtained via the courtesy of the Kamakura Corporation

⁵ The t^* in expression (5) should really be indexed by the firm in question, that is, t_A^* . We need this condition so that t_A^* is approximately equal to t_B^* for a distinct firm B

B. Default event correlations

Correlations in default probabilities					Auto industry				Airline industry					Financial services industry			
Industry	Name	Symbol	Avg. KDP	Avg. KDP std. dev.	General		Volks- wagen	British Delta Japan Sing.					Bar- clays	Citi- group	Deutsche Bank	JP Morgan	
					Ford	Motors		Toyota	AMR	Airways	Airlines	Airlines					Airlines
					F	GM	7203	VOW	AMR	BAY	DAL	9205	S55	BARC	C	XPTH	JPM
					0.019	0.016	0.003	0.011	0.047	0.023	0.038	0.012	0.002	0.004	0.003	0.011	0.009
Auto industry	Ford	F	0.019	0.013	1.000	0.583	0.694	0.229	0.060	0.022	0.036	0.011	0.003	0.006	0.005	0.010	0.011
	General Motors	GM	0.016	0.011	0.583	1.000	0.547	0.626	0.665	0.677	0.523	0.506	0.554	0.478	0.641	0.456	0.789
	Toyota	7203	0.003	0.003	0.694	0.547	1.000	0.388	0.547	0.488	0.414	0.375	0.568	0.434	0.446	0.476	0.443
	Volkswagen	VOW	0.011	0.008	0.229	0.626	0.388	1.000	0.805	0.802	0.707	0.690	0.792	0.647	0.816	0.649	0.711
Airline industry	AMR	AMR	0.047	0.060	0.665	0.547	0.805	0.436	1.000	0.860	0.791	0.752	0.721	0.823	0.890	0.767	0.769
	British Airways	BAY	0.023	0.022	0.677	0.488	0.802	0.380	0.860	1.000	0.699	0.706	0.782	0.674	0.812	0.681	0.787
	Delta Airlines	DAL	0.038	0.036	0.523	0.414	0.707	0.392	0.791	0.699	1.000	0.530	0.702	0.732	0.830	0.609	0.713
	Japan Airlines	9205	0.012	0.011	0.506	0.375	0.690	0.685	0.752	0.706	0.530	1.000	0.726	0.671	0.614	0.709	0.467
	Sing. Airlines	S55	0.002	0.003	0.554	0.568	0.792	0.749	0.721	0.782	0.702	0.726	1.000	0.649	0.619	0.586	0.523
Financial services	Barclays	BARC	0.004	0.006	0.478	0.434	0.647	0.547	0.823	0.674	0.732	0.671	0.649	1.000	0.810	0.807	0.551
	Citigroup	C	0.003	0.005	0.641	0.446	0.816	0.277	0.890	0.812	0.830	0.614	0.619	0.810	1.000	0.750	0.811
	Deutsche Bank	XPTH	0.011	0.010	0.456	0.476	0.649	0.521	0.767	0.681	0.609	0.709	0.586	0.807	0.750	1.000	0.591
	JP Morgan	JPM	0.009	0.011	0.789	0.443	0.711	0.019	0.769	0.787	0.713	0.467	0.523	0.551	0.811	0.591	1.000

Correlations in events of default					Auto industry				Airline industry					Financial services industry			
Industry	Name	Symbol	Avg. KDP	Avg. KDP std. dev.	General		Volks- wagen	British Delta Japan Sing.					Bar- clays	Citi- group	Deutsche Bank	JP Morgan	
					Ford	Motors		Toyota	AMR	Airways	Airlines	Airlines					Airlines
					F	GM	7203	VOW	AMR	BAY	DAL	9205	S55	BARC	C	XPTH	JPM
					0.019	0.016	0.003	0.011	0.047	0.023	0.038	0.012	0.002	0.004	0.003	0.011	0.009
Auto industry	Ford	F	0.019	0.013	1.000	0.005	0.004	0.002	0.018	0.010	0.010	0.005	0.003	0.004	0.005	0.004	0.009
	General Motors	GM	0.016	0.011	0.005	1.000	0.003	0.004	0.014	0.006	0.007	0.003	0.003	0.004	0.003	0.004	0.004
	Toyota	7203	0.003	0.003	0.004	0.003	1.000	0.001	0.012	0.006	0.007	0.004	0.002	0.003	0.004	0.003	0.004
	Volkswagen	VOW	0.011	0.008	0.002	0.004	0.001	1.000	0.009	0.004	0.005	0.005	0.003	0.004	0.002	0.004	0.000
Airline industry	AMR	AMR	0.047	0.060	0.018	0.014	0.012	0.009	1.000	0.036	0.042	0.022	0.011	0.022	0.021	0.022	0.025
	British Airways	BAY	0.023	0.022	0.010	0.006	0.006	0.004	0.036	1.000	0.020	0.011	0.006	0.009	0.010	0.010	0.013
	Delta Airlines	DAL	0.038	0.036	0.010	0.007	0.007	0.005	0.042	0.020	1.000	0.010	0.007	0.013	0.013	0.011	0.015
	Japan Airlines	9205	0.012	0.011	0.005	0.003	0.004	0.005	0.022	0.011	0.010	1.000	0.004	0.006	0.005	0.007	0.005
	Sing. Airlines	S55	0.002	0.003	0.003	0.003	0.002	0.003	0.011	0.006	0.007	0.004	1.000	0.003	0.003	0.003	0.003
Financial services	Barclays	BARC	0.004	0.006	0.004	0.004	0.003	0.004	0.022	0.009	0.013	0.006	0.003	1.000	0.006	0.008	0.006
	Citigroup	C	0.003	0.005	0.005	0.003	0.004	0.002	0.021	0.010	0.013	0.005	0.003	0.006	1.000	0.006	0.008
	Deutsche Bank	XPTH	0.011	0.010	0.004	0.004	0.003	0.004	0.022	0.010	0.011	0.007	0.003	0.008	0.006	1.000	0.007
	JP Morgan	JPM	0.009	0.011	0.009	0.004	0.004	0.000	0.025	0.013	0.015	0.005	0.003	0.006	0.008	0.007	1.000

When $T = 1$, expression (7) gives us the correlation in the yearly default events of firms A and B .

It is easy to calculate an estimate of this quantity, if we observe time-series observations of the annualised conditional default probabilities $\lambda_A(t)$ and $\lambda_B(t)$. To calculate the default correlation in expression (7), one needs to calculate the correlation of these annualised conditional default probabilities and the adjustment term (the right-hand side). These are computable using the sample moments of the hazard rate estimates $\lambda_A(t)$ and $\lambda_B(t)$. Moment estimators of these quantities make the implicit assumption that the annualised conditional default probability's means, variances and correlations are time stationary.⁶ This is a reasonable assumption, although any well-specified alternative non-time stationary structure can also be estimated.

Using the hazard rate estimation discussed in the preceding section, we compute the relevant quantities:

- Correlation of annualised default probabilities of Citigroup and FleetBoston: 0.8239280.
- Standard deviation of annualised default probabilities, Citigroup: 0.00469470.
- Standard deviation of annualised default probabilities, FleetBoston: 0.00565789.
- Average of annualised default probabilities, Citigroup: 0.00310565.
- Average of annualised default probabilities, FleetBoston: 0.00439433.

Note that the correlation in the annualised default probabilities is very close to the observed correlation of the credit spreads on bonds of the two companies. From the 0.8239280 correlation in the annualised default probabilities themselves, we can use expressions (7) and (8) to calculate the correlation in the events of default, taking care to adjust for the fact that we are using correlations, averages and standard deviations. The formula for the adjustment term is:

$$\Theta = \frac{0.00469470(0.00565789)}{\sqrt{\{0.00310565 - (0.00310565)^2\} \{0.00439433 - (0.00439433)^2\}}}$$

yielding a correlation in default events of $(0.8239280)(0.007217) = 0.005946$.

We see that the adjustment to the default probabilities' correlation is substantial, lowering the number significantly. This makes sense, because even given the correlation of the default probabilities due to common macroeconomic conditions, the event of default occurs due to firm-specific – idiosyncratic – events. The likelihood that these firm-specific events occur simultaneously (within the same year) is quite low.

In a similar manner, we can apply equations (7) and (8) to a broader range of data to better understand its implications for practical application.

⁶ For example, if $\lambda_A(t) = a_A + b_A \log X_t$ and $d \log X_t = \mu dt + \sigma dW_t$ where dW_t is a Brownian motion, then $E_{t-1}(\lambda_A(t)) = a_A + b_A \mu$, $Var_{t-1}(\lambda_A(t)) = b_A^2 \sigma^2$ and $Corr(\lambda_A(t), \lambda_B(t)) = b_A b_B \sigma^2$ satisfy the time stationarity condition

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Table B shows the average annualised default probabilities and their standard deviations for 14 firms in three industries –automobiles, airlines and financial services. The five-year monthly correlations in the annualised default probabilities for Ford, General Motors, Toyota and Volkswagen range from 0.229 to 0.694. Within the airline industry, the five-year monthly correlations in the annualised default probabilities are higher, with a range from 0.530 to 0.860 for five major airlines – American Airlines (AMR), British Airways, Delta, Japan Airlines and Singapore Airlines. The correlations in the monthly annualised default probabilities are similarly high for four major international services firms, ranging from 0.551 to 0.811. The firms included in table B are Barclays, Citigroup, Deutsche Bank and JP Morgan. This high correlation among the historical annualised default probabilities is due to the dependence of the default probabilities on common macroeconomic factors that have both a direct impact on the default probabilities and an indirect impact on the other model inputs such as stock prices and financial accounting ratios.

We can transform the correlations in the annualised default probabilities to correlations in the events of yearly default in a manner similar to the Citigroup/FleetBoston example. These results are also shown in table B.⁷ As in the Citigroup/FleetBoston example, the correlation in the yearly default events is much lower than the correlations in the annualised default probabilities themselves. One can see that theta equals approximately 0.01 if both firms have annualised default probabilities of 0.01 and standard deviations of the annualised default probabilities that are approximately equal in magnitude to the annualised default probabilities themselves, a common occurrence. In the automobile industry, for example, the pair-wise correlations in the events of yearly default are all below 1%. In the airline industry, some of the correlations in the yearly default events range up to 4%. For the financial services industry, the correlations in the events of yearly default are also less than 1%.

To readers used to viewing default correlations for copula usage, these correlations may appear low. Note that there is no reason why the correlations in the event of default derived in this manner should be of the same magnitude as correlations used in copula calculations. This is because a copula assumes a particular form for the joint distribution function of the yearly default events, and this assumed distribution function may be different from that implied by the reduced-form model expression (2). For example, if one assumes a normal distribution function for the copula, then:

$$\Pr\{(\tau_A \leq 1) \cap (\tau_B \leq 1)\} = \Phi(z_A, z_B, \rho)$$

where $z_A \equiv \Phi^{-1}(\Pr\{\tau_A \leq 1\})$, $z_B \equiv \Phi^{-1}(\Pr\{\tau_B \leq 1\})$, $\Phi(\cdot)$ is the univariate standard normal distribution function, ρ is the correlation of z_A and z_B , and $\Phi(z_A, z_B, \rho)$ is the standard bivariate normal distribution function.⁸ If the normal copula does not equal expression (2), the correlations will not be equal.

Conclusion

This paper shows how to estimate default correlations using reduced-form models. Given these estimates, it is an easy exercise to utilise the parameter estimates to value credit derivatives on baskets and CDOs. Standard procedures apply in this regard. ■

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⁷ We note that equations (7) and (8) do not apply to the case where correlation in the default events equals one because of a discontinuity in the formula for identical random variables (see the Appendix)

⁸ In these models, the events z_A and z_B usually correspond to each firm's asset value exceeding a default barrier at year end

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Appendix

$[t, t + T]$ is the horizon over which the default probabilities are calculated, for example, $T = 1$ corresponds to one year. The intensity process is measured on a per year basis, so that $\lambda_A(t^*)$ is approximately the probability of default over the next year. $\lambda_A(t^*)T$ is approximately the probability of default over the next T units of a year (for example, $T = 1/2$ implies a half a year).

□ Covariances.

$$\begin{aligned} & Cov_t \left(\mathbf{1}_{(\tau_A \leq t+T)}, \mathbf{1}_{(\tau_B \leq t+T)} \right) \\ &= E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \mathbf{1}_{(\tau_B \leq t+T)} \right) - E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \right) E_t \left(\mathbf{1}_{(\tau_B \leq t+T)} \right) \\ &= E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \mathbf{1}_{(\tau_B \leq t+T)} | \mathbf{X} \right) - E_t \left(E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} | \mathbf{X} \right) \right) E_t \left(E_t \left(\mathbf{1}_{(\tau_B \leq t+T)} | \mathbf{X} \right) \right) \end{aligned}$$

by iterated expectation

$$\begin{aligned} &= E_t \left(E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} | \mathbf{X} \right) E_t \left(\mathbf{1}_{(\tau_B \leq t+T)} | \mathbf{X} \right) \right) \\ &- E_t \left(E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} | \mathbf{X} \right) \right) E_t \left(E_t \left(\mathbf{1}_{(\tau_B \leq t+T)} | \mathbf{X} \right) \right) \end{aligned}$$

by conditional independence

$$\begin{aligned} &= E_t \left(\left[1 - e^{-\int_t^{t+T} \lambda_A(s) ds} \right] \left[1 - e^{-\int_t^{t+T} \lambda_B(s) ds} \right] \right) \\ &- E_t \left(\left[1 - e^{-\int_t^{t+T} \lambda_A(s) ds} \right] \right) E_t \left(\left[1 - e^{-\int_t^{t+T} \lambda_B(s) ds} \right] \right) \end{aligned}$$

by the facts above

$$= Cov_t \left(1 - e^{-\int_t^{t+T} \lambda_A(s) ds}, 1 - e^{-\int_t^{t+T} \lambda_B(s) ds} \right)$$

Using the mean value theorem, there exists a $t_A^* \in [t, t + T]$ such that $\int_t^{t+T} \lambda_A(s) ds = \lambda_A(t_A^*)T$. Assuming that T is small, so that $t_A^* \approx t_B^* \equiv t^*$, we have

$$= Cov_t \left(1 - e^{-\lambda_A(t^*)T}, 1 - e^{-\lambda_B(t^*)T} \right)$$

Using a Taylor series approximation for small $\lambda_A(t^*)$ and $\lambda_B(t^*)$, up to the first order in these default intensities, gives

$$Cov_t \left(1 - e^{-\lambda_A(t^*)T}, 1 - e^{-\lambda_B(t^*)T} \right) \approx Cov_t \left(\lambda_A(t^*), \lambda_B(t^*) \right) T^2$$

Hence

$$Cov_t \left(\mathbf{1}_{(\tau_A \leq t+T)}, \mathbf{1}_{(\tau_B \leq t+T)} \right) \approx Cov_t \left(\lambda_A(t^*), \lambda_B(t^*) \right) T^2$$

□ Variances.

Next, we calculate the variance of each process. Note that

$$\begin{aligned} & Var_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \right) = Cov_t \left(\mathbf{1}_{(\tau_A \leq t+T)}, \mathbf{1}_{(\tau_A \leq t+T)} \right) \\ &= E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \mathbf{1}_{(\tau_A \leq t+T)} \right) - E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \right) E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \right) \\ &\quad \text{because } \mathbf{1}_{(\tau_A \leq t+T)} \mathbf{1}_{(\tau_A \leq t+T)} = \mathbf{1}_{(\tau_A \leq t+T)} \\ &= E_t \left(E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} | \mathbf{X} \right) \right) - E_t \left(E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} | \mathbf{X} \right) \right) E_t \left(E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} | \mathbf{X} \right) \right) \end{aligned}$$

by iterated expectation

$$= E_t \left(1 - e^{-\int_t^{t+T} \lambda_A(s) ds} \right) - E_t \left(1 - e^{-\int_t^{t+T} \lambda_A(s) ds} \right) E_t \left(1 - e^{-\int_t^{t+T} \lambda_A(s) ds} \right)$$

Using the mean value theorem, there exists a $t^* \in [t, t + T]$ such that

$$\begin{aligned} & \int_t^{t+T} \lambda_A(X_s) ds = \lambda_A(X_{t^*})T \\ &= E_t \left(1 - e^{-\lambda_A(t^*)T} \right) - E_t \left(1 - e^{-\lambda_A(t^*)T} \right) E_t \left(1 - e^{-\lambda_A(t^*)T} \right) \end{aligned}$$

Next, for small $\lambda_A(X_{t^*})$, we do a Taylor series expansion up to the first order in the default intensity

$$\begin{aligned} &= E_t \left(1 - (1 - \lambda_A(t^*)T) \right) - E_t \left(1 - (1 - \lambda_A(t^*)T) \right) E_t \left(1 - (1 - \lambda_A(t^*)T) \right) \\ &= E_t \left(1 - 1 + \lambda_A(t^*)T \right) - E_t \left(1 - 1 + \lambda_A(t^*)T \right) E_t \left(1 - 1 + \lambda_A(t^*)T \right) \\ &= E_t \left(\lambda_A(t^*)T \right) - \left[E_t \left(\lambda_A(t^*)T \right) \right]^2 \\ &= E_t \left(\lambda_A(t^*) \right) T - E_t \left(\lambda_A(t^*)^2 \right) T^2 + var_t \left(\lambda_A(t^*) \right) T^2. \text{ Note that} \\ & E_t \left(\lambda_A(t^*) \right) T - E_t \left(\lambda_A(t^*)^2 \right) T^2 + var_t \left(\lambda_A(t^*) \right) T^2 > var_t \left(\lambda_A(t^*) \right) T \end{aligned}$$

$$\text{so that } \left[\frac{Std_t \left(\lambda_A(t^*) T \right)}{\sqrt{\left\{ E_t \left(\lambda_A(t^*) \right) T - E_t \left(\lambda_A(t^*)^2 \right) T^2 + var_t \left(\lambda_A(t^*) \right) T^2 \right\}}} \right] < 1$$

as long as $Pr_t \left(\lambda_A(t^*) T < 1 \right) = 1$

□ Correlations.

$$\begin{aligned} & Corr_t \left(\mathbf{1}_{(\tau_A \leq t+T)}, \mathbf{1}_{(\tau_B \leq t+T)} \right) = \frac{Cov_t \left(\mathbf{1}_{(\tau_A \leq t+T)}, \mathbf{1}_{(\tau_B \leq t+T)} \right)}{Std_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \right) Std_t \left(\mathbf{1}_{(\tau_B \leq t+T)} \right)} \\ &\approx \frac{Cov_t \left(\lambda_A(t^*), \lambda_B(t^*) \right) T^2}{Std_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \right) Std_t \left(\mathbf{1}_{(\tau_B \leq t+T)} \right)} = \frac{Cov_t \left(\lambda_A(t^*), \lambda_B(t^*) \right) T^2}{Std_t \left(\lambda_A(t^*) T \right) Std_t \left(\lambda_B(t^*) T \right)} \times \\ &\left[\frac{Std_t \left(\lambda_A(t^*) T \right) Std_t \left(\lambda_B(t^*) T \right)}{\sqrt{\left\{ E_t \left(\lambda_A(t^*) \right) T - \left[E_t \left(\lambda_A(t^*) \right) \right]^2 T^2 \right\} \left\{ E_t \left(\lambda_B(t^*) \right) T - \left[E_t \left(\lambda_B(t^*) \right) \right]^2 T^2 \right\}}} \right] \end{aligned}$$

Simplifying by eliminating redundant T s gives

$$\begin{aligned} &= Corr_t \left(\lambda_A(t^*), \lambda_B(t^*) \right) \times \\ &\left[\frac{Std_t \left(\lambda_A(t^*) \right) Std_t \left(\lambda_B(t^*) \right) T}{\sqrt{\left\{ E_t \left(\lambda_A(t^*) \right) - \left[E_t \left(\lambda_A(t^*) \right) \right]^2 T \right\} \left\{ E_t \left(\lambda_B(t^*) \right) - \left[E_t \left(\lambda_B(t^*) \right) \right]^2 T \right\}}} \right] \end{aligned}$$

□ Joint default probabilities

$$\begin{aligned} & Pr_t \left((\tau_A \leq t+T) \cap (\tau_B \leq t+T) \right) = E_t \left(\mathbf{1}_{(\tau_A \leq t+T) \cap (\tau_B \leq t+T)} \right) \\ &= E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \mathbf{1}_{(\tau_B \leq t+T)} \right) \\ &= Cov_t \left(\mathbf{1}_{(\tau_A \leq t+T)}, \mathbf{1}_{(\tau_B \leq t+T)} \right) + E_t \left(\mathbf{1}_{(\tau_A \leq t+T)} \right) E_t \left(\mathbf{1}_{(\tau_B \leq t+T)} \right) \end{aligned}$$

next, using the facts derived previously

$$\approx Cov_t \left(\lambda_A(t^*), \lambda_B(t^*) \right) T^2 + E_t \left(\lambda_A(t^*) \right) E_t \left(\lambda_B(t^*) \right) T^2$$