Internat. J. Math. & Math. Sci.
Vol. 24, No. 10 (2000) 673–689
S0161171200004671

AN EXAMPLE OF NONSYMMETRIC SEMI-CLASSICAL FORM OF CLASS $s = 1$; GENERALIZATION OF A CASE OF JACOBI SEQUENCE

MOHAMED JALEL ATIA

(Received 24 February 2000)

ABSTRACT. We give explicitly the recurrence coefficients of a nonsymmetric semi-classical sequence of polynomials of class $s = 1$. This sequence generalizes the Jacobi polynomial sequence, that is, we give a new orthogonal sequence $\{\hat{P}_n^{(\alpha,\alpha+1)}(x,\mu)\}$, where μ is an arbitrary parameter with $\Re(1-\mu) > 0$ in such a way that for $\mu = 0$ one has the well-known Jacobi polynomial sequence $\{\hat{P}_n^{(\alpha,\alpha+1)}(x)\}$, $n \geq 0$.

Keywords and phrases. Orthogonal polynomials, semi-classical polynomials.

2000 Mathematics Subject Classification. Primary 33C45; Secondary 42C05.

1. Introduction. Many authors [1, 2, 3] have studied semi-classical sequences of polynomials of class $s = 1$. In particular, Bachène [2, page 87] gave the system fulfilled by such sequences using the structure relation and Belmehdi [3, page 272] gave the same system (in a more simple way) using directly the functional equation. This system is not linear and has not been sorted out before. The aim of this paper is to present a method that may give us some solutions.

In Section 2, we recall the general features which are needed in what follows. Section 3 is devoted to the setting of the problem, to give an integral representation and the expressions of the moments of the form $\mathcal{J}(\alpha,\alpha+1)(\mu)$ which generalizes the form $\mathcal{J}(\alpha,\alpha+1)$, where $\mathcal{J}(\alpha,\beta)$ is the Jacobi functional.

In Section 4, the recurrence coefficients of the semi-classical sequence of polynomials orthogonal with respect to $\mathcal{J}(\alpha,\alpha+1)(\mu)$ are explicitly given using the Laguerre-Freud equation of semi-classical orthogonal sequences of class $s = 1$ given in [3, page 272].

2. Preliminaries. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle$, $n \geq 0$ the moments of u. Let us define the following operations on $\mathcal{P}'$:

- the left-multiplication of a linear functional by a polynomial

$$\langle gu, f\rangle := \langle u, gf\rangle, \quad f,g \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.1}$$

- the derivative of a linear functional

$$\langle u', f\rangle := -\langle u, f'\rangle, \quad f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.2}$$

- the homothetic of a linear functional

$$\langle h_a u, f\rangle := \langle u, h_a f\rangle, \quad a \in \mathbb{C} - \{0\}, \tag{2.3}$$

where

$$(h_a f)(x) = f(ax), \quad f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.4}$$

- the translation of a linear functional

$$\langle \tau_b u, f\rangle := \langle u, \tau_{-b} f\rangle, \quad b \in \mathbb{C}, \tag{2.5}$$

where

$$(\tau_b f)(x) = f(x-b), \quad f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.6}$$

- the division of a linear functional by a polynomial of first degree

$$\langle (x-c)^{-1} u, f\rangle := \langle u, \theta_c f\rangle, \quad c \in \mathbb{C}, \tag{2.7}$$

where

$$(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}, \quad f \in \mathcal{P},\ u \in \mathcal{P}', \tag{2.8}$$

- using (2.1) and (2.2) we can easily prove

$$(fu)' = f'u + fu', \quad f \in \mathcal{P},\ u \in \mathcal{P}'. \tag{2.9}$$

Definition 2.1 (see [4]). A sequence of polynomials $\{\hat{P}_n\}_{n\geq 0}$ is said to be a monic orthogonal polynomial sequence with respect to the linear functional u if

(i) $\deg \hat{P}_n = n$ *and the leading coefficient of* $\hat{P}_n(x)$ *is equal to* 1.

(ii) $\langle u, \hat{P}_n \hat{P}_m\rangle = r_n \delta_{n,m}$, $n, m \geq 0$, $r_n \neq 0$, $n \geq 0$.

It is well known that a sequence of monic orthogonal polynomial satisfies a three-term recurrence relation

$$\begin{gathered}\hat{P}_0(x) = 1, \qquad \hat{P}_1(x) = x - \beta_0, \\ \hat{P}_{n+2}(x) = (x - \beta_{n+1})\hat{P}_{n+1}(x) - \gamma_{n+1}\hat{P}_n(x), \quad n \geq 0,\end{gathered} \tag{2.10}$$

with $(\beta_n, \gamma_{n+1}) \in \mathbb{C} \times \mathbb{C} - \{0\}$, $n \geq 0$.

In such conditions, we say that u is regular or quasi-definite (see [4]). In what follows, we assume that the linear functionals are regular.

A shifting leaves invariant the orthogonality for the sequence $\{\tilde{P}_n\}_{n\geq 0}$. In fact, $\tilde{P}_n(x) = a^{-n}\hat{P}_n(ax+b)$, $n \geq 0$, fulfills the recurrence relation [6] and [8, page 265]

$$\begin{gathered}\tilde{P}_0(x) = 1, \qquad \tilde{P}_1(x) = x - \tilde{\beta}_0, \\ \tilde{P}_{n+2}(x) = (x - \tilde{\beta}_{n+1})\tilde{P}_{n+1}(x) - \tilde{\gamma}_{n+1}\tilde{P}_n(x), \quad n \geq 0\end{gathered} \tag{2.11}$$

with $\tilde{\beta}_n = (\beta_n - b)/a$, $\tilde{\gamma}_{n+1} = (\gamma_{n+1})/a^2$, $n \geq 0$, $a \in \mathbb{C} - \{0\}$.

Definition 2.2 (see [4]). $\{\hat{P}_n\}_{n\geq 0}$ (respectively, the linear functional u) is semi-classical of class s, if and only if the following statement holds: [6] and [7, pages 143–144].

There exist two polynomials ψ of degree $p \geq 1$ and ϕ of degree $t \geq 0$, such that

$$(\phi u)' + \psi u = 0,$$
$$\prod_{c\in Z_\phi} \left(|\psi(c) + \phi'(c)| + |\langle u, \theta_c(\psi) + \theta_c^2(\phi)\rangle| \right) \neq 0, \tag{2.12}$$

where Z_ϕ is the set of zeros of ϕ. The class of $\{P_n\}_{n\geq 0}$ or u is given by $s = \max(p-1, t-2)$ [7, pages 143–144].

If u is a semi-classical functional of class s, then $v = (h_{a^{-1}} \circ \tau_{-b})u$ is also semi-classical of the same class and it verifies the equation $(\phi_1 v)' + \psi_1 v = 0$, where

$$\phi_1(x) = a^{-t}\phi(ax+b), \qquad \psi_1(x) = a^{1-t}\psi(ax+b). \tag{2.13}$$

3. Generalization of $\mathcal{J}(\alpha, \alpha+1)$ as a semi-classical sequence of class $s = 1$

3.1. Problem setting. If u is a classical linear function, that is,

$$(\phi(x)u)' + \psi(x)u = 0, \quad \deg\phi \leq 2, \ \deg\psi = 1, \tag{3.1}$$

from (2.9) the multiplication by x gives

$$(x\phi(x)u)' - \phi(x)u + x\psi(x)u = 0, \quad \deg(x\phi) \leq 3, \ \deg(x\psi - \phi) \leq 2. \tag{3.2}$$

If we consider the following perturbed equation

$$\begin{gathered}(x\phi(x)u(\mu))' + ((\mu-1)\phi(x) + x\psi(x))u(\mu) = 0, \\ \deg(x\phi) \leq 3, \qquad \deg\left(x\psi + (\mu-1)\phi\right) \leq 2,\end{gathered} \tag{3.3}$$

we obtain, under some conditions on μ, a linear functional $u(\mu)$ of class $s = 1$ which generalizes the classical linear functional u.

Examples

(1) The Hermite case. One knows that the functional equation for the Hermite linear functional, noted $\mathcal{H}$, is [6, page 117]

$$\mathcal{H}' + 2x\mathcal{H} = 0 \tag{3.4}$$

multiplied by x gives

$$(x\mathcal{H})' + (2x^2 - 1)\mathcal{H} = 0. \tag{3.5}$$

Thus, we consider the functional equation

$$(x\mathcal{H}(\mu))' + (2x^2 - 2\mu - 1)\mathcal{H}(\mu) = 0 \tag{3.6}$$

which is the functional equation of the well-known generalized-Hermite linear functional, noted $\mathcal{H}(\mu)$, which is regular for $\mu \neq -n-1/2$, $n \geq 0$, and semi-classical of class $s=1$ for $\mu \neq 0$ [4] and [5, page 243]. Notice that $\mathcal{H}(0)=\mathcal{H}$.

(2) The Jacobi case. Let us consider the functional equation for the Jacobi form, $\mathcal{J}(\alpha,\beta)$:

$$((x^2-1)\mathcal{J}(\alpha,\beta))' + (-(\alpha+\beta+2)x+\alpha-\beta)\mathcal{J}(\alpha,\beta) = 0 \tag{3.7}$$

multiplication by x gives the following equation:

$$((x^3-x)\mathcal{J}(\alpha,\beta))' - (x^2-1)\mathcal{J}(\alpha,\beta) + (-(\alpha+\beta+2)x^2+(\alpha-\beta)x)\mathcal{J}(\alpha,\beta) = 0. \tag{3.8}$$

Thus, consider

$$((x^3-x)\mathcal{J}(\alpha,\beta)(\mu))' + ((\mu-\alpha-\beta-3)x^2+(\alpha-\beta)x+1-\mu)\mathcal{J}(\alpha,\beta)(\mu) = 0. \tag{3.9}$$

Notice that $\mathcal{J}(\alpha,\beta)(0)=\mathcal{J}(\alpha,\beta)$.

(a) The Gegenbauer case ($\alpha=\beta$). In this case (3.9) becomes

$$((x^3-x)\mathcal{J}(\alpha,\alpha)(\mu))' + ((\mu-2\alpha-3)x^2+1-\mu)\mathcal{J}(\alpha,\alpha)(\mu) = 0 \tag{3.10}$$

which is the functional equation of the symmetric semi-classical functional, regular for $\mu \neq 2n+2\alpha+1$, $\mu \neq 2n+1$, $n \geq 0$, of class $s=1$ for $\mu \neq 0$, and $\mathcal{J}(\alpha,\alpha)(0)=\mathcal{J}(\alpha,\alpha)$.

In fact, in [1, page 317], we have

$$((x^3-x)u)' + 2(-(\tilde{\alpha}+\tilde{\beta}+2)x^2+\tilde{\beta}+1)u = 0 \tag{3.11}$$

and if we denote by $\{P_n\}_{n\geq 0}$ the sequence of monic polynomials orthogonal with respect to u, then $\{P_n\}_{n\geq 0}$ fulfills (2.10) such that

$$\begin{aligned} \beta_n &= 0, \\ \gamma_{2n+1} &= \frac{(n+\tilde{\beta}+1)(n+\tilde{\alpha}+\tilde{\beta}+1)}{(2n+\tilde{\alpha}+\tilde{\beta}+1)(2n+\tilde{\alpha}+\tilde{\beta}+2)}, \\ \gamma_{2n+2} &= \frac{(n+1)(n+\tilde{\alpha}+1)}{(2n+\tilde{\alpha}+\tilde{\beta}+2)(2n+\tilde{\alpha}+\tilde{\beta}+3)}, \end{aligned} \tag{3.12}$$

for $n \geq 0$. Put

$$-2(\tilde{\alpha}+\tilde{\beta}+2) = \mu-(2\alpha+3), \qquad 2(\tilde{\beta}+1) = 1-\mu, \tag{3.13}$$

we obtain $((x^3-x)u)' + ((\mu-2\alpha-3)x^2+1-\mu)u = 0$ with

$$\begin{aligned} \beta_n &= 0, \\ \gamma_{2n+1} &= \frac{(2n+2\alpha+1-\mu)(2n+1-\mu)}{(4n+2\alpha+1-\mu)(4n+2\alpha+3-\mu)}, \\ \gamma_{2n+2} &= \frac{4(n+1)(n+\alpha+1)}{(4n+2\alpha+3-\mu)(4n+2\alpha+5-\mu)}, \end{aligned} \tag{3.14}$$

for $n \geq 0$.

(b) $\mathcal{J}(\alpha,\alpha+1)$ case. If in (3.9), $\beta=\alpha+1$ we get

$$((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))'+((\mu-2\alpha-4)x^2-x+1-\mu)\mathcal{J}(\alpha,\alpha+1)(\mu)=0. \tag{3.15}$$

In what follows, we will look for the regular linear functional, $\mathcal{J}(\alpha,\alpha+1)(\mu)$ which is a solution of (3.15) and we denote by $\{P_n\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to $\mathcal{J}(\alpha,\alpha+1)(\mu)$ and by $\beta_n,\gamma_n,\ n\geq 0$ the recurrence coefficients of P_n.

Remark 3.1. The solutions of the functional equation (3.15) depend on the value of $(\mathcal{J}(\alpha,\alpha+1)(\mu))_1=\beta_0$, in fact,

$$\langle((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))'+((\mu-2\alpha-4)x^2-x+1-\mu)\mathcal{J}(\alpha,\alpha+1)(\mu),1\rangle=0, \tag{3.16}$$

then, using (2.2), one has

$$\begin{aligned}&\langle((\mu-2\alpha-4)x^2-x+1-\mu)\mathcal{J}(\alpha,\alpha+1)(\mu),1\rangle\\&\quad=\langle\mathcal{J}(\alpha,\alpha+1)(\mu),((\mu-2\alpha-4)x^2-x+1-\mu)\rangle\\&\quad=(\mu-2\alpha-4)(\mathcal{J}(\alpha,\alpha+1)(\mu))_2-(\mathcal{J}(\alpha,\alpha+1)(\mu))_1+1-\mu=0,\end{aligned} \tag{3.17}$$

but $(\mathcal{J}(\alpha,\alpha+1)(\mu))_2=\gamma_1+\beta_0^2$ and $(\mathcal{J}(\alpha,\alpha+1)(\mu))_1=\beta_0$ then

$$(\mu-2\alpha-4)\gamma_1+(\mu-2\alpha-4)\beta_0^2-\beta_0+1-\mu=0. \tag{3.18}$$

First we search an integral representation in order to obtain β_0.

3.2. An integral representation

Proposition 3.2. *An integral representation of a linear functional* $\mathcal{J}(\alpha,\alpha+1)(\mu)$ *is*

$$\langle\mathcal{J}(\alpha,\alpha+1)(\mu),f(x)\rangle=\frac{\Gamma((2\alpha+3-\mu)/2)}{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}(1-x)f(x)\,dx \tag{3.19}$$

with $\operatorname{Re}(1-\mu)>0$*, that is,* $\operatorname{Re}(-u)>-1$ *and* $\operatorname{Re}(\alpha+1)>0$*.*

Proof. A solution of (3.15) has the integral representation

$$\langle\mathcal{J}(\alpha,\alpha+1)(\mu),f\rangle=\int_C U(x)f(x)\,dx,\quad f\in\mathcal{P} \tag{3.20}$$

if the following conditions hold [5]:

$$\begin{gathered}((x^3-x)U(x))'+((\mu-2\alpha-4)x^2-x+1-\mu)U(x)=0,\\(x^3-x)U(x)f(x)]_C=0,\quad f\in\mathcal{P},\end{gathered} \tag{3.21}$$

where C is an acceptable integration path. We solve the first condition as a differential equation:

$$((x^3-x)U(x))'+((\mu-2\alpha-4)x^2-x+1-\mu)U(x)=0 \tag{3.22}$$

or, equivalently,

$$(x^3-x)U'(x)+((\mu-2\alpha-1)x^2-x-\mu)U(x)=0,$$
$$\frac{U'(x)}{U(x)}=-\frac{(\mu-2\alpha-1)x^2-x-\mu}{x^3-x}=-\frac{(\mu-2\alpha-1)x^2-x-\mu}{x(x-1)(x+1)}. \tag{3.23}$$

Thus

$$\frac{U'(x)}{U(x)}=-\frac{\mu}{x}+\frac{(\alpha+1)}{(x-1)}+\frac{\alpha}{(x+1)} \tag{3.24}$$

and

$$U(x)=\begin{cases}k|x|^{-\mu}(1-x^2)^{\alpha}(1-x), & |x|<1,\\ 0, & |x|>1.\end{cases} \tag{3.25}$$

If we assume $\mathrm{Re}(1-\mu)>0$, $\mathrm{Re}(\alpha+1)>0$, then

$$(x^3-x)U(x)f(x)]_C=k(x^3-x)|x|^{-\mu}(1-x^2)^{\alpha}(1-x)f(x)]_{-1}^{+1}=0 \tag{3.26}$$

holds.

Determination of the normalisation factor.

$$\begin{aligned}\langle\mathcal{J}(\alpha,\alpha+1)(\mu),1\rangle&=k_1\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}(1-x)\,dx\\&=k_1\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}\,dx\\&=2k_1\int_{0}^{+1}(x)^{-\mu}(1-x^2)^{\alpha}\,dx\\&=2k_1\frac{1}{2}B\left(\frac{1-\mu}{2},\alpha+1\right)=1,\end{aligned} \tag{3.27}$$

where $B(p,q)$ is the beta function. Thus, from

$$\langle\mathcal{J}(\alpha,\alpha+1)(\mu),1\rangle=k_1B\left(\frac{1-\mu}{2},\alpha+1\right)=k_1\frac{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}{\Gamma((2\alpha+3-\mu)/2)}=1 \tag{3.28}$$

we get

$$k_1=\frac{\Gamma((2\alpha+3-\mu)/2)}{\Gamma((1-\mu)/2)\Gamma(1+\alpha)}. \tag{3.29}$$

Conversely, using this integral representation, we give explicitly the expressions of the moments and the functional equation (3.15). □

3.3. The expressions of the moments. Using the integral representation we have a relation between $(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1}$ and $(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2}$ and a relation between $(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2}$ and $(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n}$. Then, using these two relations, we obtain the functional equation.

Lemma 3.3. *Using the integral representation we have*

$$(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1}=-(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2},\quad n\geq 0. \tag{3.30}$$

Proof.

$$\begin{aligned}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+1}+x^{2n+2}\rangle &= k_1\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}(1-x)(x^{2n+1}+x^{2n+2})\,dx\\ &= k_1\int_{-1}^{+1}x^{2n+1}|x|^{-\mu}(1-x^2)^{\alpha+1}\,dx=0\end{aligned} \tag{3.31}$$

because $x^{2n+1}|x|^{-\mu}(1-x^2)^{\alpha+1}$ is an odd function. □

Lemma 3.4. *Using the integral representation we have*

$$(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2}=\frac{\Gamma((2n+3-\mu)/2)\Gamma(\alpha+1)}{\Gamma((2n+2\alpha+5-\mu)/2)} \tag{3.32}$$

and, in particular,

$$(2n+2\alpha+3-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2}=(2n+1-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n},\quad n\geq 0. \tag{3.33}$$

Proof. From

$$\begin{aligned}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+2}\rangle &= k_1\int_{-1}^{+1}|x|^{-\mu}(1-x^2)^{\alpha}(1-x)x^{2n+2}\,dx\\ &= k_1\int_{-1}^{+1}x^{2n+2}|x|^{-\mu}(1-x^2)^{\alpha}\,dx\end{aligned} \tag{3.34}$$

taking into account that $x^{2n+3}|x|^{-\mu}(1-x^2)^{\alpha}$ is an odd function,

$$\begin{aligned}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+2}\rangle &= 2k_1\int_{0}^{+1}x^{2n+2-\mu}(1-x^2)^{\alpha}\,dx\\ &= 2k_1\frac{1}{2}B\Big(\frac{2n+3-\mu}{2},\alpha+1\Big),\end{aligned} \tag{3.35}$$

where $B(p,q)$ is the beta function

$$\begin{aligned}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+2}\rangle &= \frac{\Gamma((2n+3-\mu)/2)\Gamma(\alpha+1)}{\Gamma((2n+2\alpha+5-\mu)/2)}\\ &= \frac{2n+1-\mu}{2n+2\alpha+3-\mu}\,\frac{\Gamma((2n+1-\mu)/2)\Gamma(\alpha+1)}{\Gamma((2n+2\alpha+3-\mu)/2)}\\ \langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+2}\rangle &= \frac{2n+1-\mu}{2n+2\alpha+3-\mu}\langle \mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n}\rangle,\quad n\geq 0.\end{aligned} \tag{3.36}$$

Using (3.30) and (3.33) we can find the functional equation (3.15).

From (3.33), we have, for $n\geq 0$,

$$(2n+2\alpha+3-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2}=(2n+1-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n}, \tag{3.37}$$

with (3.30), one has

$$\begin{aligned}&(2n+2\alpha+4-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2}\\ &\quad=-(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1}+(2n+1-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n},\quad n\geq 0.\end{aligned} \tag{3.38}$$

Using (2.1) and (2.2), we get, for $n \geq 0$,

$$\langle ((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))' + ((\mu-2\alpha-4)x^2-x-(\mu-1))\mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n}\rangle = 0. \quad (3.39)$$

From (3.15) and (3.33), we have, for $n \geq 0$,

$$(2n+2\alpha+5-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+3} = (2n+3-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1}. \quad (3.40)$$

Thus, taking into account (3.30), one has

$$\begin{aligned}&(2n+2\alpha+5-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+3}\\ &\quad = -(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2} + (2n+2-\mu)(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1}, \quad n \geq 0.\end{aligned} \quad (3.41)$$

From

$$\langle ((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))' + ((\mu-2\alpha-4)x^2-x-(\mu-1))\mathcal{J}(\alpha,\alpha+1)(\mu), x^{2n+1}\rangle = 0, \quad n \geq 0. \quad (3.42)$$

equations (3.39) and (3.42) give

$$\langle ((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))' + ((\mu-2\alpha-4)x^2-x-(\mu-1))\mathcal{J}(\alpha,\alpha+1)(\mu), x^{n}\rangle = 0, \quad n \geq 0. \quad (3.43)$$

Hence

$$((x^3-x)\mathcal{J}(\alpha,\alpha+1)(\mu))' + ((\mu-2\alpha-4)x^2-x-(\mu-1))\mathcal{J}(\alpha,\alpha+1)(\mu) = 0. \quad (3.44)$$

□

Corollary 3.5. *From (3.30) and (3.33) we deduce the expressions of the moments:*

$$\begin{aligned}(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1} &= -\prod_{i=0}^{n} \frac{(2i+1-\mu)}{(2\alpha+2i+3-\mu)}, \quad n \geq 0,\\ (\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+2} &= -(\mathcal{J}(\alpha,\alpha+1)(\mu))_{2n+1}, \quad n \geq 0.\end{aligned} \quad (3.45)$$

4. The recurrence coefficients β_n, γ_n, $n \geq 0$

4.1. The system satisfied by recurrence coefficients of semi-classical sequences of class $s = 1$. Assuming that u is semi-classical of class $s = 1$, then u satisfies

$$(\phi u)' + \psi u = 0 \quad (4.1)$$

with

$$\phi(x) = \sum_{k=0}^{3} c_k x^k, \qquad \sum_{k=0}^{3} |c_k| \neq 0, \qquad \psi(x) = \sum_{k=0}^{2} a_k x^k, \qquad |a_2| + |a_1| \neq 0 \quad (4.2)$$

(see [3, page 272]). Furthermore, the nonlinear system satisfied by the recurrence coefficients of semi-classical orthogonal sequences of class $s = 1$ is

$$\begin{gathered}(a_2-2nc_3)(\gamma_n+\gamma_{n+1}) = 4c_3\sum_{k=1}^{n-1}\gamma_k+2\sum_{k=0}^{n-1}(\theta_{\beta_n}\phi)(\beta_k)-\psi(\beta_n), \quad n\geq 2,\\ (a_2-2c_3)(\gamma_1+\gamma_2) = 2(\theta_{\beta_1}\phi)(\beta_0)-\psi(\beta_1),\\ a_2\gamma_1 = -\psi(\beta_0),\end{gathered} \tag{4.3}$$

$$\begin{aligned}(a_2-(2n+1)c_3)\gamma_{n+1}\beta_{n+1} = {} & \sum_{k=0}^{n}\phi(\beta_k)+c_3\left(2\gamma_n\left(n\beta_n+\sum_{k=0}^{n}\beta_k\right)+3\sum_{k=1}^{n}\gamma_k(\beta_k+\beta_{k-1})\right)\\ & +c_2\left((2n+1)\gamma_{n+1}+2\sum_{k=1}^{n}\gamma_k\right)-(a_2\beta_n+a_1)\gamma_{n+1}, \quad n\geq 1,\\ (a_2-c_3)\gamma_1\beta_1 = {} & \phi(\beta_0)+\gamma_1(2c_3\beta_0+c_2-a_2\beta_0-a_1).\end{aligned} \tag{4.4}$$

In our case, since $c_3 = -c_1 = 1$, $c_2 = c_0 = 0$, the first equation of (4.3) becomes

$$(\mu-2n-2\alpha-4)(\gamma_n+\gamma_{n+1}) = 4\sum_{k=1}^{n-1}\gamma_k+2\sum_{k=0}^{n-1}(\theta_{\beta_n}\phi)(\beta_k)-\psi(\beta_n), \quad n\geq 2. \tag{4.5}$$

Using (2.7), we get

$$\begin{aligned}(\mu-2n-2\alpha-4)\gamma_{n+1} &= -(\mu-2n-2\alpha-4)\gamma_n+4\sum_{k=1}^{n-1}\gamma_k+2\sum_{k=0}^{n-1}(\beta_n^2+\beta_k^2+\beta_n\beta_k-1)\\ &\quad -(\mu-2\alpha-4)\beta_n^2+\beta_n-(1-\mu)\\ &= -(\mu-2n-2\alpha-4)\gamma_n+4\sum_{k=1}^{n-1}\gamma_k+2\sum_{k=0}^{n-1}\beta_k^2+2\beta_n\sum_{k=0}^{n-1}\beta_k\\ &\quad +(2n+2\alpha+4-\mu)\beta_n^2+\beta_n+\mu-2n-1, \quad n\geq 2\end{aligned} \tag{4.6}$$

then

$$\begin{aligned}(\mu-2n-2\alpha-6)\gamma_{n+2} &= -(\mu-2n-2\alpha-6)\gamma_{n+1}+4\sum_{k=1}^{n}\gamma_k+2\sum_{k=0}^{n}\beta_k^2+2\beta_{n+1}\sum_{k=0}^{n}\beta_k\\ &\quad +(2n+2\alpha+6-\mu)\beta_{n+1}^2+\beta_{n+1}+\mu-2n-3, \quad n\geq 1.\end{aligned} \tag{4.7}$$

If we subtract both identities,

$$\begin{aligned}(\mu-2n-2\alpha-6)\gamma_{n+2} &= -(\mu-2n-2\alpha-6)\gamma_{n+1}+(\mu-2n-2\alpha-4)\gamma_{n+1}\\ &\quad +(\mu-2n-2\alpha-4)\gamma_n+4\gamma_n+2\beta_n^2\\ &\quad +2\beta_{n+1}\sum_{k=0}^{n}\beta_k-2\beta_n\sum_{k=0}^{n-1}\beta_k+(2n+2\alpha+6-\mu)\beta_{n+1}^2\\ &\quad -(2n+2\alpha+4-\mu)\beta_n^2+\beta_{n+1}-\beta_n-2, \quad n\geq 1.\end{aligned} \tag{4.8}$$

Thus the first equation of (4.3) becomes

$$\begin{aligned}(\mu-2n-2\alpha-6)\gamma_{n+2} &= 2\gamma_{n+1}+(\mu-2n-2\alpha)\gamma_n+2\beta_{n+1}\sum_{k=0}^{n}\beta_k-2\beta_n\sum_{k=0}^{n-1}\beta_k \\ &\quad+(2n+2\alpha+6-\mu)\beta_{n+1}^2-(2n+2\alpha+2-\mu)\beta_n^2 \\ &\quad+(\beta_{n+1}-\beta_n)-2, \quad n\geq 1.\end{aligned} \tag{4.9}$$

On the other hand, (4.4) becomes

$$\begin{aligned}(\mu-2n-2\alpha-5)\gamma_{n+1}\beta_{n+1} &= \sum_{k=0}^{n}\phi(\beta_k)+\left(2\gamma_n\left(n\beta_n+\sum_{k=0}^{n}\beta_k\right)+3\sum_{k=1}^{n}\gamma_k(\beta_k+\beta_{k-1})\right) \\ &\quad+c_2\left((2n+1)\gamma_{n+1}+2\sum_{k=1}^{n}\gamma_k\right)-((\mu-2\alpha-4)\beta_n-1)\gamma_{n+1} \\ &= \sum_{k=0}^{n}(\beta_k^3-\beta_k)+\left(2\gamma_n\left(n\beta_n+\sum_{k=0}^{n}\beta_k\right)+3\sum_{k=1}^{n}\gamma_k(\beta_k+\beta_{k-1})\right) \\ &\quad-((\mu-2\alpha-4)\beta_n-1)\gamma_{n+1}, \quad n\geq 1.\end{aligned} \tag{4.10}$$

Shifting the indices and subtracting, we get

$$\begin{aligned}(\mu-2n-2\alpha-7)\gamma_{n+2}\beta_{n+2} &= (\mu-2n-2\alpha-5)\gamma_{n+1}\beta_{n+1} \\ &\quad+\beta_{n+1}^3-\beta_{n+1}+3\gamma_{n+1}(\beta_{n+1}+\beta_n) \\ &\quad+\left(2\gamma_{n+2}\left((n+1)\beta_{n+1}+\sum_{k=0}^{n+1}\beta_k\right)\right) \\ &\quad-\left(2\gamma_{n+1}\left(n\beta_n+\sum_{k=0}^{n}\beta_k\right)\right)-((\mu-2\alpha-4)\beta_{n+1}-1)\gamma_{n+2} \\ &\quad+((\mu-2\alpha-4)\beta_n-1)\gamma_{n+1}, \quad n\geq 0.\end{aligned} \tag{4.11}$$

Thus, from (4.9) and (4.11) we have the following.

Proposition 4.1.

$$\begin{aligned}&(\mu-2n-2\alpha-6)\gamma_{n+2} \\ &\quad= 2\gamma_{n+1}+(\mu-2n-2\alpha)\gamma_n+2\beta_{n+1}\sum_{k=0}^{n}\beta_k-2\beta_n\sum_{k=0}^{n-1}\beta_k \\ &\qquad+(2n+2\alpha+6-\mu)\beta_{n+1}^2-(2n+2\alpha+2-\mu)\beta_n^2+(\beta_{n+1}-\beta_n)-2, \quad n\geq 1\end{aligned} \tag{4.12}$$

$$(\mu-2\alpha-6)(\gamma_1+\gamma_2) = 2(\beta_1^2+\beta_0\beta_1+\beta_0^2-1)-(\mu-2\alpha-4)\beta_1^2+\beta_1-(1-\mu) \tag{4.13}$$

$$(\mu-2\alpha-4)\gamma_1 = -(\mu-2\alpha-4)\beta_0^2+\beta_0-(1-\mu). \tag{4.14}$$

$$\begin{aligned}&(\mu-2n-2\alpha-7)\gamma_{n+2}\beta_{n+2} \\ &\quad= \beta_{n+1}^3-\beta_{n+1}+(2n+2\alpha+8-\mu)\gamma_{n+2}\beta_{n+1}+(\mu-2n-2\alpha-2)\gamma_{n+1}\beta_{n+1} \\ &\qquad+(\mu-2n-2\alpha-1)\gamma_{n+1}\beta_n+\left(2\sum_{k=0}^{n}\beta_k+1\right)(\gamma_{n+2}-\gamma_{n+1}), \quad n\geq 0\end{aligned} \tag{4.15}$$

$$(\mu-2\alpha-5)\gamma_1\beta_1 = \beta_0^3-\beta_0+\gamma_1(2\beta_0-(\mu-2\alpha-4)\beta_0+1). \tag{4.16}$$

Next, we will find the expressions of the recurrence parameters β_n, γ_n, $n \geq 0$. Since $\beta_0 = -(\mu-1)/(\mu-2\alpha-3)$ and from (4.14) we have

$$\begin{aligned}\gamma_1 &= -\frac{(\mu-2\alpha-4)\beta_0^2+\beta_0-(1-\mu)}{\mu-2\alpha-4}\\ &= -\beta_0^2+\frac{\beta_0}{\mu-2\alpha-4}+\frac{\mu-1}{\mu-2\alpha-4}\\ &= -\left(\frac{\mu-1}{\mu-2\alpha-3}\right)^2-\frac{\mu-1}{(\mu-2\alpha-3)(\mu-2\alpha-4)}+\frac{\mu-1}{\mu-2\alpha-4}\\ &= -\left(\frac{\mu-1}{\mu-2\alpha-3}\right)^2+\frac{\mu-1}{\mu-2\alpha-3}=2\frac{(\alpha+1)(1-\mu)}{(2\alpha+3-\mu)^2}.\end{aligned} \tag{4.17}$$

Using (4.16), (4.17) gives

$$\beta_1 = \frac{\beta_0^3-\beta_0+\gamma_1(-(\mu-2\alpha-6)\beta_0+1)}{(\mu-2\alpha-5)\gamma_1} = \frac{\mu(\mu-2\alpha-4)-(2\alpha+1)}{(2\alpha+3-\mu)(2\alpha+5-\mu)}. \tag{4.18}$$

With β_0, β_1, and γ_1, (4.13) gives

$$\gamma_2 = -\gamma_1+\frac{2(\beta_1^2+\beta_0\beta_1+\beta_0^2-1)-(\mu-2\alpha-4)\beta_1^2+\beta_1-(1-\mu)}{\mu-2\alpha-6} = \frac{2(2\alpha+3-\mu)}{(2\alpha+5-\mu)^2}. \tag{4.19}$$

With β_0, β_1, γ_1, and γ_2, (4.15) and some easy computations

$$\beta_2 = -\frac{\mu(\mu-2\alpha-6)+(2\alpha+1)}{(2\alpha+5-\mu)(2\alpha+7-\mu)}. \tag{4.20}$$

Proposition 4.2. *Assuming*

$$\begin{aligned}\beta_0 &= -\frac{\mu-1}{\mu-2\alpha-3},\\ \beta_{n+1} &= (-1)^n\frac{\mu(\mu-2n-2\alpha-4)+(-1)^{n+1}(2\alpha+1)}{(2n+2\alpha+3-\mu)(2n+2\alpha+5-\mu)},\\ \gamma_{2n+1} &= 2\frac{(n+\alpha+1)(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2},\\ \gamma_{2n+2} &= \frac{(2n+2)(2n+2\alpha+3-\mu)}{(4n+2\alpha+5-\mu)^2},\end{aligned} \tag{4.21}$$

for $n \geq 0$ *and assume* $\mu \neq 2n+1$, $\mu \neq 2n+2\alpha+1$, $\alpha \neq -n-1$, $n \geq 0$.

Lemma 4.3. *If* $E_n = \sum_{k=0}^{n}\beta_k$, $n \geq 0$, *then*

$$E_{2n} = -\left(\frac{2n+1-\mu}{4n+2\alpha+3-\mu}\right), \quad E_{2n+1} = -\frac{2n+2}{4n+2\alpha+5-\mu}, \quad n \geq 0. \tag{4.22}$$

Proof. $E_0 = \beta_0$. For $n \geq 0$, we have

$$
\begin{aligned}
E_{2n+1} &= \sum_{k=0}^{n} (\beta_{2k} + \beta_{2k+1}) \\
&= \sum_{k=0}^{n} -\frac{\mu(\mu-4k-2\alpha-2)+2\alpha+1}{(4k+2\alpha+1-\mu)(4k+2\alpha+3-\mu)} \\
&\quad + \frac{\mu(\mu-4k-2\alpha-4)-2\alpha-1}{(4k+2\alpha+3-\mu)(4k+2\alpha+5-\mu)} \\
&= \sum_{k=0}^{n} -\frac{1}{2}\frac{\mu-2\alpha-1}{(-4k-2\alpha-1+\mu)} - \frac{1}{2}\frac{\mu+2\alpha+1}{(-4k-2\alpha-3+\mu)} \\
&\quad + \frac{1}{2}\frac{\mu+2\alpha+1}{(-4k-2\alpha-3+\mu)} + \frac{1}{2}\frac{\mu-2\alpha-1}{(-4k-2\alpha-5+\mu)} \qquad (4.23) \\
&= \sum_{k=0}^{n} -\frac{1}{2}\frac{\mu-2\alpha-1}{(-4k-2\alpha-1+\mu)} + \frac{1}{2}\frac{\mu-2\alpha-1}{(-4k-2\alpha-5+\mu)} \\
&= -\frac{1}{2}\frac{\mu-2\alpha-1}{(-2\alpha-1+\mu)} + \frac{1}{2}\frac{\mu-2\alpha-1}{(-4n-2\alpha-5+\mu)} \\
&= -\frac{\mu-2\alpha-1}{2}\left(\frac{1}{-2\alpha-1+\mu} - \frac{1}{-4n-2\alpha-5+\mu}\right) \\
&= -\frac{(\mu-2\alpha-1)(-4n-2\alpha-5+\mu+2\alpha+1-\mu)}{2(-2\alpha-1+\mu)(-4n-2\alpha-5+\mu)} \\
&= -\frac{\mu-2\alpha-1}{2}\left(\frac{-4n-4}{(-2\alpha-1+\mu)(-4n-2\alpha-5+\mu)}\right)
\end{aligned}
$$

$$
E_{2n+1} = -\frac{2n+2}{(4n+2\alpha+5-\mu)}, \quad \mu \neq 4n+2\alpha+5,\ n \geq 0. \qquad (4.24)
$$

Calculus of

$$
\begin{aligned}
E_{2n+2} &= E_{2n+1} + \beta_{2n+2} \\
&= -\frac{2n+2}{4n+2\alpha+5-\mu} - \frac{\mu(\mu-4n-2\alpha-6)+2\alpha+1}{(4n+2\alpha+5-\mu)(4n+2\alpha+7-\mu)} \\
&= -\frac{1}{4n+2\alpha+5-\mu}\left(2n+2+\frac{\mu(\mu-2\alpha-4n-6)+2\alpha+1}{4n+2\alpha+7-\mu}\right) \\
&= -\frac{1}{4n+2\alpha+5-\mu} \\
&\quad \times \left(\frac{(2n+2)(4n+2\alpha+7)-(2n+2)\mu+\mu(\mu-2\alpha-4n-6)+2\alpha+1}{4n+2\alpha+7-\mu}\right) \\
&= -\frac{1}{4n+2\alpha+5-\mu}\left(\frac{\mu^2-(6n+2\alpha+8)\mu+(2n+2)(4n+2\alpha+7)+2\alpha+1}{4n+2\alpha+7-\mu}\right) \\
&= -\frac{1}{4n+2\alpha+5-\mu}\left(\frac{(4n+2\alpha+5-\mu)(2n+3-\mu)}{4n+2\alpha+7-\mu}\right),
\end{aligned} \qquad (4.25)
$$

$$
E_{2n+2} = -\frac{2n+3-\mu}{4n+2\alpha+7-\mu}, \quad \mu \neq 4n+2\alpha+7,\ n \geq 0. \qquad (4.26)
$$

□

Proof of Proposition 4.2. Suppose that we have

$$\begin{aligned}
\beta_0 &= -\frac{\mu-1}{\mu-2\alpha-3},\\
\beta_{2k+1} &= \frac{\mu(\mu-4k-2\alpha-4)-(2\alpha+1)}{(4k+2\alpha+3-\mu)(4k+2\alpha+5-\mu)}, \quad 0\le k\le n,\\
\beta_{2k} &= -\frac{\mu(\mu-4k-2\alpha-2)+(2\alpha+1)}{(4k+2\alpha+1-\mu)(4k+2\alpha+3-\mu)}, \quad 1\le k\le n,\\
\gamma_{2k+1} &= 2\frac{(k+\alpha+1)(2k+1-\mu)}{(4k+2\alpha+3-\mu)^2}, \quad 0\le k\le n,\\
\gamma_{2k+2} &= \frac{(2k+2)(2k+2\alpha+3-\mu)}{(4k+2\alpha+5-\mu)^2}, \quad 0\le k\le n-1,
\end{aligned} \tag{4.27}$$

and, using (4.10), (4.13), we prove by induction β_{2n+2}, β_{2n+3}, γ_{2n+2}, and γ_{2n+3}. The substitution $n\to 2n$ in (4.10) gives

$$\begin{aligned}
(\mu-2\alpha-4n-6)\gamma_{2n+2} = {}& 2\gamma_{2n+1}+(\mu-2\alpha-4n)\gamma_{2n}+2\beta_{2n+1}E_{2n}-2\beta_{2n}E_{2n-1}\\
&+(4n-\mu+2\alpha+6)\beta_{2n+1}^2-(4n-\mu+2\alpha+2)\beta_{2n}^2\\
&+(\beta_{2n+1}-\beta_{2n})-2, \quad n\ge 1.
\end{aligned} \tag{4.28}$$

We suppose known γ_{2n+1}, γ_{2n}, β_{2n+1}, β_{2n}, E_{2n}, and E_{2n-1} and then we evaluate γ_{2n+2} for the proof by recurrence; because of cumbersome computation, using Maple. The substitution $n\to 2n+1$ in (4.10) gives (see appendix)

$$\begin{aligned}
(\mu-2\alpha-4n-8)\gamma_{2n+3} = {}& 2\gamma_{2n+2}+(\mu-2\alpha-4n-2)\gamma_{2n+1}+2\beta_{2n+2}E_{2n+1}-2\beta_{2n+1}E_{2n}\\
&+(4n-\mu+2\alpha+8)\beta_{2n+2}^2-(4n-\mu+2\alpha+4)\beta_{2n+1}^2\\
&+(\beta_{2n+2}-\beta_{2n+1})-2, \quad n\ge 0.
\end{aligned} \tag{4.29}$$

The substitution $n\to 2n+1$ in (4.13) gives (see appendix)

$$\begin{aligned}
(\mu-2\alpha-4n-7)\gamma_{2n+2}\beta_{2n+2} = {}& \beta_{2n+1}^3-\beta_{2n+1}+(-\mu+2\alpha+4n+5)\beta_{2n+1}\gamma_{2n+2}\\
&-(-\mu+2\alpha+4n+2)\beta_{2n+1}\gamma_{2n+1}\\
&-(-\mu+2\alpha+4n+1)\beta_{2n}\gamma_{2n+1}\\
&+(2E_{2n}+1)(\gamma_{2n+2}-\gamma_{2n+1}), \quad n\ge 0.
\end{aligned} \tag{4.30}$$

Finally, the substitution $n\to 2n+2$ in (4.13) gives (see appendix)

$$\begin{aligned}
(\mu-2\alpha-4n-5)\gamma_{2n+3}\beta_{2n+3} = {}& \beta_{2n+2}^3-\beta_{2n+2}+(-\mu+2\alpha+4n+10)\beta_{2n+2}\gamma_{2n+3}\\
&-(-\mu+2\alpha+4n+4)\beta_{2n+2}\gamma_{2n+2}\\
&-(-\mu+2\alpha+4n+3)\beta_{2n+1}\gamma_{2n+2}\\
&+(2E_{2n+1}+1)(\gamma_{2n+3}-\gamma_{2n+2}), \quad n\ge 0.
\end{aligned} \tag{4.31}$$

□

Remarks. (1) An homothetie of rapport −1 gives a generalization of $\mathcal{J}(\alpha+1,\alpha)$, with (2.11), (2.13) we have

$$((x^3-x)u)' + ((\mu-2\alpha-4)x^2 + x - (\mu-1))u = 0, \tag{4.32}$$

$$\begin{aligned}
\beta_0 &= \frac{\mu-1}{\mu-2\alpha-3},\\
\beta_{n+1} &= (-1)^{n+1}\frac{\mu(\mu-2n-2\alpha-4)+(-1)^{n+1}(2\alpha+1)}{(2n+2\alpha+3-\mu)(2n+2\alpha+5-\mu)},\\
\gamma_{2n+1} &= 2\frac{(n+\alpha+1)(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2},\\
\gamma_{2n+2} &= \frac{(2n+2)(2n+2\alpha+3-\mu)}{(4n+2\alpha+5-\mu)^2},
\end{aligned} \tag{4.33}$$

for $n \geq 0$.

(2) For $\mu = 2\alpha+4$, we have an apparent particular case

$$((x^3-x)u)' + (x-(2\alpha+3))u = 0, \tag{4.34}$$

$$\begin{aligned}
\beta_0 &= -(2\alpha+3),\\
\beta_{n+1} &= (-1)^n\frac{(2\alpha+4)(-2n)+(-1)^{n+1}(2\alpha+1)}{(2n-1)(2n+1)},\\
\gamma_{2n+1} &= 2\frac{(n+\alpha+1)(2n-2\alpha-3)}{(4n-1)^2},\\
\gamma_{2n+2} &= \frac{(2n+2)(2n-1)}{(4n+1)^2},
\end{aligned} \tag{4.35}$$

for $n \geq 0$.

5. Appendix. In this appendix, we give both the input and output of the Maple programme used to carry out the computations of Section 4.

```
> restart;
> beta0:=-(mu-1)/(mu-2*alpha-3);
```

$$\beta 0 := -\frac{\mu-1}{\mu-2\alpha-3}$$

```
> gamma1:=factor(simplify(1/(mu-2*alpha-4)*((2*alpha+4-mu)*beta0^2
  +beta0+mu-1)));
```

$$\gamma 1 := -2\frac{(1+\alpha)(\mu-1)}{(\mu-2\alpha-3)^2}$$

```
> beta1:=collect(factor(simplify(1/((mu-2*alpha-5)*gamma1)*(beta0^3
  -beta0+gamma1*(-(mu-2*alpha-6)*beta0+1)))),mu);
  E1:=collect(simplify(beta0+beta1),mu);
```

$$\beta 1 := \frac{\mu^2+(-2\alpha-4)\mu-2\alpha-1}{(\mu-2\alpha-3)(\mu-2\alpha-5)}$$

$$E1 := \frac{2}{\mu-2\alpha-5}$$

```
> gamma2:=factor(simplify(-gamma1+1/(mu-2*alpha-6)*(2*beta1^2
  +2*beta0*beta1+2*beta0^2-2-(mu-2*alpha-4)*beta1^2+beta1-1+mu)));
```

$$\gamma 2 := -2\,\frac{\mu-2\alpha-3}{(\mu-2\alpha-5)^2}$$

```
> beta2:=collect(factor(simplify(1/((mu-2*alpha-4-3)*gamma2)
  *(beta1^3-beta1+(4-mu+2*alpha+4)*beta1*gamma2+(2+mu-2*alpha-4)
  *beta1*gamma1+(3+mu-2*alpha-4)*beta0*gamma1+(2*beta0+1)
  *(gamma2-gamma1)))),mu);E2:=collect(simplify(beta2+E1),mu);
```

$$\beta 2 := -\frac{\mu^2+(-2\alpha-6)\mu+2\alpha+1}{(\mu-2\alpha-5)(\mu-2\alpha-7)}$$

$$E2 := -\frac{\mu-3}{\mu-2\alpha-7}$$

```
> gamma3:=factor(simplify(1/(mu-2*alpha-8)*(2*gamma2+(mu-2*alpha-2)
  *gamma1+2*beta2*E1-2*beta1*beta0+(8-mu+2*alpha)*beta2^2
  -(2*alpha+4-mu)*beta1^2+beta2-beta1-2)));
```

$$\gamma 3 := -2\,\frac{(\alpha+2)(\mu-3)}{(\mu-2\alpha-7)^2}$$

```
> beta3:=collect(factor(simplify(1/((mu-2*alpha-4-5)*gamma3)
  *(beta2^3-beta2+(6-mu+2*alpha+4)*beta2*gamma3+(mu-2*alpha-4)
  *beta2*gamma2+(1+mu-2*alpha-4)*beta1*gamma2+(2*E1+1)
  *(gamma3-gamma2)))),mu);E3:=collect(simplify(beta3+E2),mu);
```

$$\beta 3 := \frac{\mu^2+(-2\alpha-8)\mu-2\alpha-1}{(\mu-2\alpha-7)(\mu-2\alpha-9)}$$

$$E3 := \frac{4}{\mu-2\alpha-9}$$

```
> gamma4:=factor(simplify(1/(mu-2*alpha-10)*(2*gamma3+(mu-2*alpha-4)
  *gamma2+2*beta3*E2-2*beta2*E1+(10-mu+2*alpha)*beta3^2
  -(2*alpha+6-mu)*beta2^2+beta3-beta2-2)));
```

$$\gamma 4 := -4\,\frac{\mu-2\alpha-5}{(\mu-2\alpha-9)^2}$$

```
> beta4:=collect(factor(simplify(1/((mu-2*alpha-4-4*1-3)*gamma4)
  *(beta3^3-beta3+(4*1+4-mu+2*alpha+4)*beta3*gamma4
  +(-4*1+2+mu-2*alpha-4)*beta3*gamma3+(-4*1+3+mu-2*alpha-4)
  *beta2*gamma3+(2*E2+1)*(gamma4-gamma3)))),mu);
```

$$\beta 4 := -\frac{\mu^2+(-10-2\alpha)\mu+2\alpha+1}{(\mu-2\alpha-9)(\mu-2\alpha-11)}$$

```
> gamma2n:=2*n*(2*n+2*alpha+1-mu)/(4*n+2*alpha+1-mu)^2;
```

$$\gamma 2n := 2\,\frac{n(2n+2\alpha+1-\mu)}{(4n+2\alpha+1-\mu)^2}$$

```
> gamma2np1:=2*(n+alpha+1)*(2*n+1-mu)/(4*n+2*alpha+3-mu)^2;
```

$$\gamma 2np1 := 2\,\frac{(n+\alpha+1)\,(2n+1-\mu)}{(4n+2\alpha+3-\mu)^2}$$

```
> beta2n:=-(mu*(mu-4*n-2*alpha-2)+2*alpha+1)/((4*n+2*alpha+1-mu)
  *(4*n+2*alpha+3-mu));
```

$$\beta 2n := -\frac{\mu(\mu-4n-2\alpha-2)+2\alpha+1}{(4n+2\alpha+1-\mu)\,(4n+2\alpha+3-\mu)}$$

```
> convert(beta2n, parfrac,n);
```

$$-1/2\,\frac{-2\alpha-1+\mu}{-4n-2\alpha-1+\mu}-1/2\,\frac{2\alpha+1+\mu}{-4n-2\alpha-3+\mu}$$

```
> beta2np1:=(mu*(mu-4*n-2*alpha-4)-2*alpha-1)/((4*n+2*alpha+3-mu)
  *(4*n+2*alpha+5-mu));
```

$$\beta 2np1 := \frac{\mu(\mu-4n-2\alpha-4)-2\alpha-1}{(4n+2\alpha+3-\mu)\,(4n+2\alpha+5-\mu)}$$

```
> convert(beta2np1, parfrac,n);
```

$$1/2\,\frac{2\alpha+1+\mu}{-4n-2\alpha-3+\mu}+1/2\,\frac{-2\alpha-1+\mu}{-4n-2\alpha-5+\mu}$$

```
> E2n:=-(2*n+1-mu)/(4*n+2*alpha+3-mu);
```

$$E2n := -\frac{2n+1-\mu}{4n+2\alpha+3-\mu}$$

```
> E2np1:=-(2*n+2)/(4*n+2*alpha+5-mu);
  E2nm1:=-(2*n)/(4*n+2*alpha+1-mu);
```

$$E2np1 := -\frac{2n+2}{4n+2\alpha+5-\mu}$$
$$E2nm1 := -2\,\frac{n}{4n+2\alpha+1-\mu}$$

```
> gamma2np2:=factor(simplify(1/(mu-2*alpha-4*n-6)
  *(2*gamma2np1+(mu-2*alpha-4*n)*gamma2n+2*beta2np1*E2n-2
  *beta2n*E2nm1+(4*n+6-mu+2*alpha)*beta2np1^2
  -(4*n+2*alpha+2-mu)*beta2n^2+beta2np1-beta2n-2)));
```

$$\gamma 2np2 := -2\,\frac{(n+1)\,(-2n+\mu-3-2\alpha)}{(-4n-2\alpha-5+\mu)^2}$$

```
> beta2np2:=factor(simplify(1/((mu-2*alpha-4-4*n-3)*gamma2np2)
  *(beta2np1^3-beta2np1+(4*n+4-mu+2*alpha+4)*beta2np1*gamma2np2
  +(-4*n+2+mu-2*alpha-4)*beta2np1*gamma2np1+(-4*n+3+mu-2*alpha-4)
  *beta2n*gamma2np1+(2*E2n+1)*(gamma2np2-gamma2np1))));
```

$$\beta 2np2 := -\frac{\mu^2-2\mu\alpha-4\mu n-6\mu+1+2\alpha}{(-4n-2\alpha-5+\mu)\,(\mu-2\alpha-7-4n)}$$

```
> gamma2np3:=factor(simplify(1/(mu-2*alpha-4*n-8)
  *(2*gamma2np2+(mu-2*alpha-4*n-2)*gamma2np1+2*beta2np2*E2np1-2
  *beta2np1*E2n+(4*n+8-mu+2*alpha)*beta2np2^2-(4*n+2*alpha+4-mu)
  *beta2np1^2+beta2np2-beta2np1-2)));
```

$$\gamma 2np3 := -2\frac{(n+2+\alpha)(\mu-2n-3)}{(\mu-2\alpha-7-4n)^2}$$

```
> beta2np3:=(factor(simplify(1/((mu-2*alpha-4-4*n-5)*gamma2np3)
  *(beta2np2^3-beta2np2+(4*n+6-mu+2*alpha+4)*beta2np2*gamma2np3
  +(-4*n+mu-2*alpha-4)*beta2np2*gamma2np2+(-4*n+1+mu-2*alpha-4)
  *beta2np1*gamma2np2+(2*E2np1+1)*(gamma2np3-gamma2np2)))));
```

$$\beta 2np3 := \frac{\mu-4\mu n-8\mu-2\mu\alpha-2\alpha-1}{(\mu-2\alpha-7-4n)(\mu-2\alpha-9-4n)}$$

References

[1] J. Alaya and P. Maroni, *Symmetric Laguerre-Hahn forms of class* $s = 1$, Integral Transform. Spec. Funct. **4** (1996), no. 4, 301–320. MR 98m:42032. Zbl 865.42021.

[2] M. Bachene, *Les polynômes orthogonaux semi-classiques de classe zéro et de classe un*, Tech. report, Universite de Paris (Pierre et Marie Curie), Paris, France, 1985, Thèse de troisième cycle.

[3] S. Belmehdi, *On semi-classical linear functionals of class* $s = 1$*. Classification and integral representations*, Indag. Math. (N.S.) **3** (1992), no. 3, 253–275. MR 94e:33038. Zbl 783.33003.

[4] T. S. Chihara, *An introduction to orthogonal polynomials*, Mathematics and its Applications, vol. 13, Gordon and Breach Science Publishers, New York, London, Paris, 1978. MR 58#1979. Zbl 389.33008.

[5] P. Maroni, *Sur la suite de polynômes orthogonaux associée à la forme* $u = \delta_c + \lambda(x-c)^{-1}L$ *[On the sequence of orthogonal polynomials associated with the form* $u = \delta_c + \lambda(x-c)^{-1}L$*]*, Period. Math. Hungar. **21** (1990), no. 3, 223–248 (French). MR 92c:42025. Zbl 732.42015.

[6] ______, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques [An algebraic theory of orthogonal polynomials. Application to semiclassical orthogonal polynomials]*, Orthogonal Polynomials and their Applications (Erice, 1990) (Basel), IMACS Ann. Comput. Appl. Math., vol. 9, Baltzer, 1991, pp. 95–130 (French). MR 95i:42018. Zbl 950.49938.

[7] ______, *Variations around classical orthogonal polynomials. Connected problems*, J. Comput. Appl. Math. **48** (1993), no. 1-2, 133–155, Proceedings of the Seventh Spanish Symposium on Orthogonal Polynomials and Applications (VII SPOA) (Granada, 1991). MR 94k:33013. Zbl 790.33006.

[8] ______, *Modified classical orthogonal polynomials associated with oscillating functions—open problems*, Appl. Numer. Math. **15** (1994), no. 2, 259–283, Innovative methods in numerical analysis (Bressanone, 1992). MR 96d:42037. Zbl 826.42020.

Mohamed Jalel Atia: Faculté des Sciences de Gabès, 6029 Route de Mednine Gabès, Tunisia

E-mail address: jalel.atia@fsg.rnu.tn

Journal of Applied Mathematics and Decision Sciences

Special Issue on Intelligent Computational Methods for Financial Engineering

Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems)

This special issue will include (but not be limited to) the following topics:

- **Computational methods**: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning
- **Application fields**: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management
- **Implementation aspects**: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

Authors should follow the Journal of Applied Mathematics and Decision Sciences manuscript format described at the journal site http://www.hindawi.com/journals/jamds/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Lean Yu, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; yulean@amss.ac.cn

Shouyang Wang, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; sywang@amss.ac.cn

K. K. Lai, Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong; mskklai@cityu.edu.hk