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A sieve for all primes of the form $x^2+(x+1)^2$

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Abstract: All composite numbers of the form $x^2+(x+1)^2$ are determined in terms of suitable (non-homogeneous) linear recurrence sequences of order 2 (Theorem 4.12). As a consequence, all primes of the same form in a given interval can be determined by a sieving procedure (Theorem 4.13).

Introduction

The object of this study are the prime and composite numbers of the form $x^2+(x+1)^2$. Their study depends heavily on the following

Theorem 1.1. (SIERPINSKI) [3]) *The number $x^2+(x+1)^2$ is composite if and only if there exist natural numbers y, z such that:*

$$T(x) = T(y) + T(z). \tag{T}$$

(*Here $T(x), T(y), T(z)$ denote triangular numbers.*)

The description of all composite numbers of the form $x^2+(x+1)^2$ is reduced to the study of the integral solutions of the following family of Diophantine equations of Fermat-Pell type:

$$X^2 - 2Y^2 = 2k^2 - 1, \quad k = 0, 1, 2, \dots. \tag{F_k}$$

Thus the study of equation (T) is reduced to the study of the family of equations (F_k) in terms of Gauss type transformations.

The detailed study of all solutions of (F_k) is carried on via Nagell's method of equivalence classes, thus avoiding any reference to fundamental units.

We will consider the Diophantine equation

$$\xi^2 - d\eta^2 = -1 \qquad (d \neq \square) \tag{1.1}$$

where $d \neq \square$ (non-square) is a natural number. The sequence of non-negative (that is $\xi_{2n+1} \geq 0$ and $\eta_{2n+1} \geq 0$) integral solutions of (1.1) is determined by the following recursive formulae:

$$\begin{aligned} &\xi_{2n+3} = 2x_1\xi_{2n+1} - \xi_{2n-1}, \text{ where } \xi_1 = \xi_1 \text{ and } \xi_3 = \xi_1^3 + 3d\xi_1\eta_1^2 \\ &\eta_{2n+3} = 2x_1\eta_{2n+1} - \eta_{2n-1}, \text{ where } \eta_1 = \eta_1 \text{ and } \eta_3 = 3\xi_1^2\eta_1 + d\eta_1^3, \end{aligned} \tag{1.2}$$

($n = 1, 2, \dots$) where $\xi_1 + \eta_1\sqrt{d}$ is the fundamental solution of (1.1) and $x_1 + y_1\sqrt{d}$ is the fundamental solution of

$$(P) \qquad x^2 - dy^2 = 1 \quad (d \neq \square).$$

The following Theorems can be found in [5] (cf. also [4]).

Theorem 1.2. *Consider the Diophantine equation*

$$(F) \qquad X^2 - dY^2 = C. \quad (d \neq \square,\ C > 0).$$

Let $X_r^* + Y_r^*\sqrt{d}$ be the fundamental solution of a class A_r of integral solutions of (F) with $X_r^* > 0$ Let $x_n + y_n\sqrt{d}$, where $n = 0, 1, \dots$, be the sequence of all non-negative integral solutions of (P). Let

$$X_n + Y_n\sqrt{d} \equiv (X_r^* + Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \text{ for all } n = 0, 1, \dots,$$
$$X_n' + Y_n'\sqrt{d} \equiv (X_r^* - Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}) \text{ for all } n = 1, 2, \dots$$

(for a typical r).

Then the following hold true:

(i) $Y_{n+1} > Y_n \geq 0$ *for every* $n = 0, 1, \dots$.

(ii) Let $Y_r^* > 0$. *Then* $Y_{n+1}' \geq Y_n > Y_n' > 0$ *for every* $n = 1, 2, \dots$.

(iii) Let $Y_r^* = 0$. *Then* $Y_n = Y_n'$ *for every* $n = 0, 1, \dots$.

(iv) Let A_r be genuine (= non-ambiguous). Then

$$Y_{n+1}' > Y_n > Y_n' > 0 \text{ for all } n = 1, 2, \dots.$$

(v) Let A_r be ambiguous. Then for every m there exist n such that:

$$X_m' = X_n \text{ and } Y_m' = Y_n.$$

(vi) Let $X_r^* + Y_r^*\sqrt{d}$, where $r = 1, 2, \dots, m$, be the only integral solutions of (F) such that

$$0 < X_r^* \leq \sqrt{(x_1 + 1)C/2} \text{ and } 0 \leq Y_r^* \leq y_1\sqrt{C}/\sqrt{2(x_1 + 1)}.$$

Then the set of all non-negative integral solutions of (F) consists of all pairs (X_n, Y_n) together with all pairs (X_n', Y_n') for all respective genuine classes A_r in addition to all pairs (X_n, Y_n) for all respective ambiguous classes

B_r. Moreover, $X_n, Y_n, X_n^{'}$ and $Y_n^{'}$ are determined by the following recursive formulae:

$$(1.3)\quad \begin{aligned} X_{n+1} &= 2x_1X_n - X_{n-1} \text{ for } n = 1,2,\dots \\ \text{with } X_0 &= X_r^*,\ X_1 = x_1X_r^* + dy_1Y_r^* \text{ and } r = 1,2,\dots,m. \\ Y_{n+1} &= 2x_1Y_n - Y_{n-1} \text{ for } n = 1,2,\dots \\ \text{with } Y_0 &= Y_r^*,\ Y_1 = y_1X_r^* + x_1Y_r^* \text{ and } r = 1,2,\dots,m. \end{aligned}$$

$$(1.4)\quad \begin{aligned} X_{n+1}^{'} &= 2x_1X_n^{'} - X_{n-1}^{'} \text{ for } n = 1,2,\dots \\ \text{with } X_0^{'} &= X_r^*,\ X_1^{'} = x_1X_r^* - dy_1Y_r^* \text{ and } r = 1,2,\dots,m. \\ Y_{n+1}^{'} &= 2x_1Y_n^{'} - Y_{n-1}^{'} \text{ for } n = 1,2,\dots \\ \text{with } Y_0^{'} &= -Y_r^*,\ Y_1^{'} = y_1X_r^* - x_1Y_r^* \text{ and } r = 1,2,\dots,m. \end{aligned}$$

Theorem 1.3. *Consider the Diophantine equation (F), $C \neq 0$. Let $X_r^* + Y_r^*\sqrt{d}$ be the fundamental solution of a class A_r of integral solutions of (F). Let $x_1 + y_1\sqrt{d}$ be the fundamental solutions of (P) and*

$$\begin{aligned} X_n + Y_n\sqrt{d} &\equiv (X_r^* + Y_r^*\sqrt{d})(x_1 + y_1\sqrt{d})^n \equiv (X_r^* + Y_r^*\sqrt{d})(x_n + y_n\sqrt{d}), \\ X_n^{'} + Y_n^{'}\sqrt{d} &\equiv (X_r^* - Y_r^*\sqrt{d})(x_1 + y_1\sqrt{d})^n \quad \text{for all } \ n = 0,1,\dots. \end{aligned}$$

Let $R_n \equiv Y_n^2 + k^2$ and $R_n^{'} \equiv Y_n^{'2} + k^2$, where k is a fixed integer. Then the numbers R_n and $R_n^{'}$ are determined by the following recursive formulae:

$$R_{n+1} = 2x_2R_n - R_{n-1} - 2k^2(x_2 - 1) + 2y_1^2C,$$

where $R_0 = {Y_r^*}^2 + k^2$ and $R_1 = (y_1X_r^* + x_1Y_r^*)^2 + k^2$.

$$R_{n+1}^{'} = 2x_2R_n^{'} - R_{n-1}^{'} - 2k^2(x_2 - 1) + 2y_1^2C,$$

where $R_0^{'} = {Y_r^*}^2 + k^2$ and $R_1^{'} = (y_1X_r^* - x_1Y_r^*)^2 + k^2$.

2. Reduction of the Diophantine equation $x(x+1) = y(y+1) + z(z+1)$ to a family of Fermat equations

Theorem 2.1 below aims at reducing the problem of solving the Diophantine equation

$$(E)\qquad x(x+1) = y(y+1) + z(z+1)$$

to that of solving each one of the Diophantine equations (F_k).

Theorem 1.3. *Consider the Diophantine equations (E) and (F_k). Then the following hold true:*
$(i)_1$ *Let (x, y, z) be an integral solution of (E) with $y \geq z$. Let*

$$X \equiv 2x + 1 \text{ and } Y \equiv 2y - (k-1), \text{ where } k \equiv y - z.$$

Then $X + Y\sqrt{2}$ is an integral solution of (F_k).
$(i)_2$ *If $y \neq 0, -1$ and $z \neq 0, -1$ then $|Y| \neq k \pm 1$.*
$(ii)_1$ *Let $X + Y\sqrt{2}$ be an integral solution of (F_k). Let*

$$x = (X-1)/2, \; y = (Y + k - 1)/2 \;\text{ and }\; z = (Y - k - 1)/2. \tag{2.1}$$

Then (x, y, z) is an integral solution of (E).
$(ii)_2$ *If $|Y| \neq k \pm 1$, then $y \neq 0, -1$ and $z \neq 0, -1$.*

Proof. $(\mathrm{i})_1$ By direct computation.

$(\mathrm{i})_2$ Clear because $|Y| = k \pm 1$ implies $(y = 0, -1)$ or $(z = 0, -1)$.

$(\mathrm{ii})_1$ Let $X + Y\sqrt{2}$ be an integral solution of (F_k). Then it is easily proved by parity considerations that the numbers (2.1) are integers. Also

$$X = 2x + 1, \; Y = 2y - (k-1) \;\text{ and }\; k = y - z,$$

whence (F_k) implies

$$(2x+1)^2 - 2(2y - (y - z - 1))^2 = 2(y-z)^2 - 1,$$

that is

$$x(x+1) = y(y+1) + z(z+1).$$

$(\mathrm{ii})_2$ Is proved in a way similar to the proof of $(\mathrm{i})_2$, namely $(y = 0, -1)$ or $(z = 0, -1)$ imply $|Y| = k \pm 1$.

Note. The transformation leading from (E) to (F_k) emanate from Gauss (Art. 216 in [1])

3. Determination of all integral solutions of the equation $X^2 - 2Y^2 = 2k^2 - 1$, where $k = 0, 1, \ldots$

Proposition 3.1 is crucial for the location of the fundamental solutions of (F_k). Further, Theorem 3.4 characterizes the classes of solutions of (F_k), (as regards genuineness or ambiguity) in terms of their representing fundamental solutions. Special attention is given to the case of $2k^2 - 1$ being a square number (cf. Theorem 3.5). The set of all non-negative solutions of (F_k) is determined recursively by Theorem 3.6 together with Corollary 3.7.

Proposition 3.1. *Consider the Diophantine equation (F_k) where k is a natural number. Let $X^*+Y^*\sqrt{2}$ be a solution of (F_k). Then $X^*+Y^*\sqrt{2}$ is the fundamental solution of a class of integral solutions of (F_k) if and only if the following (equivalent) inequalities are satisfied:*

$$0 < |X^*| \le 2k - 1, \tag{3.1}$$
$$0 \le Y^* \le k - 1. \tag{3.2}$$

Proof. By using Theorem 109 in [2].

Note. The fundamental solution of (F_0) is $X^* + Y^*\sqrt{2} = 1 + \sqrt{2}$.

Proposition 3.2. *Let k be a natural number. Then $2k-1+(k-1)\sqrt{2}$ is the fundamental solution of a class of integral solutions of (F_k).*

Proof. Evident by Proposition 3.1.

Proposition 3.3. *Let A be a class of integral solutions of the Diophantine equation (F), $C \neq 0$. Let $X + Y\sqrt{d}$ be a representative of A and*

$$L = (-X^2 - dY^2)/C \quad \text{and} \quad M = -2XY/C.$$

Then the following hold true:
(i) A is a genuine if and only if at least one of the numbers L, M is not integral.
(ii) A is ambiguous if and only if both numbers L and M are integral.

Proof. Immediate by using Nagell's criterion (p. 205, [2]).

Theorem 3.4. *Let $X^* + Y^*\sqrt{2}$ be the fundamental solution of a class A of integral solutions of (F_k), where $k = 1, 2, \ldots$. Then the following hold true:*
(i) A is genuine if and only if $Y^* > 0$.
(ii) A is ambiguous if and only if $Y^* = 0$.

Proof. (i) (a) If A is genuine, then the previous Proposition 3.3 easily implies $Y^* > 0$.

(b) Let now $Y^* > 0$ and assume that A is ambiguous. Then, by the same Proposition, the numbers

$$L = (-{X^*}^2 - 2{Y^*}^2)/(2k^2 - 1) \quad \text{and} \quad M = -2X^*Y^*/(2k^2 - 1)$$

are integers. In particular, because L is an integer it follows that

$$(2k^2-1) \mid {X^*}^2 + 2{Y^*}^2 = 4{Y^*}^2 + 2k^2 - 1.$$

Thus

$$(2k^2-1) \mid 4{Y^*}^2.$$

Also, $Y^* \le \sqrt{(2k^2-1)/2}$, i.e.

$$4{Y^*}^2 < 2(2k^2-1).$$

Hence

$$2k^2 - 1 < 4{Y^*}^2 = h(2k^2-1) < 2(2k^2-1),$$

where h is a natural number. Hence $1 < h < 2$, which is impossible. Hence A is genuine.

(ii) Immediate by (i).

Note: (F_0) has only one class of integral solutions, which is ambiguous.

Theorem 3.5. *Let k be a natural number. Then the following are equivalent:*

(i) $2k^2 - 1$ is a square number.

(ii) The totality of ambiguous classes of integral solutions of (F_k) consists of a single class.

In consequence, if $2k^2 - 1$ is not a square number, then every class of integral solutions of (F_k) is genuine.

Proof. By using Proposition 3.1 and Theorem 3.4.

Theorem 3.6. *Consider the Diophantine equation (F_k), where k is a natural number. Let $x_n + y_n\sqrt{2}$, where $n = 0, 1, 2, \ldots$, be the sequence of all non-negative integral solutions of*

$$x^2 - 2y^2 = 1.$$

Let $X_r^* + Y_r^*\sqrt{2}$, (where $r = 1, 2, \ldots, m$), be the only integral solutions of (F_k) such that:

$$0 < X_r^* \le 2k - 1 \ \text{ and } \ 0 \le Y_r^* \le k - 1.$$

Let

$$X_n + Y_n\sqrt{2} \equiv (X_r^* + Y_r^*\sqrt{2})(x_n + y_n\sqrt{2}) \text{ for all } n = 0, 1, \ldots,$$
$$X_n' + Y_n'\sqrt{2} \equiv (X_r^* - Y_r^*\sqrt{2})(x_n + y_n\sqrt{2}) \text{ for all } n = 1, 2, \ldots,$$

(for a typical r). Then the following hold true:

(i) Let $Y_r^* > 0$ and $k \geq 2$. (Case of genuine classes of integral solutions of (F_k)). Then the pairs (X_n, Y_n) and $(X_n^{'}, Y_n^{'})$ are determined by (1.3) and (1.4) (for $x_1 = 3$, $y_1 = 2$ and $d = 2$).

(ii) Let $Y_r^* = 0$. (Case of ambiguous classes). Then the pairs (X_n, Y_n) are determined by (1.3).

Moreover, in case (i) all pairs (X_n, Y_n) together with all pairs $(X_n^{'}, Y_n^{'})$ constitute the set of all non-negative integral solutions of (F_k) which belong to the class with typical fundamental solution $X_r^* + Y_r^*\sqrt{2}$. Also, in case (ii) all pairs (X_n, Y_n) constitute the set of all non-negative integral solutions of (F_k) which belong to the class with typical fundamental solution $X_r^* + 0\sqrt{2}$.

Proof. By using Theorems 3.4, 3.5, 1.2(vi) and Proposition 3.1.

Corollary 3.7. *The sequence of all positive integral solutions (X_n, Y_n) of (F_0) is determined by (1.2) (for $X_n \equiv \xi_{2n+1}, Y_n \equiv \eta_{2n+1}, \xi_1 = 1, \xi_3 = 7, \eta_1 = 1$ and $\eta_3 = 5$).*

4. Determination of all prime and composite numbers of the form $x^2 + (x+1)^2$.

In Theorem 4.2 it is shown that every positive (integral) solution of (T) leads to a non-negative solution of a certain (F_k) and vice-versa. Theorems 4.6, 4.7 together with Corollary 4.8 determine all (F_k) whose non-negative solutions (taken together) lead to all positive solutions of (T).

In Theorem 1.1 a primality criterion is given for numbers of the form $N(x) = x^2+(x+1)^2$. Composite numbers of the form $N(x)$ are characterized (in terms of a suitable solution of (F_k)) in Theorem 4.9. The recursive determination of all composite numbers of the form $N(x)$ is given by Theorems 4.10, 4.11 and 4.12. This leads to our final Theorem 4.13, which constitutes an algorithm (sieve) for the determination of all primes of the form $N(x)$.

Lemma 4.1. *Let $X + Y\sqrt{2}$ be a non-negative integral solution of (F_k). Let*

$$x \equiv (X-1)/2, \; y \equiv (Y+k-1)/2 \;\; \text{and} \;\; z \equiv (Y-k-1)/2.$$

Then x, y, z are natural numbers if and only if $Y > k+1$.

Proof. Easy and so omitted.

Theorem 4.2. *Consider the Diophantine equations (F_k) and (T). Then the following hold true:*

(i) *Let* $X+Y\sqrt{2}$ *be a (non-negative) integral solution of* (F_k)*, with* $Y > k+1$*. Let*

$$x \equiv (X-1)/2,\ y \equiv (Y+k-1)/2 \ \text{ and } \ z \equiv (Y-k-1)/2.$$

Then (x,y,z) *is a triad of positive integral solutions of* (T)*.*

(ii) *Let* (x,y,z) *be a triad of positive integral solutions of* (T) *with* $y \geq z$*. Let*

$$k \equiv y-z,\ X \equiv 2x+1 \ \text{ and } \ Y \equiv 2y-(k-1).$$

Then $X+Y\sqrt{2}$ *is a (non-negative) integral solution of* (F_k) *with* $Y > k+1$*.*

Proof. By using Theorem 2.1, Lemma 4.1 and the fact that the Diophantine equation (T) is equivalent to the equation (E).

Proposition 4.3. *Let* k *be a natural number. Let* $X+Y\sqrt{2}$ *be a non-negative integral solution of* (F_k)*. Then the following hold true:*

(i) *Let* $0 \leq Y \leq k-1$*. Then* $X+Y\sqrt{2}$ *is a fundamental solution of a class of integral solutions of* (F_k)*.*

(ii) $Y \neq k$*.*

(iii) *Let* $Y = k+1$*. Then* $X = 2k+1$*. Moreover,* $X+Y\sqrt{2} = (2k+1)+(k+1)\sqrt{2}$ *is obtained from the fundamental solution* $(X^* = 2k-1, Y^* = k-1)$ *as follows:*

$$X+Y\sqrt{2} = (2k-1+(k-1)\sqrt{2})(3+2\sqrt{2}) \text{ for } k=1 \text{ and}$$
$$X+Y\sqrt{2} = (2k-1-(k-1)\sqrt{2})(3+2\sqrt{2}) \text{ for } k>1.$$

Proof. By direct computations.

Proposition 4.4. *Consider the Diophantine equation* (F_k)*, where* $k > 1$*. Let* $X^*+Y^*\sqrt{2}$ *be the fundamental solution of a class* A *of* (F_k) *with* $X^* > 0$*. Let* $3+2\sqrt{2}$ *be the fundamental solution of the equation*

$$x^2-2y^2=1.$$

Let

$$Z_n \equiv X_n+Y_n\sqrt{2} \equiv (X^*+Y^*\sqrt{2})(3+2\sqrt{2})^n \text{ for all } n=0,1,\ldots, \text{ and}$$
$$Z_n^{'} \equiv X_n^{'}+Y_n^{'}\sqrt{2} \equiv (X^*-Y^*\sqrt{2})(3+2\sqrt{2})^n \text{ for all } n=1,2,\ldots.$$

Then the following hold true:

(i) *Let* A *be genuine. Then the only (non-negative) integral solutions* $X+Y\sqrt{2}$ *of* (F_k) *which belong to* A *or to* $\overline{A}$ *and satisfy the inequality* $Y > k+1$ *are the following:*

(a) $Z_n \in A$ *and* $Z_n^{'} \in \overline{A}$ *for all* $n \geq 1$ *if and only if* $Y^* < k-1$.
(b) $Z_n \in A$ *for all* $n \geq 1$ *and* $Z_n^{'} \in \overline{A}$ *for all* $n \geq 2$ *if and only if* $Y^* = k-1$.

(ii) Let A *be ambiguous, (whence* $Y^* = 0$*, while* $2k^2-1$ *is a square number). Then the only (non-negative) integral solutions* $X + Y\sqrt{2}$ *of* (F_k) *which belong to* A *and satisfy the inequality* $Y > k+1$ *are all* Z_n *for every* $n \geq 1$.

Proof. (i) By Theorem 1.2 (iv) we have:

$$Y_{n+1}^{'} > Y_n > Y_n^{'} > 0 \quad \text{for all } n \geq 1, \text{ where } Y_1^{'} = 2X^* - 3Y^*.$$

(a) Hence, we have $Y_1^{'} = 2X^* - 3Y^* > k+1$ if and only if $(2X^*)^2 > (3Y^* + k + 1)^2$, that is if and only if $(Y^* - (k-1))(Y^* + 7k + 5) < 0$, and so if and only if $Y^* < k-1$.

Consequently, by Proposition 4.3, the only (non-negative) integral solution $X + Y\sqrt{2}$ of (F_k), which belong to A or $\overline{A}$ and satisfy the inequality $Y > k+1$ are all $Z_n \in A$ and all $Z_n^{'} \in \overline{A}$, $n = 1, 2, \ldots$, for which $Y^* < k-1$.

(b) Hence, $Y_1^{'} = 2X^* - 3Y^* = k+1$ if and only if $Y^* = k-1$.

Thus, the only (non-negative) integral solutions $X+Y\sqrt{2}$ of (F_k), which belong to A or $\overline{A}$ and satisfy the inequality $Y > k+1$ are all $Z_n \in A$ for all $n \geq 1$ and all $Z_n^{'} \in \overline{A}$ for all $n \geq 2$ if and only if $Y^* = k-1$.

(ii) By Theorem 1.2 (i) the following hold true: $Y_{n+1} > Y_n \geq 0$ for all $n = 0, 1, \ldots$, while $Y^* = Y_0 = 0$ and $Y_1 = 2\sqrt{2k^2-1}$.

Also, (by direct computations) we show that $Y_1 > k+1$. Consequently, the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k), which belong to A and satisfy the inequality $Y > k+1$ are all Z_n for every $n \geq 1$.

Proposition 4.5. *Consider the Diophantine equation* (F_1)*. Let*

$$X_n + Y_n\sqrt{2} \equiv (1 + 0\sqrt{2})(3 + 2\sqrt{2})^n \quad \textit{for all } n = 0, 1, \ldots.$$

Then the only (non-negative) integral solutions $X + Y\sqrt{2}$ *of* (F_1)*, such that* $Y > 2$*, are all* $X_n + Y_n\sqrt{2}$ *for every* $n \geq 2$.

Proof. By using Theorem 1.2 (i).

Theorem 4.6. *Let* k *be a natural number. Consider the Diophantine equation* (F_k)*. Let* $Z_r^* \equiv X_r^* + Y_r^*\sqrt{2}$*, (where* $r = 1, 2, \ldots, m$*) be the only integral solutions of* (F_k) *such that:*

$$X_r^* > 0 \quad \textit{and} \quad 0 \leq Y_r^* \leq k-1.$$

Let A_r be the corresponding classes of integral solutions of (F_k) with fundamental solutions Z_r^*. Let

$$Z_n \equiv X_n + Y_n\sqrt{2} \equiv (X_r^* + Y_r^*\sqrt{2})(3+2\sqrt{2})^n \text{ for all } n = 0,1,\dots,$$
$$Z_n^{'} \equiv X_n^{'} + Y_n^{'}\sqrt{2} \equiv (X_r^* - Y_r^*\sqrt{2})(3+2\sqrt{2})^n \text{ for all } n = 1,2,\dots.$$

for an (arbitrary) typical r. Then the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k), which satisfy the inequality $Y > k+1$, are the following:

(i) All $Z_n \in A_r$ and all $Z_n^{'} \in \overline{A}_r$ for every $n \geq 1$ if and only if $0 < Y_r^* < k-1$.

(ii) All $Z_n \in A_r$ for every $n \geq 1$ and all $Z_n^{'} \in \overline{A}_r$ for every $n \geq 2$ if and only if $0 < Y^* = k-1$.

(iii) All $Z_n \in A_r$ for every $n \geq 1$ if and only if $Y_r^* = 0$ for $k \geq 2$.

(iv) All $Z_n \in A_r$ for every $n \geq 2$ if and only if $Y_r^* = 0$ for $k = 1$.

Proof. By using Propositions 4.4, 4.5 and Theorem 3.6.

Theorem 4.7. *Let k be a natural number. Consider the Diophantine equation (F_k). Let $X_r^* + Y_r^*\sqrt{2}$, (where $r = 1,2,\dots,m$) be the only integral solutions of (F_k) such that:*

$$X_r^* > 0 \ \text{ and } \ 0 \leq Y_r^* \leq k-1.$$

Let

$$X_n + Y_n\sqrt{2} \equiv (X_r^* + Y_r^*\sqrt{2})(3+2\sqrt{2})^n \text{ for all } n = 0,1,\dots \ \text{ and } r = 1,2,\dots,m,$$
$$X_n^{'} + Y_n^{'}\sqrt{2} \equiv (X_r^* - Y_r^*\sqrt{2})(3+2\sqrt{2})^n \text{ for all } n = 1,2,\dots \ \text{ and } r = 1,2,\dots,m.$$

Then the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_k) such that $Y > k+1$ are the following:

(i) All $X_n + Y_n\sqrt{2}$ and all $X_n^{'} + Y_n^{'}\sqrt{2}$ (with $n \geq 1$) for every Y_r^* with $0 < Y_r^* < k-1$, when $k \geq 2$.

(ii) All $X_n + Y_n\sqrt{2}$ (with $n \geq 1$) and all $X_n^{'} + Y_n^{'}\sqrt{2}$ (with $n \geq 2$) for $0 < Y_r^* = k-1$, when $k \geq 2$.

(iii) All $X_n + Y_n\sqrt{2}$ (with $n \geq 1$) for $Y_r^* = 0$, when $k \geq 2$.

(iv) All $X_n + Y_n\sqrt{2}$ (with $n \geq 2$) for $Y_r^* = 0$, when $k = 1$.

Proof. By using Theorems 3.6 and 4.6.

By Corollary 3.7 it follows that

Corollary 4.8. *The only non-negative integral solutions $X + Y\sqrt{2}$ of (F_0) such that $Y > 1$ are:*

$$X_n + Y_n\sqrt{2} \ \text{ for every } n = 1,2,\dots.$$

Theorem 4.9. *Consider the Diophantine equation (F_k), $k = 0, 1, \ldots$. Let $X+Y\sqrt{2}$ be a non-negative integral solution of (F_k). Let $x \equiv (X-1)/2$ and $N(x) \equiv x^2 + (x+1)^2$. Then $N(x) = Y^2 + k^2$. Moreover, the following are equivalent:*
(i) $N(x)$ is composite.
(ii) $Y > k+1$.

Proof. The equality $N(x) = Y^2 + k^2$ follows by direct computations, while the equivalence of (i) and (ii) follows from Theorems 4.2 and 1.1.

Theorem 4.10. *Let $N(x) \equiv x^2 + (x+1)^2$. Consider the Diophantine equation (F_k), $k = 0, 1, \ldots$. Let $X_r^* + Y_r^*\sqrt{2}$, (where $r = 1, 2, \ldots, m$) be the only non-negative integral solutions of (F_k) such that:*

$$0 \leq Y_r^* \leq k-1 \quad \text{for } k \geq 1,$$

while, for $k = 0$ we have: $X_r^* = Y_r^* = 1$ for all $r = 1, 2, \ldots, m$. Let

$$X_n + Y_n\sqrt{2} \equiv (X_r^* + Y_r^*\sqrt{2})(3+2\sqrt{2})^n,$$
$$X_n^{'} + Y_n^{'}\sqrt{2} \equiv (X_r^* - Y_r^*\sqrt{2})(3+2\sqrt{2})^n \text{ for all } n = 0, 1, \ldots,$$

(for a typical r). Let $\tilde{x}_n \equiv (X_n - 1)/2$ and $\tilde{x}_n^{'} \equiv (X_n^{'} - 1)/2$ for every $n = 0, 1, \ldots$. Let R_n, $R_n^{'}$, where $n = 0, 1, \ldots$, be the sequences defined by the recursive formmulae:

$$R_{n+1} = 34R_n - R_{n-1} - 8(2k^2+1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0 = {Y_r^*}^2 + k^2$, $R_1 = (2X_r^* + 3Y_r^*)^2 + k^2$ (for a typical r).

$$R_{n+1}^{'} = 34R_n^{'} - R_{n-1}^{'} - 8(2k^2+1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0^{'} = {Y_r^*}^2 + k^2$, $R_1^{'} = (2X_r^* - 3Y_r^*)^2 + k^2$ (for a typical r).
Then the following hold true:
(i) Let $k = 0$. The for every integer n there exists an integer m such that:

$$R_n = R_m^{'} = N(\tilde{x}_n) \quad \text{for every } n \geq 0.$$

Moreover, the numbers $R_1, R_2, \ldots$, are all composite.
(ii) Let $k = 1$, whence $X_r^* = 1$, $Y_r^* = 0$ for every $r = 1, 2, \ldots, m$. Then

$$R_n = R_n^{'} = N(\tilde{x}_n) \quad \text{for every } n \geq 0.$$

Moreover, the numbers $R_2, R_3, \ldots$, are all composite.

(iii) Let $k \geq 2$ and $Y_r^* = 0$ Then

$$R_n = R_n^{'} = N(\tilde{x}_n) \ \text{ for every } n \geq 0.$$

Moreover, the numbers $R_1, R_2, \ldots$, are all composite.

(iv) Let $k \geq 2$ and $Y_r^* = k-1$. Then

$$R_n = N(\tilde{x}_n) \ \text{ and } \ R_n^{'} = N(\tilde{x}_n^{'}) \ \text{ for every } n \geq 0.$$

Moreover, the numbers $R_1, R_2, \ldots$, and also the numbers $R_2^{'}, R_3^{'}, \ldots$, are all composite.

(v) Let $k \geq 2$ and $0 < Y_r^* < k-1$. Then

$$R_n = N(\tilde{x}_n) \ \text{ and } \ R_n^{'} = N(\tilde{x}_n^{'}) \ \text{ for every } n \geq 0.$$

Moreover, the numbers $R_1, R_2, \ldots$, and also the numbers $R_1^{'}, R_2^{'}, \ldots$, are all composite.

Note: For the cases (iv) and (v) we have:

$$R_m \neq R_n^{'} \ \text{ for any } m, n.$$

Proof. (i) The unique class of integral solutions of (F_0) is ambiguous. By Theorem 2.4 in [5] and Corollary 4.8 we have:

$$X_n + Y_n\sqrt{2} \equiv \xi_{2n+1} + \eta_{2n+1}\sqrt{2} = (1+\sqrt{2})(x_n + y_n\sqrt{2}) = (1+\sqrt{2})^{2n+1}$$

for all $n = 0, 1, \ldots$.

Hence, by the definition of ambiguous class and Theorem 1.3, for every integer n there exists an integer m such that:

$$R_n = R_m^{'} = N(\tilde{x}_n), \text{ where } \tilde{x}_n = (\xi_{2n+1} - 1)/2.$$

According to Corollary 4.8, the only (non-negative) integral solutions $X + Y\sqrt{2}$ of (F_0) such that $Y > 1$ are all $Y_{n+1} = \eta_{2n+3}$ for every $n \geq 0$. Hence by Theorem 4.9, the numbers $R_1, R_2, \ldots$ are all composite.

(ii) Obviously $X_r^* = 1$, $Y_r^* = 0$ for every $r = 1, 2, \ldots, m$ because $k = 1$. Hence, $R_n = R_n^{'}$ for all $n = 0, 1, \ldots$. Now, Theorem 1.3 implies

$$R_n = N(\tilde{x}_n) = Y_n^2 + k^2 = Y_n^2 + 1 \ \text{ for all } n \geq 0.$$

Also, by Theorem 4.7 (iv), we deduce that $X_{n+1}+Y_{n+1}\sqrt{2}$, where $n \geq 1$, are the only (non-negative) integral solutions of (F_1) such that $Y_{n+1} > k+1 = 2$. Hence, according to Theorem 4.9, the numbers $R_2, R_3, \ldots$ are all composite.

(iii) We have $R_n = R_n^{'}$ for every $n = 0, 1, \ldots$ because $Y_r^* = 0$. By Theorem 4.7 (iii) the numbers $X_{n+1} + Y_{n+1}\sqrt{2}$, where $n \geq 0$, are the only (non-negative) integral solutions of (F_k) such that $Y_{n+1} > k+1$. This completes the proof by invoking Theorems 1.3 and 4.9.

(iv) By Theorem 4.7 (ii) the numbers $X_{n+1}+Y_{n+1}\sqrt{2}$ with $n \geq 0$, together with the numbers $X_{n+1}^{'} + Y_{n+1}^{'}\sqrt{2}$, with $n \geq 1$, are the only (non-negative) integral solutions of (F_k) such that $Y_{n+1} > k+1$ and $Y_{n+1}^{'} > k+1$. Thus the proof is completed by Theorem 1.3 and 4.9.

(v) By Theorem 4.7 (i), the numbers $X_{n+1} + Y_{n+1}\sqrt{2}$ together with the numbers $X_{n+1}^{'} + Y_{n+1}^{'}\sqrt{2}$, where $n \geq 0$, are the only (non-negative) integral solutions of (F_k) such that $Y_{n+1} > k+1$ and $Y_{n+1}^{'} > k+1$. This finishes the proof of the whole Theorem, again in view of Theorems 1.3 and 4.9.

Theorem 4.11. *Consider the Diophantine equation (F_k), $k = 0, 1, \ldots$. Let $X_r^* + Y_r^*\sqrt{2}$, (where $r = 1, 2, \ldots, m$) be the only non-negative integral solutions of (F_k) such that:*

$$0 \leq Y_r^* \leq k-1 \quad \text{for } k \geq 1,$$

While, for $k = 0$ we have: $X_r^* = Y_r^* = 1$ for all $r = 1, 2, \ldots, m$. Let $R_n, R_n^{'}$ be the sequences, defined by the recursive formulae:

$$R_{n+1} = 34R_n - R_{n-1} - 8(2k^2+1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0 = {Y_r^*}^2 + k^2$, $R_1 = (2X_r^* + 3Y_r^*)^2 + k^2$ (for a typical r).

$$R_{n+1}^{'} = 34R_n^{'} - R_{n-1}^{'} - 8(2k^2+1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0^{'} = {Y_r^*}^2 + k^2$, $R_1^{'} = (2X_r^* - 3Y_r^*)^2 + k^2$ (for a typical r).

Suppose that the number $N(x) \equiv x^2 + (x+1)^2$ is composite. Then $N(x)$ is equal to some of the composite numbers R_n or $R_n^{'}$, for a suitable index, as stated in cases (i)–(v) of Theorem 4.10 (for some value of k).

Proof. Since $N(x)$ is composite it follows from Theorem 1.1 that there exist natural numbers y, z such that

$$T(x) = T(y) + T(z).$$

Let $y \geq z$. Let also $k \equiv y - z$, $X \equiv 2x + 1$ and $Y \equiv 2y - (k - 1)$. Then, according to Theorem 4.2 (ii), $X + Y\sqrt{2}$ is a (non-negative) integral solution of (F_k), with $Y > k + 1$. Hence, $X + Y\sqrt{2}$ is a solution of type (i) or (ii) or (iii) or (iv) of Theorem 4.7 or it is a solution $X + Y\sqrt{2}$ of (F_0) with $Y > 1$ (see Corollary 4.8). Also, $N(x) = Y^2 + k^2$. Hence, by Theorem 1.3 $N(x)$ is equal to some R_n or some $R_n^{'}$. Finally, the appropriate index n for which $N(x) = R_n$ or $N(x) = R_n^{'}$ is obtained by applying Theorem 4.6 to the respective case as in (i)-(v) of Theorem 4.10. This ends the proof of the Theorem.

Theorem 4.12. (*Determination of all composites of the form* $N(x) \equiv x^2 + (x+1)^2$) *Consider the Diophantine equations*

$$(F_k) \qquad X^2 - 2Y^2 = 2k^2 - 1, \quad \text{where } k = 0, 1, \ldots .$$

Let $X_r^* + Y_r^*\sqrt{2}$, *(where* $r = 1, 2, \ldots, m$*), be the only non-negative integral solutions of* (F_k) *such that:*

$$0 \leq Y_r^* \leq k - 1 \quad \text{for } k \geq 1,$$

While, for $k = 0$ *we have:* $X_r^* = Y_r^* = 1$ *for all* $r = 1, 2, \ldots, m$. *Let* $R_n, R_n^{'}$ *be the sequences defined by the recursive formulae:*

$$R_{n+1} = 34R_n - R_{n-1} - 8(2k^2 + 1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0 = {Y_r^*}^2 + k^2$, $R_1 = (2X_r^* + 3Y_r^*)^2 + k^2$ *(for a typical* r*).*

$$R_{n+1}^{'} = 34R_n^{'} - R_{n-1}^{'} - 8(2k^2 + 1) \quad \text{for all } n = 1, 2, \ldots,$$

where $R_0^{'} = {Y_r^*}^2 + k^2$, $R_1^{'} = (2X_r^* - 3Y_r^*)^2 + k^2$ *(for a typical* r*).*

Then, the only composite numbers of the form $N(x) \equiv x^2 + (x+1)^2$ *are the following:*

(i) $R_1, R_2, \ldots$ *(for* $k = 0$*).*
(ii) $R_2, R_3, \ldots$ *(for* $k = 1$ *and* $Y_r^* = 0$*).*
(iii) $R_1, R_2, \ldots$ *(for* $k \geq 2$ *and* $Y_r^* = 0$*).*
(iv) $R_1, R_2, \ldots$ *together with* $R_2^{'}, R_3^{'}, \ldots$ *(for* $k \geq 2$ *and* $Y_r^* = k - 1$*).*
(v) $R_1, R_2, \ldots$ *together with* $R_1^{'}, R_2^{'}, \ldots$ *(for* $k \geq 2$ *and for all* Y_r^* *such that* $0 < Y_r^* < k - 1$*).*

Proof. By using Theorems 4.10 and 4.11.

Theorem 4.13. (*Sieve-algorithm for the determination of all primes of the form* $N(x) \equiv x^2 + (x+1)^2$ *in an Interval [5,M], where* M *is a (positive) integer*)

Step 1: Determine all numbers $N(x)$ *for* $x = 1, 2, \dots, \left[(-1+\sqrt{2M-1})/2\right]$.

Step 2: Determine all R_n *and* R_n'*, as in Theorem 4.12 obtained from the Diophantine equations*

$$X^2 - 2Y^2 = 2k^2 - 1, \quad \textit{where } k = 0, 1, \dots, \left[\sqrt{M}\right].$$

Step 3: Delete from the table of the numbers in Step 1, all numbers of Step 2. The remaining numbers are the only prime numbers of the form $N(x)$ *in the interval [5,M].*

Proof. By using Theorem 4.12.

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