ON 14-DIMENSIONAL QUADRATIC FORMS, THEIR SPINORS, AND THE DIFFERENCE OF TWO OCTONION ALGEBRAS

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This text originates from an E-mail to in May 1994. It has been revised on September 18, 1996 and March 9, 1999 (but still may contain some inaccuracies).

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Let me give a short report on some thoughts on low dimensional quadratic forms. "low" means here up to dimension 14.

1. Introduction.

One may start with the following: Consider a quadratic form q of dimension d and with trivial e^1- and e^2-invariants. Then q comes from a torsor $\tilde{q} \in H^1\big(\mathrm{Spin}(d)\big)$. Consider a fundamental spinor representation S of $\mathrm{Spin}(d)$ and let $S(\tilde{q})$ be the corresponding "twisted" representation.

It turns out that if d is smaller or equal to 14, then the dimension of S is smaller than the dimension of the group $\mathrm{Spin}(d)$. These dimensions are "exceptional" in this sense. The significance is as follows: If you take any nontrivial vector $s \in S(\tilde{q})$, it will have a nontrivial isotropy group, and one may hope that this gives an interesting reduction of the torsor $\tilde{q} \in H^1\big(\mathrm{Spin}(d)\big)$ to H^1 of a subgroup of $\mathrm{Spin}(d)$.

This is indeed the case: The isotropy group of a generic spinor $s \in S$ has the 1-component

- $G_2 \times G_2$ (for $d = 14$),
- $\mathrm{SL}(6)$ (for $d = 12$),
- semi-direct product of an additive group with $\mathrm{Spin}(7)$ (for $d = 10$),
- $\mathrm{Spin}(7)$ (for $d = 8$, 9),
- G_2 (for $d = 7$).

Moreover if $d = 10$, then the (1-component of the) isotropy group of a generic vector in $S \oplus S$, (i.e., the intersection of the isotropy group of two generic s) is G_2.

Let me note that for $d = 9$ the embedding of the isotropy group $\mathrm{Spin}(7) \to \mathrm{Spin}(9)$ is not the usual one: it is in $\mathrm{Spin}(8)$ twisted by a triality, so that the center of $\mathrm{Spin}(7)$ does not map to the center of $\mathrm{Spin}(9)$. Similarly for $\mathrm{Spin}(7) \to \mathrm{Spin}(10)$ in the case $d = 10$.

The computations (together with computations of the normalizers, e.g., in the 14-dimensional case the normalizer is $(G_2 \times G_2) \rtimes \mu_8$) yield the following:

- for a 14-dimensional form q in I^3 there is a quadratic extension $L = F(\sqrt{a})$ and a 3-fold Pfister form p over L, such that q is the trace of $\sqrt{a}p'$, where p' is the pure subform of p. (This was not known to me before and I don't know any other proof than the one described below.)
- a 12-dimensional form in I^3 is the trace of a 6-dimensional hermitian form with trivial determinant. (A known fact.)

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- a 10-dimensional form in I^3 is isotropic. (A well-known fact.)
- etc.

I will discuss here only the cases $d = 10$, 14. The case $d = 12$ is of different nature and will not be considered.

2. A construction of "the difference of two composition algebras"

The goal of the following considerations is to give a certain description of the (64-dimensional) spinor representation of Spin(14). The advantage of this description is, that it makes it easy to determine the dimension of the main orbit and to compute its isotropy group.

I assume $\operatorname{char}(F) \neq 2$, or even $\operatorname{char}(F) = 0$.

Let C be a composition algebra over a field F with norm form N and unit e and let

$$C = eF \oplus V$$

be the orthogonal decomposition.

Let D be another composition algebra over F with norm form M and unit f and let

$$D = fF \oplus W$$

be the orthogonal decomposition.

Put

$$R = C \otimes D$$

and

$$L = \textstyle\bigwedge^2(V \oplus W) = \bigwedge^2 V \oplus \bigwedge^2 W \oplus V \otimes W.$$

I define a map

$$\Theta\colon L \longrightarrow \operatorname{End}(R)$$

by the following formulas:

$$\begin{aligned}
\Theta(v \wedge \bar{v})(c \otimes d) &= [v \cdot (\bar{v} \cdot c) - \bar{v} \cdot (v \cdot c)] \otimes d,\\
\Theta(w \wedge \bar{w})(c \otimes d) &= c \otimes [w \cdot (\bar{w} \cdot d) - \bar{w} \cdot (w \cdot d)],\\
\Theta(v \wedge w)(c \otimes d) &= (v \cdot c) \otimes (w \cdot d).
\end{aligned}$$

Here $x \cdot y$ is the product in the composition algebras.

Lemma 1. *The map Θ is injective and its image is a Lie subalgebra.*

Proof. This follows from standard rules for composition algebras. □

Let

$$p\colon V \longrightarrow F, \quad p(v) = N(v)$$

be the restriction of N. I identify $\bigwedge^2 V$ with the Lie algebra so(p). (In the natural way as subspaces of the even Clifford algebra of p.)

Let

$$J\colon \textstyle\bigwedge^2 V \longrightarrow V, \quad J(v \wedge w) = v \cdot w - w \cdot v.$$

- Suppose $\dim C = 4$. Then C is a quaternion algebra and J is bijective.
- Suppose $\dim C = 8$. Then C is a octonion algebra. In this case J is surjective and its kernel is the Lie algebra of G_2.

Let

$$q\colon W \longrightarrow F, \quad q(v) = M(v)$$

be the restriction of M. I identify $\bigwedge^2 W$ with the Lie algebra $\mathrm{so}(q)$.

Let

$$U = V \oplus W$$

and consider the quadratic form

$$[p-q]\colon U \longrightarrow F,$$
$$[p-q](v, w) = p(v) - q(w).$$

Note that $L = \bigwedge^2 U$. I identify L with the Lie algebra of $\mathrm{so}([p-q])$. In this way the inclusion

$$\bigwedge^2 V \oplus \bigwedge^2 W \longrightarrow L$$

is identified with the inclusion

$$\mathrm{so}(p) \oplus \mathrm{so}(q) \longrightarrow \mathrm{so}([p-q]).$$

Lemma 2. *With these identifications (with the Lie algebra structures eventually altered by a scalar factor), the map* Θ *is a homomorphism of Lie algebras.*

Proof. By carefully comparing both Lie brackets. □

3. The main orbit

We have now established R as $\mathrm{Spin}([p-q])$-module. The tangent space of the $\mathrm{Spin}([p-q])$-orbit of some vector r of R is $\Theta(L)(r)$.

We now assume that $\dim C$, $\dim D$ are equal to 4 or to 8. This means that the maps J are in both cases surjective.

Let us consider the specific element $r = e \otimes f$. One finds for the tangent space of the orbit through r:

$$\Theta(L)(e \otimes f) = e \otimes W \oplus V \otimes f \oplus V \otimes W.$$

This is a subspace of codimension 1, transversal to r. So, if we extend the group by scalar multiplication, we see that the orbit of r is open. Therefore r generates a generic line and the orbit of r has codimension 1.

Let us compute the Lie algebra of the isotropy group of r. This is the sum of the Lie algebras of the isotropy groups of e and f. These are the kernels of the maps J on $\mathrm{so}(p) = \bigwedge^2 V$, and on $\mathrm{so}(q) = \bigwedge^2 W$. These are trivial if C, D have dimension 4 and are the Lie algebras of G_2 if C, D have dimension 8.

Since the main orbit has codimension 1 it follows that there is an invariant form

$$\psi\colon R \longrightarrow F$$

on R such that the varieties

$$\{\, \psi = \text{nonzero constant} \,\}$$

are the main orbits.

4. On the case $\dim C = \dim D = 8$

Consider the above construction for split C and D of dimension 8. Then we are given a representation

$$\text{split-}\operatorname{Spin}(14) \to \operatorname{End}(R).$$

Now given a 14-dimensional quadratic form $h\colon X \longrightarrow F$ with trivial e^1- and e^2-invariants, we may twist this representation to a representation

$$\operatorname{Spin}(h) \longrightarrow \operatorname{End}(R).$$

leaving invariant a certain form $\rho\colon R \longrightarrow F$, with ρ the twist of ψ.

If you take a vector r in R such that $\rho(r)$ is nontrivial, then there comes along a decomposition $X = Y \oplus Z$, such that h restricted to Y and h restricted to Z are similar to the pure subform of 8-dimensional composition algebras C and D, respectively. Moreover $R = C \otimes D$.

Well, this is not quite true: The decomposition $X = Y \oplus Z$ might be defined only after passing to a quadratic extension E/F, whose Galois group interchanges the two factors. In fact, there are elements in the normalizer of $G_2 \times G_2$ which interchange the two factors.

To make this more precise we compute the normalizer of $G_2 \times G_2$:

Lemma 3. *The normalizer of $G_2 \times G_2$ in $\operatorname{Spin}(7,7)$ is $(G_2 \times G_2) \rtimes \mu_8$. Here a generator $\omega \in \mu_8$ acts on $G_2 \times G_2$ via $(g,h) \mapsto (h,g)$.*

Proof. Every automorphism of G_2 is inner and $G_2 \to \operatorname{GL}(7)$ is an irreducible representation. Therefore the normalizer of $G_2 \times G_2$ in $\operatorname{GL}(14)$ is

$$\big((G_2 \times \mathbf{G}_\mathrm{m}) \times (G_2 \times \mathbf{G}_\mathrm{m})\big) \rtimes \mathbf{Z}/2.$$

Intersecting this group with $\operatorname{SO}(7,7)$ yields the normalizer of $G_2 \times G_2$ in $\operatorname{SO}(7,7)$:

$$(G_2 \times G_2) \rtimes \mu_4$$

where μ_4 is considered as a subgroup of $\operatorname{SO}(7,7)$ via

$$\zeta \mapsto \tau(\zeta) := \zeta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The extension $\mu_2 \to \operatorname{Spin}(7,7) \to \operatorname{SO}(7,7)$ is nontrivial on the subgroup $\tau(\mu_4) \subset \operatorname{SO}(7,7)$, whence the claim.

To be more specific, we describe μ_8 as a subgroup of $\operatorname{Spin}(7,7)$ explicitly. Let $v_1, \dots, v_7, w_1, \dots, w_7$ be an orthogonal basis with $\langle v_i, v_i \rangle = 1$ and $\langle w_i, w_i \rangle = -1$. Consider the element

$$\omega = \prod_{i=1}^{7} \left(\frac{1 + \zeta v_i w_i}{\sqrt{2}} \right) \in \operatorname{Spin}(7,7) \subset C_0(7,7)$$

where ζ is a primitive 4-th root of unity. Then ω is of order 8 and its image in $\operatorname{SO}(7,7)$ is $\tau(\zeta)$. □

One concludes that the map

$$H^1\big(F, (G_2 \times G_2) \rtimes \mu_8\big) \to H^1\big(F, \operatorname{Spin}(7,7)\big)$$

is surjective.

This gives the mentioned result on 14-dimensional quadratic forms.

5. On the cases $\dim D = 4$, $\dim C = 4,\ 8$.

5.1. **The case** $\dim C = 8$ **and** $\dim D = 4$**.** Here the isotropy group is G_2 with normalizer $G_2 \times \mathrm{Spin}(3) \cdot \mu_4$.

The associativity of D shows that the group action is compatible with the right D-module structure of $R = C \otimes D$. Moreover $R = S \oplus S$, where S is a fundamental spinor representation of $\mathrm{Spin}([p-q])$.

Suppose that D is split, $D = \mathrm{End}(K)$ where K is a 2-dimensional vector space. Then S is the tensor product over D of R with K so that $S = C \otimes K$. A computation similar as above shows that the $\mathrm{Spin}([p-q])$-group action on S has a dense orbit.

5.2. **The case** $\dim C = \dim D = 4$**.** Here the isotropy group is trivial. If $\dim D = \dim C = 4$, then R is an algebra and the group action is given by multiplication from the left with $\mathrm{Spin}([p-q]) = \mathrm{SL}(R)$. In this case the form ψ is the reduced norm on R.

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