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Generalizations involving Maltitudes

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This article presents a generalization of the concurrency of the maltitudes of a cyclic quadrilateral, as well as a generalization of the Euler line to cyclic n-gons. The role of computer exploration and proof in this discovery is also briefly discussed.

The maltitudes of a cyclic quadrilateral

Sometime ago the author came across a particular geometric diagram in a Polish high school mathematics journal, which apparently had to do with proving that the intersections of the semi-circles on adjacent sides of a cyclic quadrilateral with each other, lay on the diagonals of the cyclic quadrilateral. This was not difficult to prove (and in fact is true for *any* quadrilateral), but further analysis of the diagram led to the rediscovery of the following interesting, but not so well-known theorem, and other related results.

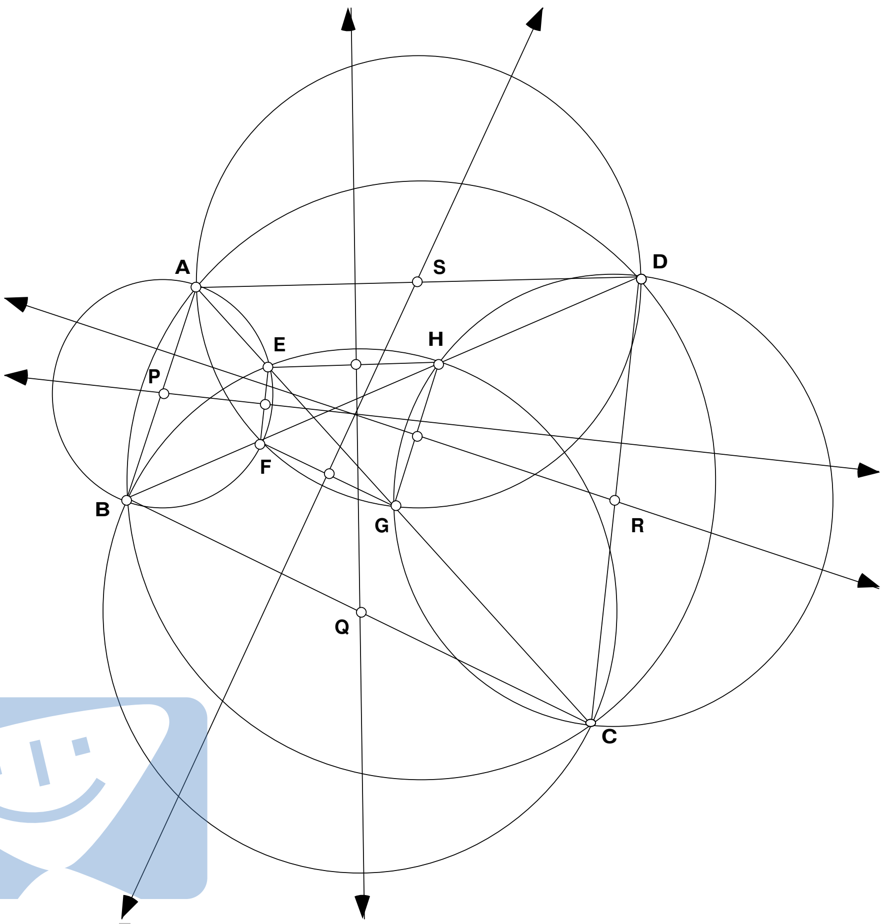

Figure 1

[1] As from January 2004, the University of Durban-Westville and the University of Natal have merged into a new institution, the University of KwaZulu-Natal.

Theorem 1

The perpendiculars from the midpoints of the sides of a cyclic quadrilateral to their corresponding opposite sides are concurrent.

Although this result is listed in [1], no proof is given. A proof is therefore provided below.

Proof

Consider Figure 1 where *ABCD* is the given cyclic quadrilateral with *P*, *Q*, *R* and *S* the midpoints of the sides as shown. Construct circles with centers at these midpoints and the sides as diameters as shown. Label the intersection of circles *P* and *Q* as *E*, the intersection of circles *P* and *S* as *F*, the intersection of circles *R* and *S* as *G* and the intersection of circles *Q* and *R* as *H*.

Connect *E* with *A* and *C*. Since $\angle AEB = 90°$ in semi-circle *P* and $\angle BEC = 90°$ in semi-circle *Q*, it follows that $\angle AEC = 180°$. Thus *AEC* is a straight line and *E* lies on the diagonal *AC*. Similarly, it follows that *G* lies on *AC*, and *F* and *H* lie on the diagonal *BC*.

Since *ABCD* is cyclic, we have $\angle ABD = \angle ACD$ on chord *AD*. But $\angle FEG = \angle ABD$ (exterior angle of cyclic quadrilateral *ABFE*) and $\angle FHG = \angle ACD$ (exterior angle of cyclic quadrilateral *DCGH*). Therefore $\angle FEG = \angle FHG$ which implies that *EFGH* is cyclic.

From the preceding paragraph, we also have $\angle FEG =$ alternate $\angle ACD$ which implies *EF* // *DC*. Similarly, *GH* // *BA*. Since *BCHE* is cyclic, $\angle EHB = \angle ECB$ on chord *EB*. But since *EFGH* is cyclic $\angle EHB = \angle EGF$ on chord *EF*. Therefore, $\angle ECB =$ corresponding $\angle EGF$ which implies *FG* // *BC*. In a similar fashion can be shown that *EH* // *AD*.

The perpendicular bisector of chord *EF* passes through midpoint *P* (center of circle) according to an elementary theorem, and is perpendicular to *CD* (since *EF* // *CD*). It therefore follows that the perpendicular from the midpoint *P* to the opposite side *CD*, coincides with the perpendicular bisector of *EF* (the perpendicular from *P* to *CD* is unique). Similar conclusions follow for the perpendicular bisectors of the other three sides of *EFGH*. But since *EFGH* is cyclic, its perpendicular bisectors are concurrent, and therefore also the perpendiculars from the midpoints of the sides of *ABCD* to its corresponding opposite sides.

Since the perpendicular from the midpoint of a side of a cyclic quadrilateral to its opposite side is analogous to the concept of an altitude for a triangle, it is called a "*maltitude*". If we further consider the special case where the cyclic quadrilateral degenerates into a triangle by letting two vertices coincide, it is interesting to note that the perpendiculars from the midpoints of two adjacent sides onto each other, are concurrent with the perpendicular from their common vertex to the third side.

A generalization of the maltitudes

Theorem 1, involving the concurrency of the maltitudes of a cyclic quadrilateral, can be generalized as follows.

Theorem 2

For any cyclic quadrilateral *ABCD*, draw circle *AEFB* with center *P* and label its intersection with *AC* and *BD* respectively as *E* and *F*. Next draw circle *BEC* with center *Q* and label its intersection with *BD* as *H*, circle *CHD* with center *R* and label its intersection with *AC* as *G*, and circle *DGA* with center *S*.

Then *EFGH* is cyclic, *F* lies on the circle *DGA* with center *S* and the perpendiculars from *P*, *Q*, *R* and *S* respectively to the opposite sides of *ABCD* are concurrent.

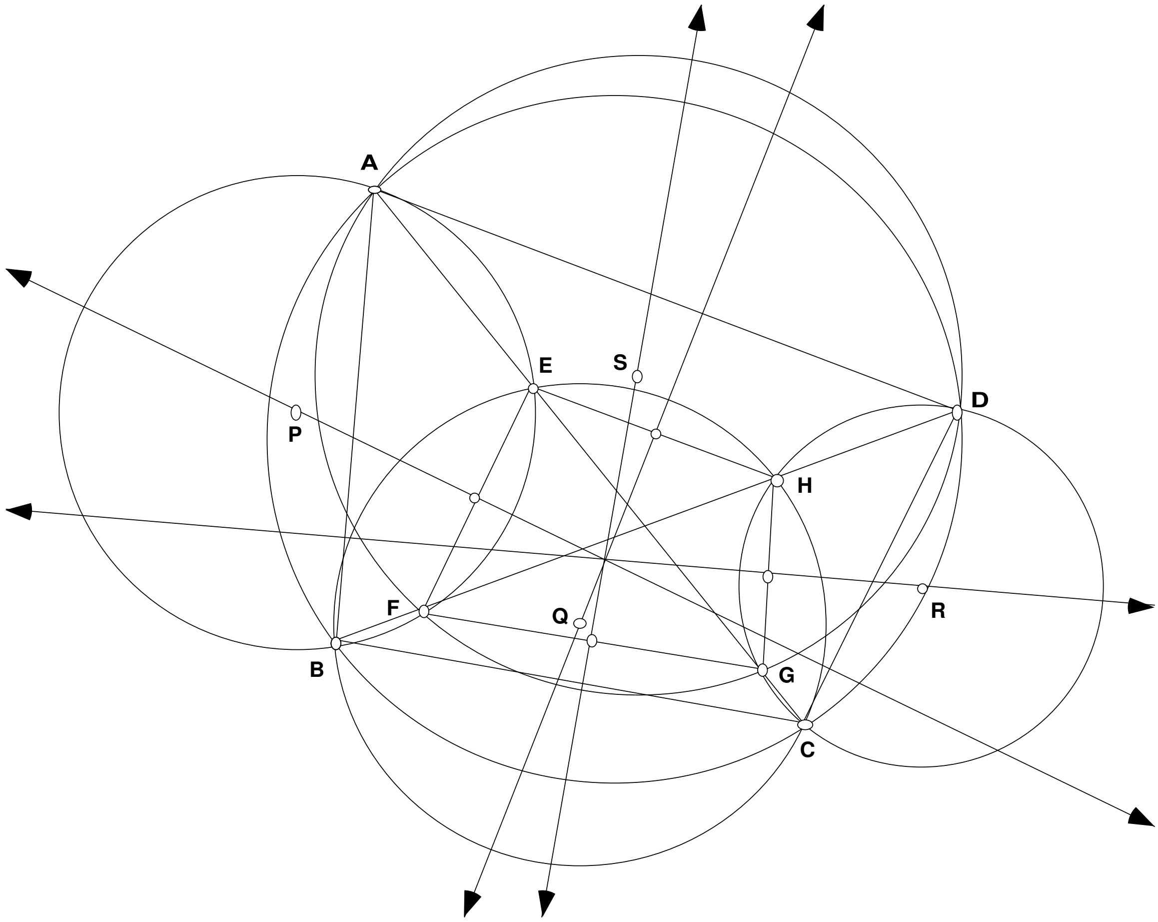

Figure 2

Proof

Consider Figure 2 where *ABCD* is cyclic and the circles *P*, *Q*, *R* and *S* have been constructed as described above. To prove that *F* lies on circle *DGA* with center *S*, it is sufficient to prove that *AFGD* is cyclic.

It is left to the reader to check that exactly as before in Proof 1, it follows that *EFGH* is cyclic, and that $EF // DC$ and $GH // BA$. Also $\angle ECB =$ corresponding $\angle EGF$ (which implies that $FG // BC$). But $\angle ADF = \angle ECB$ on chord *AB*; therefore $\angle ADF = \angle EGF = \angle AGF$ on segment *AF*, which implies that *AFGD* is cyclic. In the same way as before, it now follows that $EH // AD$ and that the perpendiculars from *P*, *Q*, *R* and *S* to the corresponding opposite sides of *ABCD* are concurrent.

Generalizing the Euler line to cyclic quadrilaterals

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Given that the orthocenter, circumcenter and centroid of a triangle are collinear on the Euler line, it seemed reasonable to conjecture that for a cyclic quadrilateral the point of concurrency of the maltitudes (its "*orthocenter*"), its circumcenter and its centroid would also be collinear. Subsequent investigation with the dynamic geometry programme *Sketchpad* showed that this was indeed the case. This "*a priori*" conviction then provided the motivation to start looking for a proof, which is presented further on. Contrary to the way in which proof is normally presented during teaching as a prerequisite for conviction, this episode demonstrated that in mathematical research, conviction is often a prerequisite for proof.

Doug Hofstadter in [2] has similarly emphasized as follows that conviction can be reached by other means than proof:

"*By the way, note that I just referred to my screen-based observation as a "fact" and a "theorem". Now any redblooded mathematician would scream bloody murder at me for referring to a "fact" or "theorem" that I had not proved. But that is not my attitude at all, and never has been. To me, this result was so clearly true that I didn't have the slightest doubt about it. I didn't need proof. If this sounds arrogant, let me explain. The beauty of Geometer's Sketchpad is that it allows you to discover instantly whether a conjecture is right or wrong - if it's wrong, it will be immediately obvious when you play around with a construction dynamically on the screen. If it's right, things will stay "in synch" right on the button no matter how you play with the figure. The degree of certainty and confidence that this gives is downright amazing. It's not a proof, of course, but in some sense, I would argue, this kind of direct contact with the phenomenon is even more convincing than a proof, because you really see it all happening right before your eyes. None of this means that I did not want a proof. In the end, proofs are critical ingredients of mathematical knowledge, and I like them as much as anyone else does. I am just not one who believes that certainty can come* ***only*** *from proofs.*"

Why does one still feel a need to prove a result like that above if one is already convinced of its truth from investigation on computer? It seems that it is precisely because one is convinced of its truth that one feels challenged to find a deductive proof, not because one doubts the result. Why? Well, here was a result that was clearly true from experimental exploration on computer, but the intriguing question of *why* it was true remained unanswered. If an exploration like that above had not shown these points to be collinear, one would certainly not have wasted one's time trying to find a proof (eg. one would have had counter-examples). In such cases, it would seem that the search for, and eventual construction, of a deductive proof (explanation) should be viewed as an intellectual challenge, definitely not as an epistemological exercise in trying to establish its "*truth*".

As the concept of the centroid of a quadrilateral is perhaps not that well-known, it will now first be discussed in relation to the following theorem.

Theorem 3

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Given any quadrilateral *ABCD*, then the respective centroids *P*, *Q*, *R* and *S* of triangles *ABD*, *ABC*, *BCD* and *CDA* form a quadrilateral *RSPQ*, similar to the original, with lines *AR*, *BS*, *CP* and *DQ* concurrent. (This point of concurrency (center of similarity) is defined as the centroid of the quadrilateral).

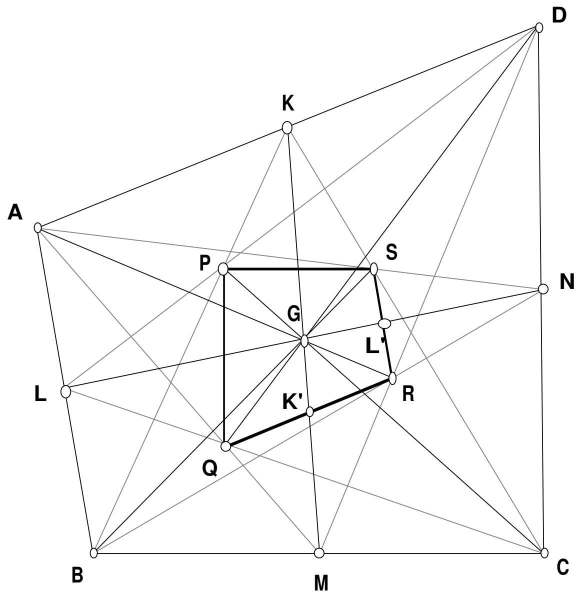

Figure 3

Proof

Consider Figure 3 with *P*, *Q*, *R* and *S* the given centroids and *K*, *L*, *M* and *N* the midpoints of the sides of *ABCD* as shown. Then $KP = {}^1/_3\ KB$ and $KS = {}^1/_3\ KC$. Therefore *SP* // *BC* and $SP = {}^1/_3\ BC$. Similarly, it follows that *PQ* //= $^1/_3$ *CD*, *QR* //= $^1/_3$ *DA* and *RS* //= $^1/_3$ *AB*. Since corresponding sides are parallel, it follows that angles *R*, *S*, *P* and *Q* are respectively equal to angles *A*, *B*, *C* and *D*. Therefore, *RSPQ* is similar to *ABCD* (corresponding angles equal, and corresponding sides in same ratio). Since two similar polygons are called homothetic if the corresponding sides are parallel, we can further say that *RSPQ* is homothetic to *ABCD*. Then from a theorem that the lines joining corresponding vertices of two homothetic polygons are concurrent (eg. see [3] or [4]), it follows that lines *AR*, *BS*, *CP* and *DQ* are concurrent (at the centroid *G*).

Corollary

Another interesting result related to Figure 3 that we will use in Theorem 4 below, is that the lines *LN* and *KM* (the diagonals of the Varignon parallelogram *KLMN*) are concurrent with the centroid *G*. This can be proved as follows.

In Δ*MAD*, *QR* // *DA* and since *K* bisects *DA*, *KM* bisects *QR* in *K'*. Similarly, *LN* bisects *RS* in *L'*. Now since the same similarity which maps *ABCD* to *RSPQ* respectively map *L* to *L'* and *K* to *K'*, the lines *LL'* and *KK'* are concurrent at the same center *G*. Therefore *G* coincides with the intersection of *LN* and *KM*, the Varignon center of parallelogram *LKMN*.

Theorem 4

The orthocenter (*H*), circumcenter (*O*) and centroid (*G*) of a cyclic quadrilateral are collinear, and the centroid bisects the segment *OH*.

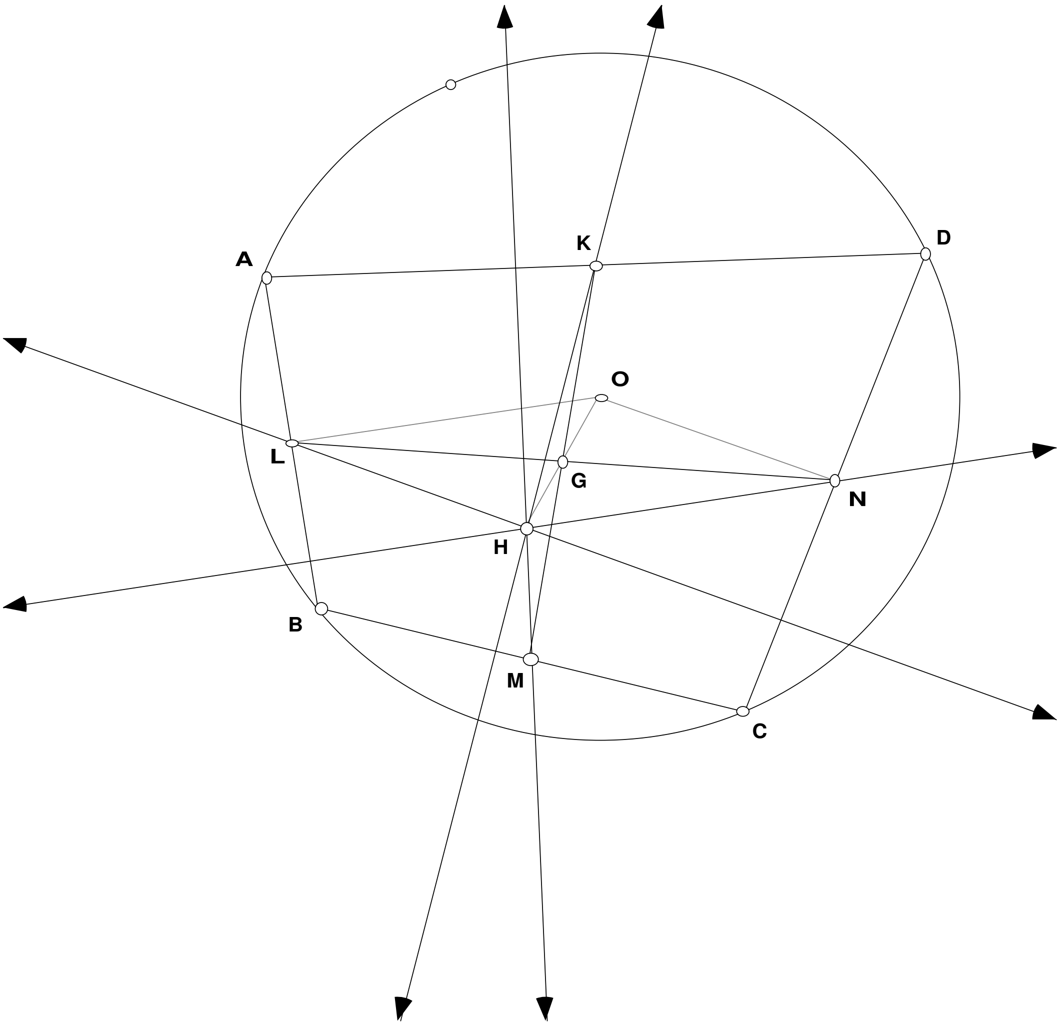

Figure 4

Proof

Consider the cyclic quadrilateral *ABCD* given in Figure 4, with *O* as the circumcenter, *G* as the centroid and *H* as the orthocenter. *K*, *L*, *M* and *N* are the respective midpoints of the sides.

Consider quadrilateral *LHNO*. Since both *LH* and *ON* are both perpendicular to *CD* by construction, it follows that *LH* // *ON*. Similarly, *OL* // *NH* which implies that *LHNO* is a parallelogram. Since *G* coincides with the midpoint of *LN* as we saw in the previous result, it follows that *G* must lie on diagonal *OH* of parallelogram *LHNO*; i.e. *O*, *G* and *H* are collinear. (Also note that *GH* = *OG*, whereas for a triangle *GH* = 2 *OG*).

It should be noted that the prior exploration on *Sketchpad* showed that *OG* = *GH*, which suggested that *LHNO* was a parallelogram and easily led to the eventual proof given above. This example therefore shows that experimentation can also be useful in providing valuable clues for the production of deductive proofs.

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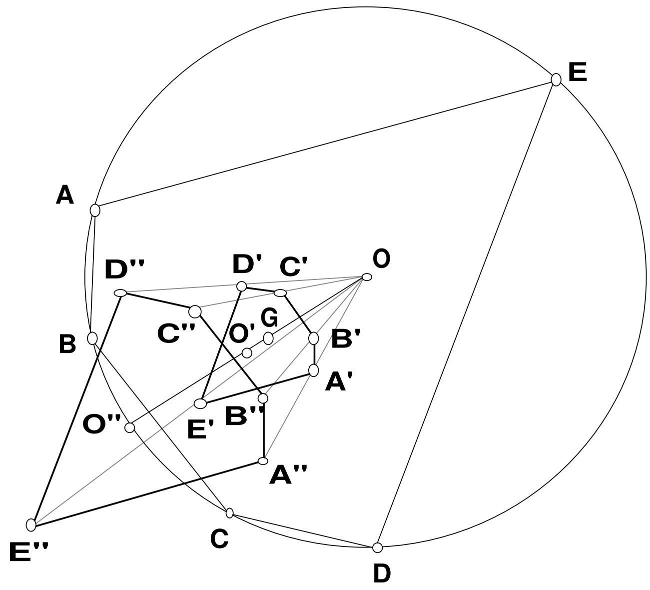
Generalizing the Euler line to cyclic pentagons

Since any cyclic polygon has a circumcenter and centroid, it seemed natural to further suspect that Theorems 1 and 4 could be generalized to cyclic polygons. Subsequent investigation of a cyclic pentagon with *Sketchpad* provided experimental confirmation, and the motivation to look for a proof.

Theorem 5

The orthocenter (H), circumcenter (O) and centroid of a cyclic pentagon are collinear, and the centroid divides the segment OH in the ratio 2:3.



Proof

Construct the respective centroids A', B', C', D' and E' for each of the five quadrilaterals (*BCDE*, *CDEA*, *DEAB*, *EABC*, and *ABCD*) into which the pentagon *ABCDE* can be divided (see Figure 5). In the same way as in Theorem 4, it now follows that $A'B'C'D'E'$ is homothetic to *ABCDE* with a scale factor of $\frac{1}{4}$, with center of similarity at G. As before, G is defined as the centroid of the whole pentagon.

Construct the respective orthocenters A'', B'', C'', D'' and E'' for the same five quadrilaterals into which the pentagon can be divided. From Theorem 4, we have for each of these cyclic quadrilaterals that its respective centroid bisects the segment connecting its orthocenter with the common circumcenter O. Therefore, $A''B''C''D''E''$ is homothetic to $A'B'C'D'E'$ with a scale factor of 2, with center of similarity at O. The circumcenter O'' of $A''B''C''D''E''$ is now defined as the orthocenter of the whole pentagon.

Under the first similarity with G as center, O maps to O' (the circumcenter of $A'B'C'D'E'$); therefore OGO' is a straight line. However, under the second similarity with O as center, O' maps to O''; therefore $OO'O''$ is a straight line, and are O, G and O'' collinear.

From the first similarity, we have $GO = 4GO'$; therefore $OO' = \frac{5}{4}GO$. From the second similarity, we have $OO'' = 2\ OO' = 2(\frac{5}{4}GO) = \frac{5}{2}GO$. Therefore, $GO'' = \frac{3}{2}GO$ which is what was required to prove.

In general, it is possible in the same way to prove that the centroid of any cyclic n-gon is collinear with its circumcenter and orthocenter, and divides the segment joining the circumcenter to the orthocenter in the ratio 2 : (n - 2). (Please see correction below) .

Although Theorem 2 is probably not original, it does not seem to appear in the mainstream mathematical literature. Theorems 4 and 5 appear as problems in [5], but the proof of Theorem 4 in this article is distinctly different. To myself as author, however, they were original discoveries, which demonstrated the dynamic interplay between conjecture, experimentation and proof. In themselves, the results are also interesting enough to be better known.

Note: Zipped *Sketchpad* sketches illustrating some of the results discussed in this article can be downloaded from http://mzone.mweb.co.za/residents/profmd/maltitudes.zip

References

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[2] Hofstadter, D., 1997, In King, J. & Schattschneider, D. *Geometry turned on: Dynamic software in learning, teaching, and research.* (Washington, DC: MAA), p. 10.

[3] Coxeter, H.S.M., 1961, *Introduction to Geometry*, (New York: Wiley).

[4] De Villiers, M., 1996, *Pythagoras*, **41**, 33-34.

[5] Yaglom, I.M., 1968, *Geometric Transformations II*, (Washington, DC: MAA), pp.24; 108-109.

Correction: with the advantage of hindsight, what I've defined here as the 'orthocenters'of a cyclic quadrilateral and cyclic pentagon respectively, more appropriately correspond to the ninepoint/Euler center E of a triangle, with the orthocenters H then simply defined as being collinear with O, G and E so that $HE = EO$.