MATHEMATICAL STATISTICS QUALIFYING EXAM
Fall 2004

NAME: ______________________________ **ID:** ______________________________

Instructions: Work four of the following six problems. You may not use notes or any other assistance.

The four problems you have attempted: ____________________________________

1. Let Ω be a set. For $A \subset \Omega$ and $B \subset \Omega$, write $A^C = \{\omega \in \Omega : \omega \notin A\}$. Let R be a σ-ring of subsets of Ω, i.e., if $A_1 \in R$ and $A_2 \in R$, then $A_1 \cap {A_2}^C \in R$ and if $A_1, A_2, \ldots \in R$, then $\bigcup_{i=1}^{\infty} A_i \in R$. For this problem, let R be a nonempty σ-ring that is not a σ-algebra and let $R^1 = \{B : B^C \in R\}$.

   (a) Prove that if $A \in R$ and $B \in R^1$, then $A \cup B \in R^1$.

   (b) Prove that if $A_1 \in R$ and $A_2 \in R$, then $A_1 \cap A_2 \in R$.

   (c) Prove that if $A_1, A_2, \ldots \in R$, then $\bigcap_{i=1}^{\infty} A_i \in R$.

2. A bivariate population of (X, Y) is sampled independently on three occasions. On the first, a random sample of size n_0 is taken and only $T = \min\{X, Y\}$ is observed for each pair. On the second, a random sample of size n_1 is taken, and only the X-marginal is observed for each pair. Finally, a random sample of size n_2 is taken, and only the Y-marginal is observed for each pair. Therefore, the combined set of observations is of the form $(\boldsymbol{T}, \boldsymbol{X}, \boldsymbol{Y})$, where $\boldsymbol{T} = (T_1, \ldots, T_{n_0})$, $\boldsymbol{X} = (X_{11}, \ldots, X_{1n_1})$ and $\boldsymbol{Y} = (Y_{21}, \ldots, Y_{2n_2})$. Assume the following two-parameter probability model for (X, Y):

$$P(X > x, Y > y) = \exp\left[-\frac{1}{\theta}(x^{1/\delta} + y^{1/\delta})^{\delta}\right],$$

   $x > 0,\ y > 0,\ \theta > 0,\ 0 < \delta \le 1$ with unknown parameters θ and δ.

   (a) Find the joint pdf of $(\boldsymbol{T}, \boldsymbol{X}, \boldsymbol{Y})$.

   (b) Identify the distributions of T_1, X_{11} and Y_{21}.

3. Let $Y_1, Y_2, \ldots, Y_n$ be independent observations with each having the gamma density

$$f(y) = y^{\alpha-1}e^{-y}/\Gamma(\alpha), \quad 0 < y < \infty,\ 0 < \alpha < \infty.$$

   (a) Show that for unbiased estimation of α, the Cramer-Rao lower bound is not attained.

   (b) A sequence of estimators $\{T_n\}$ is asymptotically efficient if

$$\lim_{n\to\infty} \frac{\text{Var}(T_n)}{\text{Cramer-Rao Lower Bound}} = 1,$$

   and the limit is the asymptotic efficiency. Show that $\overline{Y}$ is an unbiased estimator of α, but it is not asymptotically efficient. Find its asymptotic efficiency.

4. Let $X_1, \ldots, X_n$ be iid Bernoulli random variables with $P(X_i = 1) = 1 - P(X_i = 0) = p$, $0 < p < 1$. A function $g(p)$ is called estimable if and only if there is a statistic T such that for any p in the interval (0, 1), $\mathrm{E}_p(T) = g(p)$.

   (a) Show that $g(p)$ is estimable if and only if g is a polynomial in p of degree less than or equal to n. You may assume this in part (b).

   (b) Find the UMVUE of an arbitrary estimable function $g(p)$.

5. Let $X_1, \ldots, X_n$ be a random sample from a $N(\theta, \sigma^2)$ population, and suppose that the prior distribution on θ is $N(\mu, \tau^2)$.

   (a) Find the marginal distribution of $\overline{X}$.

   (b) Find $\mathrm{E}(\theta|\overline{x})$ and $\mathrm{Var}(\theta|\overline{x})$.

   (c) Find the Bayes estimator of θ under the squared error loss.

6. Let $X_1, \ldots, X_{n_1}$ be a random sample from the $N(\mu_1, \sigma^2)$ and let $Y_1, \ldots, Y_{n_2}$ be a random sample from the $N(\mu_2, \sigma^2)$ distribution which is independent of the first random sample. Consider the likelihood ratio level-α test for testing $H_0 : \mu_1 = \mu_2 = \mu$ versus $H_1 : \mu_1 \neq \mu_2$. Show that the likelihood ratio test is

$$\phi(x) = \begin{cases} 1 & \text{if } F > c, \\ 0 & \text{if } F < c, \end{cases}$$

   $P_{H_0}(F > c) = \alpha$, where

$$F = \frac{n_1(\overline{x} - \hat{\mu})^2 + n_2(\overline{y} - \hat{\mu})^2}{[\sum_{i=1}^{n_1}(x_i - \overline{x})^2 + \sum_{i=1}^{n_2}(y_i - \overline{y})^2]/(n_1 + n_2 - 2)}$$

   and $\hat{\mu}$ is the mle of μ.