Unit 1

Know the meanings and uses of these terms:

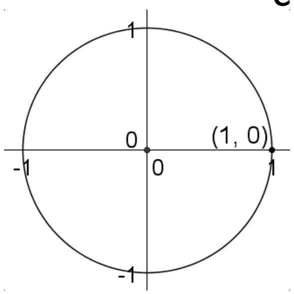
Unit circle (1-1)
Initial point of the unit circle (1-1)
Terminal point of the unit circle (1-1)
Coterminal values (1-1)
Reference number (1-1)
Identity statement (1-2)
Period *(the value)* (1-3)
Period *(the interval)* (1-3)
Amplitude (1-3)

Review the meanings and uses of these terms:

Domain of a function (1-2)
Range of a function (1-2)
Translation of a graph (1-3)
Reflection of a graph (1-3)
Dilation of a graph (1-3)
Asymptote (1-4)

Topic 1-1 The Unit Circle

Definition: The unit circle is a circle of radius 1 centered at the origin.



Thus, the unit circle is defined by the equation $x^2 + y^2 = 1$.

Example: Show that $\left(-\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ is a point on the unit circle.

Example: If P is a point on the unit circle in quadrant IV & $x = \frac{2}{5}$, find the coordinates of P.

Displacement and Terminal Points

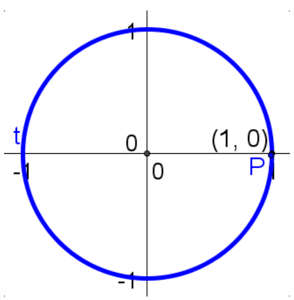
The initial point of the unit circle is (1,0).

A counterclockwise movement along the unit circle is defined to be positive. A clockwise movement along the unit circle is defined to be negative.

The displacement covered by moving around the unit circle, starting at the initial point, is defined by the variable t.

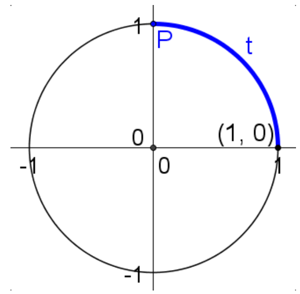
The point where t concludes is called the terminal point $P(x,y)$ of t.

Since the radius of the unit circle is 1, the circumference of the unit circle is 2π.



t = ______ P(___ , ___)

Basic t-values:



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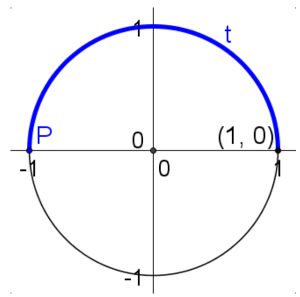

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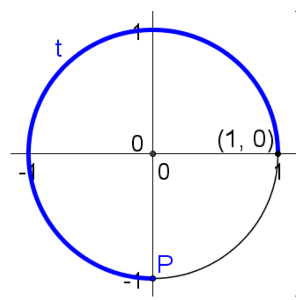

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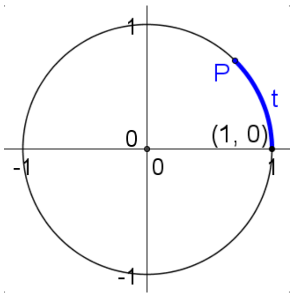

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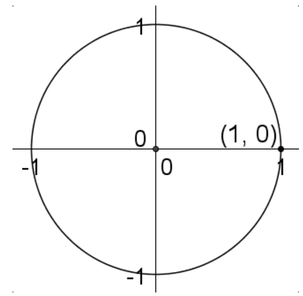
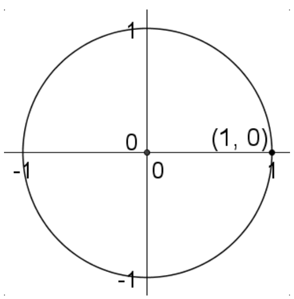
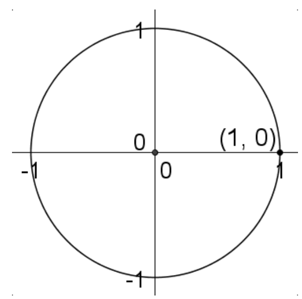
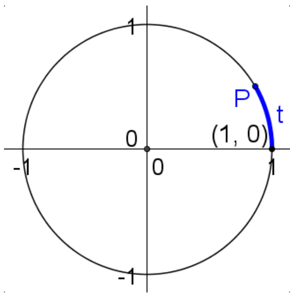
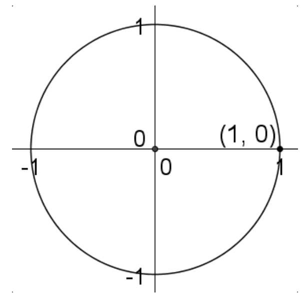


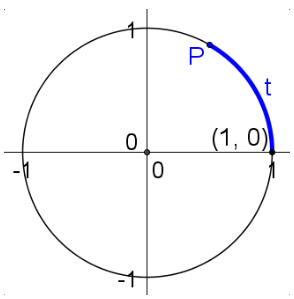






t = ______ P(____ , ____)





t = ______ P(____ , ____)

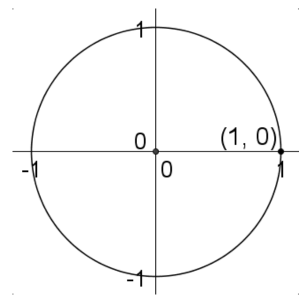
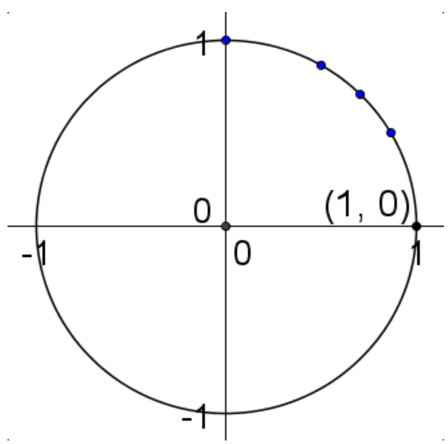


Table of Significant *t*-values

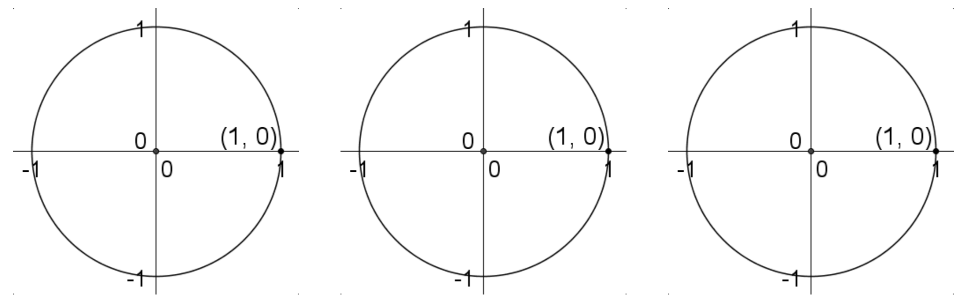
t	Terminal Point determined by t
0	$(1,0)$
$\frac{\pi}{6}$	$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$
$\frac{\pi}{4}$	$\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$
$\frac{\pi}{3}$	$\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$	$(0,1)$



Coterminal Values of *t*

Definition: Two values of t are said to be coterminal if they have the same terminal point P.

Consider the following:



t = ______ t = ______ t = ______

P(___ , ___)

If t_2 is coterminal to t_1, then $t_2 = t_1 + 2k\pi$, where k is an integer. Let t_c represent the smallest positive coterminal value for a given t; then t_c is always in the interval $[0,2\pi)$.

For each given value of t, find the coterminal value t_c.

Ex. 1: $t = \frac{19\pi}{6}$

Ex. 2: $t = -\frac{35\pi}{3}$

For each given value of t, find the coterminal value t_c.

Ex. 3: $t = \frac{29\pi}{5}$

A function can be well-defined with t as an independent variable and P as a dependent variable. The converse however cannot create a function relationship.

Reference Numbers and Terminal Points

Definition: Let t be a real number. The reference number $\bar{t}$ associated with t is the shortest distance along the unit circle between the terminal point determined by t & the x-axis.

If t is not a multiple of $\frac{\pi}{2}$, $\bar{t}$ can be found by using the smallest possible coterminal value t_c and the following table:

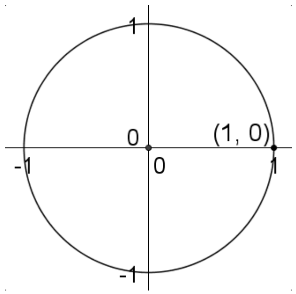
P is in quadrant	value of t_c is	formula to find $\bar{t}$
I	$0 < t_c < \frac{\pi}{2}$	$\bar{t} = t_c$
II	$\frac{\pi}{2} < t_c < \pi$	$\bar{t} = \pi - t_c$
III	$\pi < t_c < \frac{3\pi}{2}$	$\bar{t} = t_c - \pi$
IV	$\frac{3\pi}{2} < t_c < 2\pi$	$\bar{t} = 2\pi - t_c$

If t is a multiple of π, then $\bar{t} = 0$.

If t is an odd multiple of $\frac{\pi}{2}$, then $\bar{t} = \frac{\pi}{2}$.

For each value of *t*, find the reference number.

QI:



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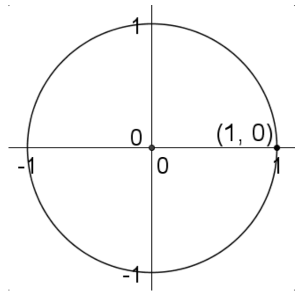

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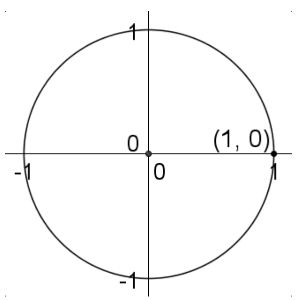
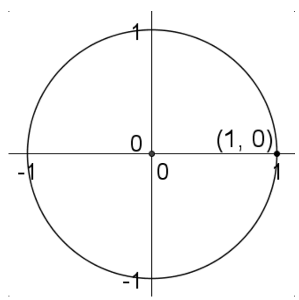


QIV:



Ex.1 $t = \frac{7\pi}{6}$

Ex.2 $t = \frac{11\pi}{3}$

For each value of t, find the reference number.

Ex. 3 $t = -\frac{17\pi}{4}$

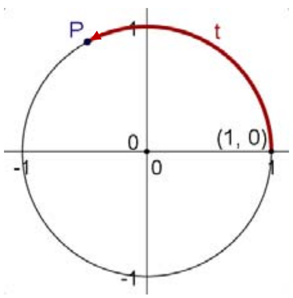
Ex. 4 $t = \frac{18\pi}{5}$

For each value of t, find the reference number and the terminal point determined by t.

Ex. 1 $t = \frac{15\pi}{4}$

Ex. 2 $t = -\frac{19\pi}{6}$

Topic 1-2 Trigonometric Functions



Definitions: Let *t* be any real number and let *P*(*x*, *y*) be the terminal point on the unit circle determined by *t*. Then:

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, x \neq 0$$

$$\csc t = \frac{1}{y}, y \neq 0 \quad \sec t = \frac{1}{x}, x \neq 0 \quad \cot t = \frac{x}{y}, y \neq 0$$

sin is the abbreviation of sine

csc is the abbreviation of cosecant

cos is the abbreviation of cosine

sec is the abbreviation of secant

tan is the abbreviation of tangent

cot is the abbreviation of cotangent

If *P* is known for a given *t*, then the six trigonometric functions are defined from *P*.

The terminal point *P*(*x*, *y*) determined by *t* is given below. Find sin *t*, cos *t*, and tan *t*.

Ex. 1: $P\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$

The terminal point $P(x, y)$ determined by t is given below. Find sin t, cos t, and tan t.

Ex. 2: $P\left(-\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$

Recall that for the t values 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$, we know the terminal point P.

Identify the terminal point for the t-value given and then find the values of the trigonometric functions.

Ex. 1: $t = \frac{\pi}{2}$

$\sin\frac{\pi}{2}$ $\cos\frac{\pi}{2}$

$\tan\frac{\pi}{2}$ $\cot\frac{\pi}{2}$

Identify the terminal point for the *t*-value given and then find the values of the trigonometric functions.

Ex. 2: $t = \frac{\pi}{3}$

$\sin\frac{\pi}{3}$

$\csc\frac{\pi}{3}$

$\cos\frac{\pi}{3}$

$\tan\frac{\pi}{3}$

Identify the terminal point for the *t*-value given and then find the values of the trigonometric functions.

Ex. 3: $t = \frac{\pi}{4}$

$\sin\frac{\pi}{4}$

$\tan\frac{\pi}{4}$

$\cos\frac{\pi}{4}$

$\sec\frac{\pi}{4}$

Quick Reference Chart

t	$\sin t$	$\cos t$	$\tan t$	$\cot t$	$\sec t$	$\csc t$
0	0	1	0	–	1	–
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	–	0	–	1

- Spaces marked by a – indicated a value for which the trigonometric value is undefined

Domains of Trigonometric Functions

$f(x) = \sin x$ and $f(x) = \cos x$

Domain: $\mathbb{R}$

$f(x) = \tan x$ and $f(x) = \sec x$

Domain: $\left\{ x \mid x \in \mathbb{R} \text{ and } x \neq n\pi + \frac{\pi}{2},\ n \text{ is an integer} \right\}$

$f(x) = \cot x$ and $f(x) = \csc x$

Domain: $\left\{ x \mid x \in \mathbb{R} \text{ and } x \neq n\pi,\ n \text{ is an integer} \right\}$

Signs of Trigonometric Functions

Since the trigonometric functions are defined off of the values of *x* and *y* of the terminal point, the sign value of a trigonometric function can be determined based on the quadrant in which the terminal point exists

P is in quadrant	Positive Functions	Negative Functions
I	all	none
II	SIN, csc	cos, sec, tan, cot
III	TAN, cot	sin, csc, cos, sec
IV	COS, sec	sin, csc, tan, cot

If a t value has a reference number of 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, or $\frac{\pi}{2}$, then it is possible to determine the trigonometric values of t using the trigonometric values of $\bar{t}$ and the quadrant in which P exists.

Find the exact value of the trigonometric functions at the given real number.

Ex. 1: $\cos\frac{8\pi}{3}$

Find the exact value of the trigonometric functions at the given real number.

Ex. 2: $\sin\frac{7\pi}{6}$

$\tan\frac{7\pi}{6}$

Find the exact value of the trigonometric functions at the given real number.

Ex. 3: $\cos\frac{23\pi}{4}$

$\cot\frac{23\pi}{4}$

Fundamental Identities

Recall the definition for an identity statement:

An identity statement is true for any value of the variable within its domain.

Reciprocal Identities:

$$\csc t = \frac{1}{\sin t} \qquad \sec t = \frac{1}{\cos t} \qquad \cot t = \frac{1}{\tan t}$$

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$

Pythagorean Identities:

$$\sin^2 t + \cos^2 t = 1$$

$$\tan^2 t + 1 = \sec^2 t$$

$$1 + \cot^2 t = \csc^2 t$$

Note: $\sin^2 t = (\sin t)^2 = (\sin t)(\sin t)$

$\sin^n t = (\sin t)^n$ for all n except $n = -1$

Even & Odd Properties of Trigonometric Functions

Recall from algebra that an even function f is a function such that $f(-x) = f(x)$ and an odd function g is a function such that $g(-x) = -g(x)$

Sine, cosecant, tangent, and cotangent are *odd* functions:

$\sin(-t) = -\sin t$ $\qquad$ $\tan(-t) = -\tan t$

$\csc(-t) = -\csc t$ $\qquad$ $\cot(-t) = -\cot t$

Cosine and secant are *even* functions:

$\cos(-t) = \cos t$ $\qquad$ $\sec(-t) = \sec t$

Various Questions

Find the sign of the expression if the terminal point determined by t is in the given quadrant.

Ex: $\tan t \sec t$, quadrant IV

From the information given, find the quadrant in which the terminal point determined by t lies.

Ex: $\tan t > 0$ and $\sin t < 0$

Determine whether the function is even, odd, or neither.

Ex. 1: $f(x) = x^3 \cos(2x)$

Determine whether the function is even, odd, or neither.

Ex. 2: $f(x) = x \sin^3 x$

Write the first expression in terms of the second if the terminal point determined by *t* is in the given quadrant.

Ex. 1: cos *t*, sin *t*; quadrant IV

Write the first expression in terms of the second if the terminal point determined by *t* is in the given quadrant.

Ex. 2: sin *t*, sec *t*; quadrant III

Find the values of the trigonometric functions of t from the given information.

Ex. 1: $\cos t = -\frac{4}{5}$, terminal point of t is in III

Find the values of the trigonometric functions of t from the given information.

Ex. 2: $\tan t = -\frac{2}{3}$, $\cos t > 0$

Topic 1-3 Graphing Trigonometric Functions, Pt. 1

Trigonometric functions are periodic.

Definition: A function f is periodic if there exists a positive number p such that $f(t + p) = f(t)$ for every t.

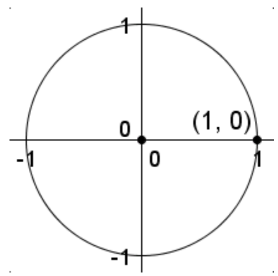
If f has period p, then the graph of f on any interval of length p is called one complete period of f.

Since sine and cosine are defined by the terminal point of t and the addition of $2n\pi$ (n is an integer) to t is coterminal to t, then periodic behavior of of sine and cosine must occur over an interval of 2π.

$\sin(t + 2\pi) = \sin t$ $\qquad$ $\cos(t + 2\pi) = \cos t$

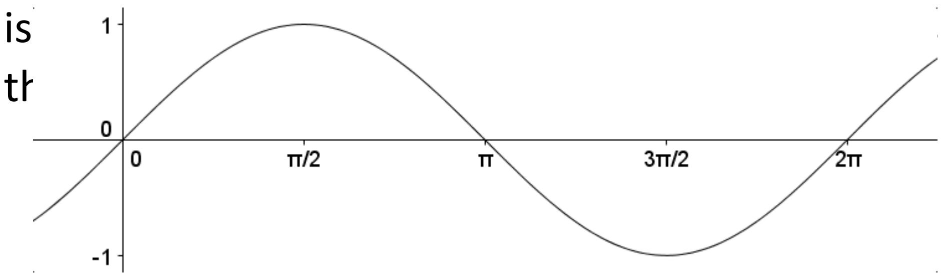
Derivation of graph of sin t

Recall that $\sin t = y$, where y is the y-value of the terminal point determined by t.



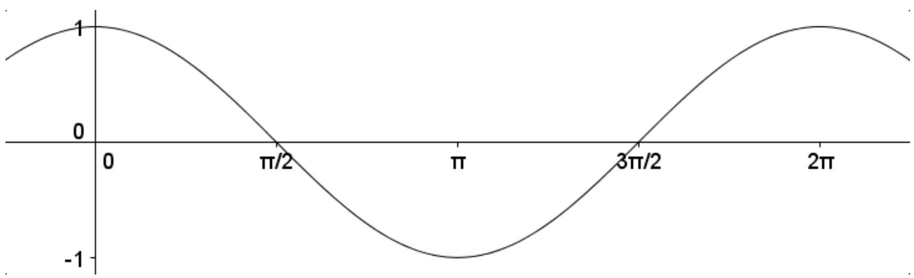
Recall the domain of sine is $\mathbb{R}$.

Observe that the maximum possible value of sine is th



Presentation of graph of cos *t*

Recall that cos *t* = *x*, where *x* is the *x*-value of the terminal point determined by *t*.



Like sine, cosine has a domain of $\mathbb{R}$.

Also, like sine, cosine has a range of [–1, 1].

In fact, cosine appears to be a shifted representation of sine.

Sine waves have amplitude; amplitude is the one half the difference of the maximum and minimum values of a periodic function.

Observe that the most basic complete period of sine or cosine is the interval $[0,2\pi]$.

Recall the topic of transformations from algebra.

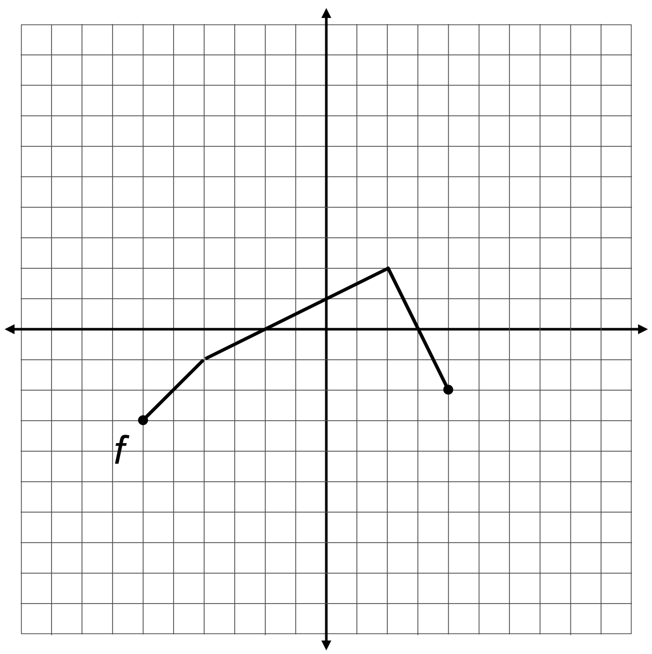
The addition of a real number to a function *translates* (shifts) the function vertically.

The addition of a real number to a variable inside a function *translates* the function horizontally.

Given f, sketch:

a: $y = f(x) - 2$

b: $y = f(x - 2)$



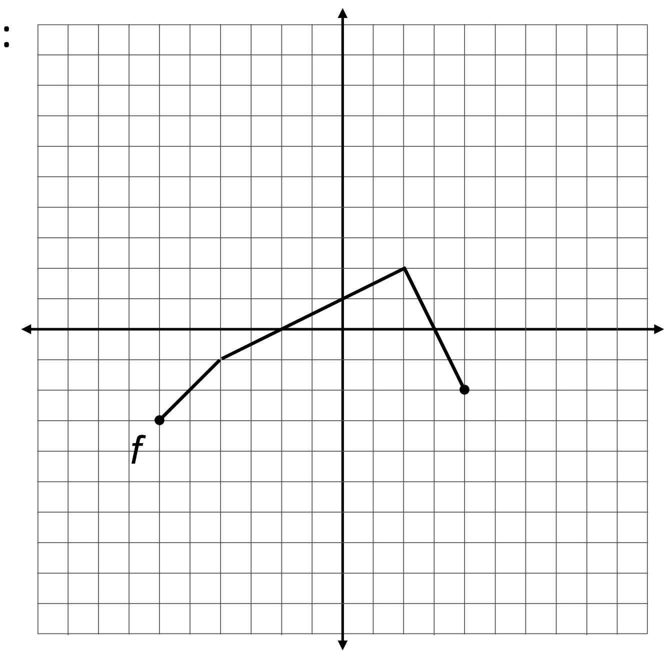
The negation of a function *reflects* (flips) the graph about the x-axis.

The negation of a variable inside a function *reflects* the graph about the y-axis.

Given f, sketch:

a: $y = -f(x)$

b: $y = f(-x)$



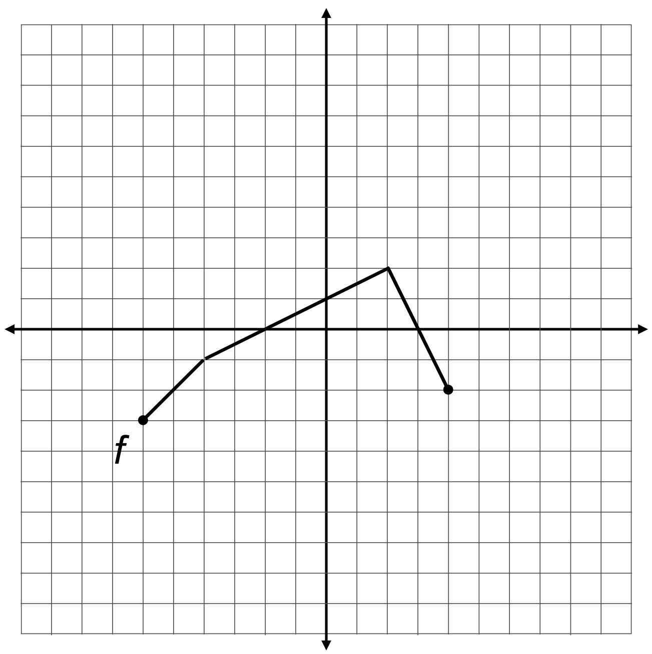
Multiplying a real number to a function *dilates* (stretches/compresses) the function with respect to the x-axis.

Multiplying a real number to the variable inside a function *dilates* the function w.r.t. the y-axis.

Given f, sketch:

a: $y = 2f(x)$

b: $y = f(2x)$



Transformations of Trigonometric Functions

$y = a \sin k(x - b) + c$ $y = a \cos k(x - b) + c$

a: If $|a| > 1$, sin/cos is stretched away from the x-axis
If $|a| < 1$, sin/cos is compressed toward the x-axis
If a is negative, sin/cos is reflected about the x-axis

k: If $|k| < 1$, sin/cos is stretched away from the y-axis
If $|\mathrm{k}| > 1$, sin/cos is compressed toward the x-axis

b: If b is positive, sin/cos is shifted to the right $(x - \#)$
If b is negative, sin/cos is shifted to the left $(x + \#)$

c: If c is positive, sin/cos is shifted upward
If c is negative, sin/cos is shifted downward

Effects on properties of a sine/cosine graph:

Dilations with respect to the *y*-axis create changes in the **period** of a trigonometric function.

Dilations with respect to the *x*-axis create changes in the **amplitude** of a trigonometric function.

Translations horizontally create a **phase shift** compared to the basic trigonometric function.

Translations vertically create a **vertical shift** compared to the basic trigonometric function.

Negations effect the location of peaks and valleys in a trigonometric function.

period = $\dfrac{2\pi}{k}$ amplitude = $|a|$ phase shift = b

Expectations for Trigonometric Graphs, pt 1:

For sine and cosine functions, these are my expectations:

1. Identify the period, amplitude, & phase shift of the sine or cosine graph.
2. Determine the domain of the primary complete period.

   For sine and cosine functions, the primary complete period will be over $\left[b,\ \frac{2\pi}{k}+b\right]$.
3. Determine the range of the graph.

   For sine and cosine functions, the range will be $\left[-|a|+c,\ |a|+c\right]$.
4. Mark and label the endpoints of the domain on the *x*-axis.
5. Mark and label the midpoint of the domain and the midpoints between an endpoint and a midpoint (which I refer to as "quarterpoints").
6. Mark and label the endpoints of the range and the midpoint of the range on the *y*-axis.
7. Evaluate the function at the five values marked on the *x*-axis. If everything has been done correctly, the value of the function at these *x*-values should correspond to one of the *y*-values marked on the *y*-axis.

Sketch a graph of the trigonometric function and identify its properties.

Ex. 1: $y = 3\sin 2x$

Period to be Graphed: [____ , ____]

Range: [____ , ____]

Period = ____

Amplitude = ____

Phase Shift = ____

Sketch a graph of the trigonometric function and identify its properties.

Period to be Graphed: [____ , ____]

Ex. 2: $y = 2\cos\frac{x}{3}$

Range: [____ , ____]

Period = ____

Amplitude = ____

Phase Shift = ____

Sketch a graph of the trigonometric function and identify its properties.

Ex. 3: $y = 2\sin x - 1$

Period to be Graphed: [____ , ____]

Range: [____ , ____]

Period = ____

Amplitude = ____

Phase Shift = ____

Sketch a graph of the trigonometric function and identify its properties.

Ex. 4: $y = \frac{1}{2}\cos\left(x - \frac{\pi}{3}\right)$

Period to be Graphed: [____ , ____]

Range: [____ , ____]

Period = ____

Amplitude = ____

Phase Shift = ____

Sketch a graph of the trigonometric function and identify its properties.

Ex. 5: $y = \sin\left(3x + \frac{\pi}{2}\right) + 2$

Period to be Graphed: $\left[\ \_\_\_\_ , \_\_\_\_\ \right]$

Range: $\left[\ \_\_\_\_ , \_\_\_\_\ \right]$

Period = ____

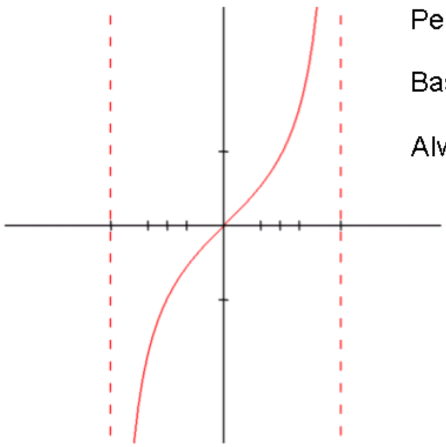
Amplitude = ____

Phase Shift = ____

Topic 1-4 Graphing Trigonometric Functions, Pt. 2

Graph of Tangent:

x	$\tan x$
$-\frac{\pi}{2}$	undefined
$-\frac{\pi}{3}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-\frac{\sqrt{3}}{3}$
0	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\frac{\pi}{2}$	undefined



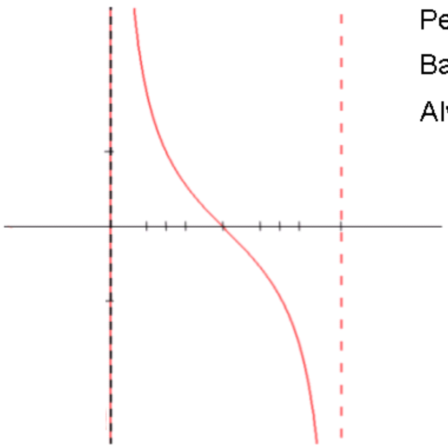
Period Length $= \pi$

Basic Period: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Always Increasing

Graph of Cotangent:

x	$\cot x$
0	undefined
$\frac{\pi}{6}$	$\sqrt{3}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	0
$\frac{2\pi}{3}$	$-\frac{\sqrt{3}}{3}$
$\frac{3\pi}{4}$	-1
$\frac{5\pi}{6}$	$-\sqrt{3}$
π	undefined



Period Length $= \pi$

Basic Period: $(0, \pi)$

Always Decreasing

General Form: $y = a \tan k(x - b) + c$

Period = $\frac{\pi}{k}$

Domain of Primary Period: $\left(-\frac{\pi}{2k} + b, \frac{\pi}{2k} + b\right)$

Period to be Graphed: $\left[-\frac{\pi}{2k} + b, \frac{\pi}{2k} + b\right]$

Range: $(-\infty, \infty)$

General Form: $y = a \cot k(x - b) + c$

Period = $\frac{\pi}{k}$

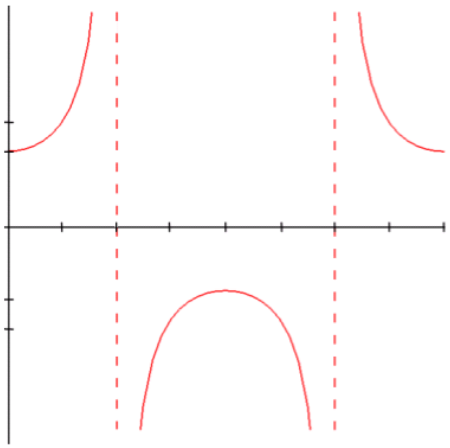
Domain of Primary Period: $\left(b, \frac{\pi}{k} + b\right)$

Period to be Graphed: $\left[b, \frac{\pi}{k} + b\right]$

Range: $(-\infty, \infty)$

Graph of Secant:

x	$\sec x$
0	1
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{2}$	undefined
$\frac{3\pi}{4}$	$-\sqrt{2}$
π	-1
$\frac{5\pi}{4}$	$-\sqrt{2}$
$\frac{3\pi}{2}$	undefined
$\frac{7\pi}{4}$	$\sqrt{2}$
2π	1



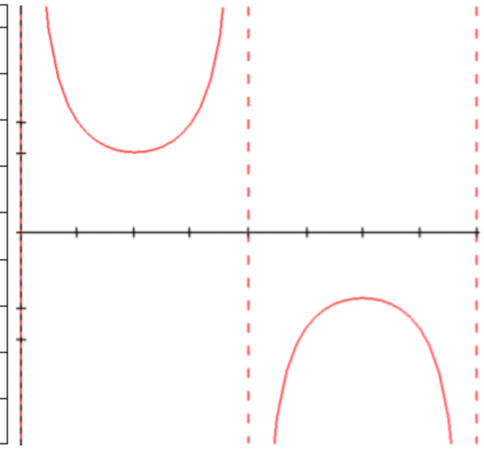
Period Length $= 2\pi$

Basic Period:

$\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right]$

Graph of Cosecant:

x	$\csc x$
0	undefined
$\frac{\pi}{4}$	$\sqrt{2}$
$\frac{\pi}{2}$	1
$\frac{3\pi}{4}$	$\sqrt{2}$
π	undefined
$\frac{5\pi}{4}$	$-\sqrt{2}$
$\frac{3\pi}{2}$	-1
$\frac{7\pi}{4}$	$-\sqrt{2}$
2π	undefined



Period Length $= 2\pi$

Basic Period:

$(0,\pi)\cup(\pi,2\pi)$

General Form: $y = a \sec k(x - b) + c$

Period $= \dfrac{2\pi}{k}$

Domain of Primary Period: $\left[b,\ \frac{2\pi}{k}+b\right]-\left\{\frac{\pi}{2k}+b,\ \frac{3\pi}{2k}+b\right\}$

Period to be Graphed: $\left[b,\ \frac{2\pi}{k}+b\right]$

Range: $(-\infty,-|a|]\cup[|a|,\infty)$

General Form: $y = a \csc k(x - b) + c$

Period $= \dfrac{2\pi}{k}$

Domain of Primary Period: $\left(b,\ \frac{2\pi}{k}+b\right)-\left\{\frac{\pi}{k}+b\right\}$

Period to be Graphed: $\left[b,\ \frac{2\pi}{k}+b\right]$

Range: $(-\infty,-|a|]\cup[|a|,\infty)$

Expectations for Trigonometric Graphs, pt 2:

For the remaining functions, these are my expectations:

1. Identify the period & phase shift of the trigonometric functions. Also note any vertical dilations or translations.
2. Mark and label the endpoints of the domain on the *x*-axis.
3. Mark and label the midpoint and the "quarterpoints".
4. Mark and label three/two points on the *y*-axis:
   $y = |a| + c, \quad y = -|a| + c, \quad y = c$ (third only for tan/cot)
5. Evaluate the function at the five values marked on the *x*-axis. The value of the function at each *x*-value should either be a value on the *y*-axis or undefined. Asymptotes will exist where the function is undefined.

Sketch a graph of the trigonometric function and identify its properties.

Ex. 1: $y = 3\tan 2x$

Period = ____

Period to be Graphed: [____ , ____]

Sketch a graph of the trigonometric function and identify its properties.

Ex. 2: $y = 4\cot\frac{x}{3}$

Period = ____

Period to be Graphed: $\left[\ \_\_\_\_ , \_\_\_\_\ \right]$

Sketch a graph of the trigonometric function and identify its properties.

Ex. 3: $y = 2\tan\left(x - \frac{\pi}{8}\right)$

Period = ____

Period to be Graphed: $\left[\ \_\_\_\_ , \_\_\_\_\ \right]$

Sketch a graph of the trigonometric function and identify its properties.

Ex. 4: $y = \cot\left(3x + \frac{\pi}{2}\right)$

Period = ____

Period to be Graphed: $\left[\ \_\_\_\_\ ,\ \_\_\_\_\ \right]$

Sketch a graph of the trigonometric function and identify its properties.

Ex. 5: $y = 4\csc 2x$

Period = ____

Period to be Graphed: $\left[\ \_\_\_\_\ ,\ \_\_\_\_\ \right]$

Sketch a graph of the trigonometric function and identify its properties.

Ex. 6: $y=\frac{1}{2}\sec 3x$

Period = ____

Period to be Graphed: $\left[\ \_\_\_\_\ ,\ \_\_\_\_\ \right]$

Sketch a graph of the trigonometric function and identify its properties.

Ex. 7: $y=-2\csc\frac{x}{5}$

Period = ____

Period to be Graphed: $\left[\ \_\_\_\_\ ,\ \_\_\_\_\ \right]$