State University of New York at Fredonia
Department of Mathematical Sciences

CERTIFICATION OF PROJECT WORK

We, the undersigned, certify that this project entitled Simplifying Structurally Comparable Expressions: A Study on Student Understanding of Algebra Versus Arithmetic by Michael P. Humbert, Candidate for the degree of Master of Science in Education, Mathematics Education (7-12), is acceptable in form and content and demonstrates a satisfactory knowledge of the field covered by this project.

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SIMPLIFYING STRUCTURALLY COMPARABLE EXPRESSIONS: A STUDY OF COLLEGE STUDENTS' UNDERSTANDING OF ALGEBRA VERSUS ARITHMETIC

by

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A Master's Project
Submitted in Partial Fulfillment
of the Requirements for the Degree of
Master of Science in Education
Mathematics Education (7-12)
Department of Mathematical Sciences
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Abstract

This study explores the connection between student understanding of arithmetic and algebra through the evaluation of numeric expressions and the simplification of structurally comparable algebraic expressions. It is hypothesized that non-major college mathematics students are more likely to correctly simplify an algebraic expression than to correctly evaluate a numeric expression of comparable structure. One hundred students from four non-major mathematics courses were given a six-problem assessment to test this hypothesis. The results suggest that students are more successful at evaluating numeric expressions than algebraic expressions. Possible correlations between the two subject areas are discussed in findings.

Table of Contents

Introduction

This research examines student understanding of the simplification of mathematical expressions. Specifically, the researcher seeks to uncover what students struggle with the most, simplifying algebraic expressions or simplifying numeric expressions. Simplifying numeric expressions is most often taught through the use of order of operations, commonly making use of the pneumonic device *PEMDAS: Parentheses, Exponents, Multiplication, Division, Addition, And Subtraction*. Despite the fact that order of operations is drilled exhaustively into the minds of middle and high school students, they often do not build meaning around that set of rules until they reach a higher level of mathematics (or sometimes not at all). Simplifying algebraic expressions, however, does not have the luxury of such simple rote techniques. The existence of a variable forces the learner to treat the different terms as objects; in order to manipulate those objects, one must understand how they work.

Interest in this topic was a bi-product of my teaching experience at a comprehensive state university in the Northeast. At this university, I taught two sections of *Survey of Precalculus*. Each section contained between 25 and 30 students and each class session lasted fifty minutes, three days a week. Both groups showed similar trends, in that students seemed very capable of simplifying single-variable algebraic expressions but were unable to properly evaluate similar expressions for a given value. It should be noted that the majority of students were not mathematics majors, although most of them had completed their high school mathematics track within the previous year or two. The informal observation of these trends led me to believe the students lacked the number sense to correctly evaluate numeric expressions. Despite evidence that suggests they were relying on *PEMDAS* to guide their work, errors were still made in the simplification process that can be attributed to a lack of understanding of the order of operations. The same errors however, were not observed as often in the process of simplifying algebraic expressions.

Rote techniques such as *PEMDAS* are always being criticized for their lack of conveying meaning. Without that sense of meaning, students are more prone to making mistakes, which has important implications for mathematics education, specifically in the development of number sense in the middle and high school curriculum. If students are incapable of simplifying numeric expressions, a relatively fundamental skill, but show understanding of more abstract concepts such as algebra, it seems that there must be a flaw in the system. This is the underlying problem that defines the hypothesis statement for this research:

> *It is hypothesized that non-major college mathematics students are more likely to correctly simplify an algebraic expression than to correctly evaluate a numeric expression of comparable structure.*

This hypothesis was tested through a formal evaluation and two surveys that were administered to a separate group of students than those who contributed to the initial observation. The first survey was given before the formal evaluation and asked students to rate their competency in simplifying algebraic expressions as well as evaluating numeric expressions. Once this survey was collected, the evaluation was administered which tested the hypothesis directly. A follow up survey again asked students to rate their level of competency in each area, requesting they take into consideration their presumed performance on the evaluation. Prior to administering these instruments, a literature review was conducted to examine the existing research that relates to the hypothesis statement.

Literature Review

The following literature review is intended to investigate the pre-existing research on student understanding of arithmetic versus algebra. Although the two topics cover a wide array of grades and curricula, this research is meant to have the largest impact on the teaching of algebra and algebra-related courses at the high school and early college level. This review is divided into four sections: From Arithmetic to Algebra, Common Errors, Why Students Fail to Conceptualize Arithmetic, and Implications of Existing Research. The first section focuses on the academic transition from arithmetic to algebra, examining the connections between the two areas and the reason that algebra usually utilizes arithmetic as a steppingstone. The Common Errors section will explore the issues that many students face while trying to master both areas. In the third section, it is discussed how students are capable of succeeding throughout many years of mathematics without fully developing a strong conceptual understanding of arithmetic. Lastly, the Implications section suggests possible changes to the existing curriculum, or at least to the manner in which we teach algebra and arithmetic.

From Arithmetic to Algebra

Addition, subtraction, multiplication, and division: The four basic building blocks of a successful education in mathematics. Arithmetic is the basis of almost everyone's mathematics career. The very first mathematics classes we take involve learning these operations, their meanings, and basic numerical facts. They follow us throughout our entire education, yet many students never seem to fully understand them. We then send our students to start learning algebra without ever mastering the arithmetic. Based on my own observations it seems as though students often succeed in algebra, but lack the ability to perform simple arithmetic tasks. Some argue that arithmetic is one of the building blocks of algebra (Nathan & Keollner, 2007; Goodson-Espy, 1995). So do students make 'silly' mistakes, or do they lack understanding? On the other hand, there are those who suggest that the transition to algebra relies little on the understanding of arithmetic (Palow, 1999; Steel, 2008). If true, this could imply that students really do understand the concept of algebra without ever developing a conceptual understanding of arithmetic.

Nathan and Keollner (2007) suggest that understanding algebra is not a natural progression of maturation. Thinking at this level of abstraction is not something that one conceptually grows into, but rather, it is a carefully engineered progression that emerges from our prior mathematical conceptions. The researchers go onto reveal their distaste for learning through problem solving, and imply that students are more likely to learn better through direct instruction (but admit that all forms of instruction have their own benefits). The authors discuss their opinion that arithmetic is, among other things, necessary to understand algebra.

It is more than likely that there are many students who show little understanding in both arithmetic and algebra, but is there evidence that might suggest any correlation between the two? Goodson-Espy (1995) examines the case of three particular subjects who were unable to perform neither arithmetic nor algebraic tasks. She states that: "these solvers were unable to conceive of arithmetic processes as objects and that their transition to algebraic methods was blocked" (p.

34). That is, thinking procedurally with variables requires the understanding of procedures involving numerical values. She goes on to explain her opinion that thinking procedurally (which is often how algebra is taught) has more advantages than thinking in abstraction, especially when attempting to solve particularly daunting problems.

Regardless of which school of thought is more beneficial to student undertstanding, the problem remains that students do seem to be capable of moving to a higher level of abstract thinking without having a good base knowledge of arithmetic processes. Steel (2008), referencing Lee and Wheeler (1998), suggests that the concepts of each area can end up being very dissociated in students' minds. The researchers do not necessarily state that students do not need arithmetic to understand algebra, but they do imply that it is possible for students to make the jump to abstract thought without developing any connection from previous knowledge. Steel pointed out that there is an approximate age in a student's life where they "are in a transitory stage" and that "some students at this stage failed to show any clear link between their understanding of arithmetic and algebra" (p. 3). If there is no link formed at this stage then it seems possible that a link never forms between the two.

Some research challenges the idea that mastery of arithmetic is necessary at all for the learning of algebra (Palow, 1999). Sixty-two beginning algebra students at a community college were given an arithmetic pretest to determine their competency in the subject area. They were divided into three treatments, which for our purposes are irrelevant. The semester final was used as the post-test. Little to no correlation was observed between the subject areas. Although this article was intended to suggest a restructuring of the mathematics curriculum at that particular college, it does have more general implications. Namely, students can easily slip through the mathematics system without fully understanding the concepts that are encountered in arithmetic.

At the very least, most college level mathematics makes use of both arithmetic and algebra, and so the proper teaching of arithmetic cannot be left unlooked. As is, the instructions of these topics leaves a lot to be desired, which results in students often making many of the same types of errors and mistakes.

Common Errors

Due to the nature of algebraic notation and how mathematics is written, teachers often place a strong emphasis on form. Students like to see form; form has order, it makes sense. This pretense should make simplification of algebraic models easy to follow, because students need simply to abide by the basic set of rules for that form. Pappanastos, Hall, and Honan (2002) and Glidden (2008) examine specific examples of students abiding by mathematical conventions. Students seem to work well with very explicit form. But when the form is missing, they show a lack of understanding of the order of operations. When working strictly with arithmetic, that form begins to become more vague. For example, once you begin to evaluate a given algebraic expression for a specific value, groupings can then be simplified and in order to proceed, you are essentially forced to follow order of operations. As groupings tend to get simplified, common symbols such as parentheses will disappear, which often results in a lessened sense of specificity.

In a study testing about 300 college level business majors on their knowledge of order of operations, Pappanastos, Hall, and Honan (2002) reported the following: Of three particular questions, 14.8% of students incorrectly reported that 6/3*2 = 1, while 33.3% of the students incorrectly reported that 10+5/5 = 3, and 88.9% of the students incorrectly reported that $-5^2 = 25$. It should be noted that a fourth question, (6/3)*2, was answered with an incorrect response the least. This suggests that students struggle mainly with lack of specificity and they do not accept common rules as truth, perhaps stemming from a lack of understanding. For example, many students fail to understand the mathematical convention of working from left to right.

In a study consisting of 381 pre-service elementary school teachers, Glidden (2008) suggests that about one fifth of them had showed signs of a weak understanding of the order of operations. He goes on to state that of "subjects who do know to perform multiplication before division [in regards to a specific example], a significant proportion of them interpret the mnemonic device *PEMDAS*…literally, that is, they perform multiplication *before* division or addition *before* subtraction rather than from left to right" (p. 6). If this is the case, then it seems likely that students are also struggling with all of the rules that are attached to order of operations, furthering the fact that they lack a conceptual understanding of them, but given the right conditions *can* follow them.

Why Students Fail to Conceptualize Arithmetic

There are likely many factors that contribute to this failure. The largest factor relates specifically to the way that arithmetic is taught today. Methods and procedures for solving arithmetic problems are taught in a very roundabout way. Rather than searching for meaning and understanding in basic operations, students are given methods of rote learning, such as mnemonic devices, which hinder conceptual understanding (McNeil, Weinberg, Hattikudur, Stephens, Asquith, Knuth, & Alibali, 2010; Merlin, 2008; Ohlsson, Rees, & Pittsburgh Univ., 1988). The most widely used mnemonic device for arithmetic is *PEMDAS*, which represents the order of operations. These articles discuss how learning mathematics in such procedural ways can cause limitations in understanding. Students lose the ability to think abstractly when they are taught in such roundabout ways.

The research by Ohlsson et al., (1988) examines arithmetic procedures that are currently being taught. Their main concern is with the all too common memorization route: "School children tend to learn arithmetic procedures by memorizing them, rather than by constructing them on the basis of their understanding of numbers" (p. 9). Ohlsson et al., (1988) go on to show how such methods impact conceptual learning:

> We have observed in our field studies children who know how to put two fractions on the same denominator and who also know how to add two fractions with equal denominators, but who nevertheless are unable to figure out how to add two fractions with unequal denominators. (p. 4)

The researchers proceed to examine the Conceptual Understanding Hypothesis, which claims that, "procedures which are derived from knowledge are more flexible and less error-prone than procedures that are learned in other ways" (p. 5).

Merlin (2008) is also concerned with dependence on roundabout methods, specifically *PEMDAS*. He believes that focusing on parsing and analysis of algebraic expressions is necessary for the development of both conceptual understanding and structural procedure. Merlin (2008) states:

> I have diagnosed student difficulties as stemming in part from the insufficient attention on the part of the traditional curriculum to the activity of parsing and to the importance of structural concepts. We have also seen students prone to err by over generalizing…likely to compete with correct structural understanding at all times when the student is doing algebra. (p. 99)

In other words, students seem to look for overgeneralized methods that will help them solve algebraic problems rather than to search for proper understanding, from which a method for solving the problem would emerge naturally. He goes on to encourage the contextualization of algebraic problems and suggests that students are less than thrilled to learn through "decontextualized symbol manipulation" (p. 99) and are more likely to develop conceptual understanding with exposure to real applications of algebra.

Implications of Existing Research

There is much research that either supports or opposes the idea of arithmetic as a necessary precursor to algebra, and warrant more research into the topic. I suspect that the outcome of more research will show a low correlation between the necessity of understanding arithmetic and a high success rate in algebra, as alluded to in Chaiklin, Lesgold, and Pittsburg Univ., (1984), Hallagan (2004), and Subramaniam and Banerjee (2004). But if students are no longer bound to an arithmetic approach to algebra, what does that mean for the arithmetic and algebra curricula? Some professionals imply that algebra and arithmetic are best suited to be taught somewhat hand-in-hand (Pierce & Stacey, 2007; Yackel, Underwood, & Elias, 2007). These researchers set out to answer that question and suggest practical improvements for teachers and curriculum planners.

Students are fully capable of understanding arithmetic: "Students are not rigid and limited in their approaches to understanding the structure of arithmetic expressions" (Chaiklin & Lesgold, 1984, p. 61). The researchers imply that our method of static memorization is the problem. Memorization should never be the focus of any teaching method. They make the following suggestion for the arithmetic curriculum, specifically, more emphasis on "the canonical interpretation of expressions from left-to-right form when mixed operations are present" (p. 66). They argue that much of the confusion over the idea of canonical interpretation stems from the fact that this is not a mathematical rule, but an adapted convention. More generally, the researchers note "widespread support for the notion that the concept of basic skills must encompass more than computational facility" (p. 66). This argument has similar roots to the debate over *teaching to the test*, which often leaves students with little real understanding of what they had learned.

The research by Yackel, Underwood, and Alias (2007) examines specific tasks designed to test the numeric abilities of select students. The researchers were able to come to the conclusion that "mathematical knowledge is not a pregiven, external body of knowledge to be acquired, but rather is built up by cognizing individuals as they engage in mathematical activity, including discussions of their own and others' mathematical actions" (p. 13). This idea suggested by the researchers implies a huge change in any arithmetic curriculum. Expecting students to build their own knowledge is the exact opposite of memorization and rote learning, on which students seem to rely the most. Furthermore, they researchers suggest more emphasis be placed on "reasoning rather than speed and accuracy, and conceptual understanding rather than procedures" (p. 1).

What changes does this research warrant for the algebra curriculum? One might suggest removing an arithmetic approach to algebra altogether, as if arithmetic were not a prerequisite to algebra. This approach seems highly unlikely as, not only is an arithmetic approach incredibly useful, it will always be placed before algebra in the curriculum. The more likely approach would be to adapt algebra so that, in addition to its current standards, it also takes a step backwards and specifically takes time to help strengthen and reinforce students' knowledge of arithmetic. Pierce and Stacey (2007) encourage a curriculum that strengthens what they call algebraic insight: (1) Understanding symbols, key features, and structure of algebraic expressions, and (2) understanding links between multiple representations. These ideas involve "the ability to make links from symbolic to numeric and graphical representations" (p. 4). I particularly enjoy this idea of using the conceptual framework of algebra to make sense of the numbers that students often mindlessly manipulate.

Despite the fact that Subramaniam and Banerjee (2004) suggest that arithmetic might be better placed after algebra in mathematics curricula (which I disagree with) the more powerful fact is that algebra just might actually help to strengthen understanding of arithmetic: "It might well be the case that learning algebra paves the way for a better understanding of arithmetic expressions since the algebraic symbolism enhances the structure of the expressions" (p. 1). Although Subramaniam and Banerjee (2004) were unable to draw any major assumptions, their research did have a very interesting side result. In a study that looked at three different groups (Groups A & C were taught a lesson on arithmetic, Group B was taught a lesson on algebra), Group B showed significant improvement on an arithmetic posttest, after failing the pretest, as seen in Figure 1. Group B was able to transfer what they had learned about brackets in their algebra lesson to a strictly arithmetic problem. Groups A and C also show significant improvement as well on this type of problem,

	Group A		Group B		Group C	
	Pre	Post	Pre	Post	Pre	Post
Comparing expressions by calculation	97	94	83	85	83	93
Equalizing expressions by filling blanks	82	86	69	65	63	80
Evaluating expressions	51	74	58	60	44	70
Removing brackets and rewriting	22	85	28	71	37	91

Figure 1. Percentage of correct responses in arithmetic.

but the arithmetic lesson covered that type of problem exactly, so those results are expected.

It is clear from the preexisting research that students struggle with the cognitive transition from arithmetic to algebra. The researcher examined in this literature review struggles with identifying whether or not there is a necessary connection between the two subject areas. The following research is intended to provide more definitive results that clarify understanding of the this idea as it relates specifically to the hypothesis statement.

Experimental Design

As stated, the hypothesis was tested through a formal evaluation and two surveys that were administered to a group of college students. The first survey was given before the formal evaluation and asked students to rate their competency in simplifying algebraic expressions as well as evaluating numeric expressions. Once this survey was collected, the evaluation was administered which tested the hypothesis directly. A follow up survey again asked students to rate their level of competency in each area, requesting they take into consideration their presumed performance on the evaluation.

Participants

This study was conducted at a comprehensive state university in the Northeast. The encompassing town has a population of around 11,000 people while the university itself has a population of around 5,500 undergraduate students, about 400 graduate students, and just over 400 academic staff members.

The participants in this study consisted of exactly 100 students enrolled in the following 4 courses: Survey of Precalculus, University Precalculus, Prize-Winning Mathematics, and Information Systems Structures. The breakdown of the number of participants per course can be found in Figure 2. The researcher was the official instructor of both Precalculus courses, while two separate colleagues instructed Prize-Winning Mathematics, and Information Systems Structures. Of all 100 students, slightly more than half were females. The ages of the participants ranged from 17 to 40, with the majority of participants being between the ages of 18 and 23.

Course Name	Survey of Precalculus	University Precalculus	Prize-Winning Mathematics	Info. Systems Structures
Number of Participants	25	16	28	31

Figure 2. The number of participants per course.

The participants came from a variety of academic backgrounds and majors. It should be noted that the majority of students were not in a mathematics or mathematics related major. Similarly, the majority of students were in their first or second undergraduate year of study. The researcher and colleagues who contributed their classes to this experiment can attest to the fact that most students from these courses have displayed a distaste for mathematics and lack any sort of passion for the subject areas that the researcher focused on.

Design

This experiment was designed to test the hypothesis that non-major college mathematics students are more capable of simplifying algebraic expressions than numeric expressions of comparable form, suggesting that they may be more likely to have a stronger understanding of algebraic structure than they do of number sense and order of operations. The experiment consisted of a short six-problem assessment both preceded and followed by a brief survey. All items in the experiment were administered to the participants by a third party other than their course instructor. All data was collected within one fifty-minute period for each of the participating classes. The assessment consisted of six expressions, ordered as numeric/algebraic pairs. Subjects were given twenty minutes to complete the assessment. The pre-assessment survey consisted of four questions, asking the highest level of mathematics work previously achieved, how long it has been since last enrolled in a mathematics course, and then to rate their own algebra and number sense skills. The post-assessment survey simply asked them to reevaluate their original rating of each skill, based on their performance on the assessment.

Instrument Items and Justification

The assessment items were comparable to problems from the prerequisite chapter of both Precalculus course textbooks. These problems cover topics that students should be familiar with when entering the course. In items 1, 3, and 5, students were asked to evaluate the following numeric expressions:

1. $3(6) + 4[8 - (5 - 2)] - [3(7) + 11]$
3. $3(16 - 4) - [4(7) + 6] + [20 - 2(9)]$
5. $3[4^2 + 3(-3) + 2] - 2[2(3^2) - 4]$.

These problems require knowledge of simple arithmetic and order of operations, which are two fundamental concepts in evaluating numeric expressions. In items 2, 4, and 6, students were asked to simplify algebraic expressions, which requires a different skillset. The even questions are listed as follows:

2. $6x + 3[2x - (1 - x)] - (3x - 18)$
4. $3(9a - 7) - (5a + 7) + (13 - 3a)$
6. $3(4y^2 + 11y + 5) - 2(3y^2 - 3)$.

These problems require an understanding of working with algebraic notation more than they do anything else.

The survey questions were designed with two different goals in mind. The first goal was to obtain an understanding of the mathematical background of the students; the first and second questions address this goal: 1) What is the highest level of mathematics training that you have?; and 2) How long has it been since any previous mathematics training? The second goal of the survey was to determine how the students perceive themselves in both number sense and algebra. On question 3 students rated their skills in number sense on a scale of one to ten (one being the lowest and ten the highest), taking special care to mention topics such as order of operations and arithmetic. On the last question students to rated their skills in algebra, noting specifically algebraic form and simplification. Questions 1 and 2 appeared only on the pre-assessment survey. Questions 3 and 4 appeared on both the pre- and post-assessment survey.

Data Collection

The data for this experiment were gathered about a third of the way into the university's Spring semester of 2012 and data collection was complete within one week. Participants were administered the assessment and the both surveys, as previously discussed. The assessment was graded as follows: A correctly evaluated/simplified expression received 1 point, and an incorrectly evaluated/simplified expression received 0 points. Simply, each response was marked as either correct or incorrect. Participants had 20 minutes to complete the assessment and no partial credit was awarded.

The surveys were evaluated as follows. The responses from the first two survey questions were used to determine statistical significance, while the responses from Questions 3 and 4 (both pre- and post-assessment) were grouped into the following three categories: An increase in rating from pre- to post-assessment survey, no change, and a decrease in rating from pre- to post-assessment survey. The prominent statistic the researcher was seeking was how many students reported a decrease in their self-perceived ratings after taking the assessment.

Methods of Data Analysis

This study was mainly quantitative in nature. The analysis compared raw scores obtained on the assessments, while mean scores were examined on an overall basis to obtain an idea of general trends for each type of question. This type of analysis allowed the researcher to evaluate cases in which students fell into trends that agree with the hypothesis, as well as those cases in which students did not fall into such trends. Another set of quantitative data came from the pre- and post-assessment surveys. Students evaluated themselves on their own skill level in each subject area. Scores from the pre-assessment survey were compared to scores from the post-assessment survey to determine how many students felt more or less confident in their skills after having taken the assessment.

Analysis of Assessment

The results were obtained through a compilation of raw scores achieved on the assessments. To determine a quantitative conclusion regarding the hypothesis, results on the assessment were analyzed as follows. First, each pair of "comparable" expressions were evaluated and placed into four groups: (1) The numeric expressions was correct while the algebraic was incorrect, (2) The numeric expression was incorrect while the algebraic was correct, (3) Both expressions were correct, (4) Both were incorrect. Recall, that each assessment contained 3 of these pairs, thus 100 participating students returned a total of 300 data points for this particular method of analysis.

Second, the total number of correct numeric expressions was compared to the total number of correct algebraic expressions. For this particular method of analysis, the 100 participating students returned a total of 300 data points *for each type of expression*. This allowed the researcher to deduce which type of expressions students as a whole were more capable of correctly answering.

Survey Analysis

The primary purpose of the survey is to determine student self-perception of skills. This was achieved by placing each student, based on their pre- and post-survey scores, into three groups as discussed in the Data Collection section of the Experimental Design: Those who scored themselves higher (than they did prior to the assessment), lower, or no change. Percentages of each of the 3 groups were examined for both the self-evaluation of numeric skills as well the self-evaluation of algebraic skills. Although this data does not particularly lend itself to a strong conclusion regarding the hypothesis, it does provide results that may impact further study into the research topic.

Lastly, an ANOVA test was used to determine *p*-values based on the significance of mathematical experience. In other words, did student responses to questions 1 and 2 on the pre-survey have any impact on their overall assessment score? The ANOVA was the only analysis item that utilized students' overall score (summing the score for both numeric and algebraic expressions on an individual basis) was used in the analysis of the data. The pedagogical importance of these particular tests will be discussed in the Implications for Teaching section, following the results.

Results

The overall analysis of the data collected from this research tends to disagree with the hypothesis statement:

- Students consistently scored higher on the numeric expressions (54.3%),
- Prior mathematics experience had little effect on student success (*p*-value: 0.012), and
- Students felt less confident in their abilities after completing the instrument.

The following bulleted sections elaborate on these conclusions.

- ***Students consistently scored higher on the numeric expressions***

Contrary to the hypothesis statement, students consistently performed better at evaluating a numeric expression as opposed to completely reducing an algebraic expression. The number of correctly reduced numeric expressions totaled 163 out of 300 (54.3%). The number of correctly simplified algebraic expressions totaled 111 out of 300 (37.0%). Figure 3 displays these results as two separate pie charts.

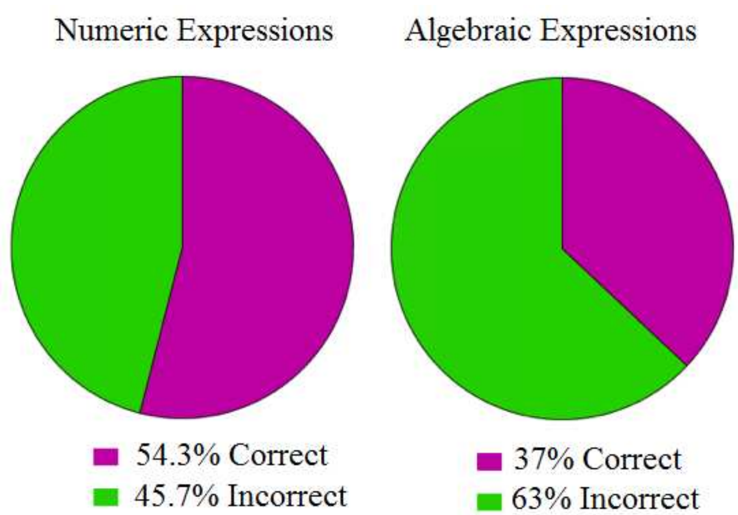

Figure 3. Pie charts of correct versus incorrect expressions.

Comparing the overall results of items 1 (numeric) and 2 (algebraic), 61% of students correctly evaluated item 1 while only 34% correctly reduced item 2. Preserving the variable order, 54% correctly evaluated item 3 while only 35% correctly reduced item 4, and 48% correctly reduced item 5 while only 42% correctly reduced item 6. Figure 4 displays those results side by side.

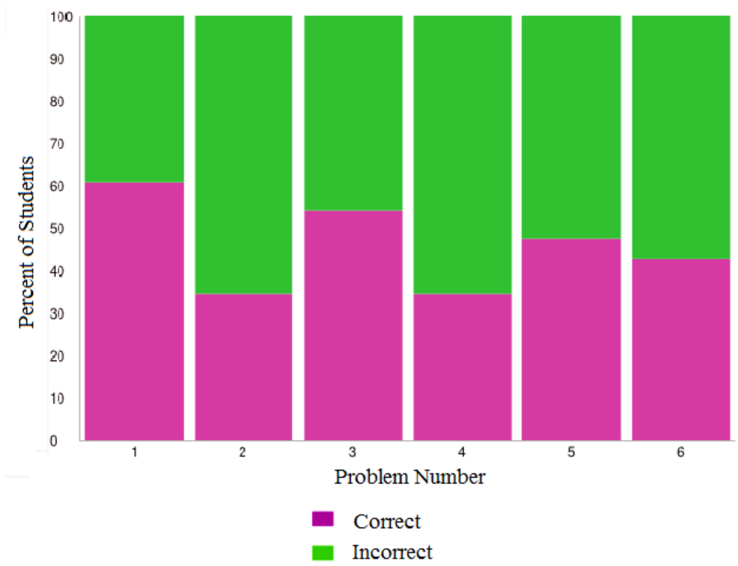

Figure 4. The number of correct and incorrect responses on each item.

In addition to the generally better scores on all of the numeric expressions, each pair of numeric/algebraic expressions were grouped as follows: 1) Both problems were correct, 2) The numeric problem was correct while the algebraic problem was incorrect, 3) The numeric problem was incorrect while the algebraic problem was correct, and 4) Both problems were incorrect. Those groupings can be found in Figure 5.

Both Correct	24.00% (72 pairs)
Numeric Correct & Algebraic Incorrect	30.33% (91 pairs)
Numeric Incorrect & Algebraic Correct	13.00% (39 pairs)
Both Incorrect	32.66% (98 pairs)

Figure 5. Comparable pairs grouped by correctness.

- ***Prior mathematics experience had little effect on student success***

An Analysis of Variance (ANOVA) Test revealed that mathematics background had very little statistical significance (*p*-value: 0.012). Similarly, the amount of time that it had been since students had previously taken a mathematics course was also insignificant (*p*-value: 0.037). It should be noted however, that this was slightly more significant than their mathematics background. Figure 6 displays a histogram of the amount of years students reported since their previous mathematics training.

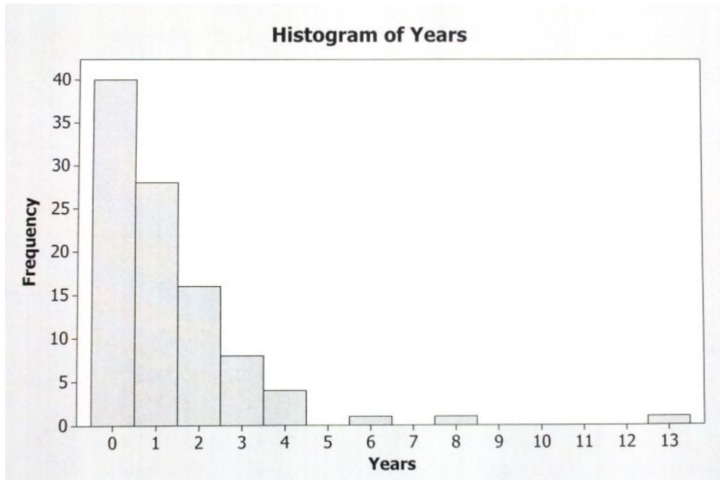

Figure 6. Frequency histogram of student age.

- ***Students felt less confident in their abilities after completing the instrument***

Having compared the results of the pre- and post-surveys, it was determined that almost two-thirds of the students had overestimated their mathematical abilities relating to each type of problem. Relating to evaluating a numeric expression, 62% of students had reported a lower self-evaluative score on the post-survey than they had on the pre-survey. Relating to simplifying an algebraic expression, 74% reported a lower score. Figures 7 displays these and other values.

	Increase from Pre- to Post-Survey	Decrease from Pre- to Post-Survey	No Change from Pre- to Post-Survey
Arithmetic Self-rating	16 students	62 student	22 students
Algebra Self-rating	11 students	74 students	15 students

Figure 7. Change in self-ratings from pre- to post-assessment surveys.

Despite the fact that students were unaware of their assessment score while completing the post-survey, these evaluations shows that most students over-estimate their mathematical abilities in both areas. Not only did students over-estimate their abilities, but many of them had made the same common errors on specific items.

Item Analysis & Common Errors

Students were asked to evaluate a given numeric expression for a single value in the odd numbered problems (1, 3, and 5). One of the most common challenges that students faced was properly following order of operations. Instead, they would often attempt to distribute multiplication across parentheses, or similarly, distribute a negative sign across groupings rather than simplify the group to a single term. Although these methods are not necessarily incorrect, they are more prone to error. These and other common errors will be discussed in the following examination of each of these items.

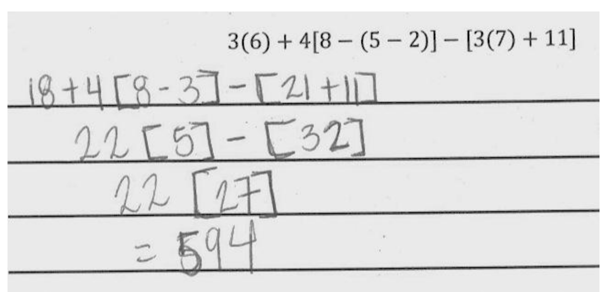

Figure 8. Student attempt at evaluating item 1.

Figure 8 shows a student's attempt at evaluating item 1. Note that from the student's first to second line of work that they ignored order of operations and added outside of the leftmost set of parentheses. This was the only error that the student made which led to a final answer that was nowhere near the correct value.

Another error that students often made was not working from left to right, when appropriate. In Figure 9, the student makes this exact error with item 3. It is evident that in order to end up with a value of zero, the student added 34 and 2 in their second line of work, but they disregarded the minus sign that precedes the 34. Errors of this type were surprisingly far more common than expected.

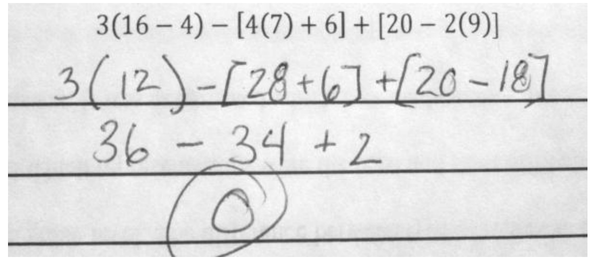

Figure 9. Student attempt at evaluating item 3.

Figure 10 shows an example of a solution in which a student distributed across parentheses instead of following order of operations. In this case, the first grouping was correctly simplified but the second grouping was not, which led to the resulting incorrect answer. This type of error was the most common in the evaluation of numeric expressions.

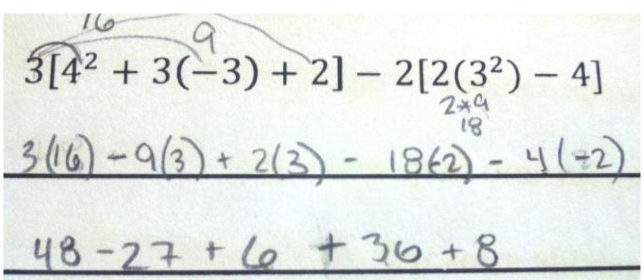

Figure 10. Student attempt at evaluating item 5.

Students were asked to completely simplify a given algebraic expression in the even numbered problems (2, 4, and 6). There were many errors similar to that of Figure 8, which show a lack of understanding of the order of operations. More so, the errors involved with these items showed a

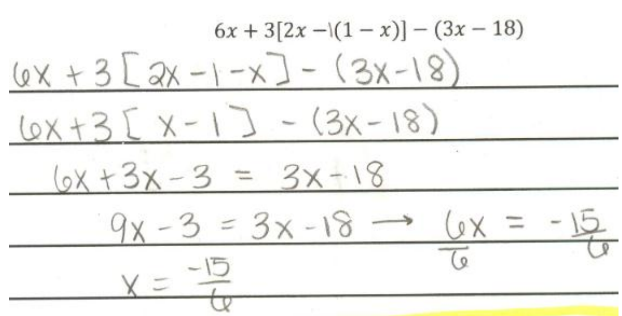

Figure 11. Student attempt at evaluating item 2.

fundamental lack of understanding of algebraic simplification. For example, one of the most common errors was to set the expression equal to zero and solve for *x*, as evident in Figure 11.

Most errors that occurred on items 4 and 6 involved the distributive property (although these two, as well, had their share of attempts at solving for the variable). Figure 12 demonstrates the most common error that ocurred on item 6. In this particular case, the student failed to distibute the minus sign in front of the second grouping. A few other noteworthy cases included attempts at using the quadratic formula to find a value for *y*, most of which ended in failure.

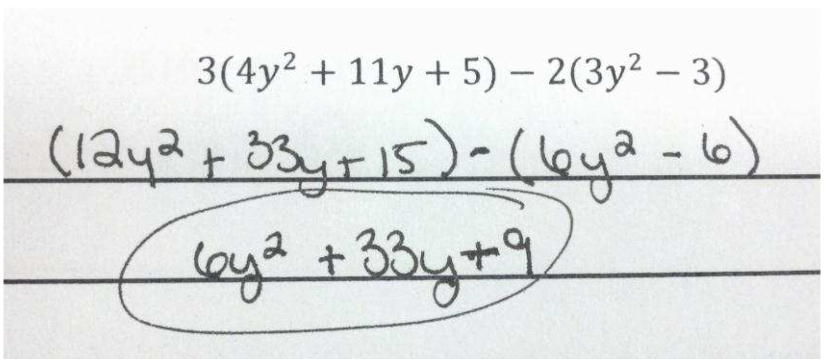

Figure 12. Student attempt at evaluating item 6.

Implications for Teaching

The results that emerged in this study have specific implications for the teaching of these topics in the mathematics classroom. The intent of this research was to show that students may lack the conceptual understanding that comes with the development of arithmetic skills, but the nature of algebra allows them to be successful without that understanding. The results do not disprove this. They do, however, support the following:

- Further emphasis should be placed on the conceptual understanding of arithmetic and its relationship to algebra,
- Students require more time focusing on arithmetic and algebra skill-building,
- Students require further development in understanding mathematical conventions,
- When solving algebraic expressions for a given value, students may be better off immediately plugging in rather than simplifying the expression first.

The following bulleted sections discuss each of these points in more details and their impact on the mathematics curriculum.

- ***Further emphasis should be placed on the conceptual understanding of arithmetic and its relationship to algebra***

The purpose of this study was to find evidence supporting the idea that technical competency provides enough skill for students to "fool" their way through algebra without gaining a real understanding of the material. Instead, the impact of the results is far more vague. Further research could show that there is no correlation at all between success rates in arithmetic and success rates in algebra, or there could very well be a strong correlation between the two (suggestions for further research are discussed in the following section). Regardless, the results do show that the participating students often displayed a lack of fundamental understanding in arithmetic; a lack of understanding that carried over to their abilities in algebra. This shows that teaching technical abilities alone is not a strong enough foundation for further development in mathematics. To abate these symptoms, emphasis on conceptual understanding is severely needed within the curriculum. Technical skill is often challenged when working with particularly daunting problems; having a conceptual understanding of the material allows students to make sense of the problem and more successfully apply those skills.

- ***Students still require more time focusing on arithmetic and algebra skill-building***

Less than half of the 600 total responses were correctly evaluated/simplified. This shows a severe lack of both understanding and technical skill from the students that participated in this study. Such skills include following order of operations, understanding mathematical conventions, and working with algebraic notation and multiple representations. Considering that the majority of these students had completed a high school mathematics curriculum within the past 1 to 2 years, the results say a lot about both what they had learned, and what they had retained from those experiences. Either way, it seems evident that the high school curriculum needs to retrace these issues to their roots. If students perform this poorly at these fundamental topics in mathematics, it is no surprise to hear them complain about being set up for failure in more advanced mathematics courses.

- ***Students require further development in understanding mathematical conventions***

This idea was examined a few times in the accompanying literature review. Students often show a lack of understanding of mathematical conventions, for example, working from left to right. Practicing rote techniques such as *PEMDAS*, if not properly taught, contradicts those conventions. In order to be truly successful in the field of mathematics, students need to understand those conventions and when they take precedence over the skill-building rules we teach (and vice-versa). In a way, this is, in and of itself, much of the battle in conceptual understanding; making sense of the relationship between rules and conventions.

- ***When solving algebraic expressions for a given value, students may be better off immediately plugging in rather than simplifying the expression first***

This is a much more specific implication for the algebra curriculum, one which often comes down to personal preference. There are really only two methods to consider, the suggested route, or the route of first completely simplifying the algebraic expression and then plugging in the given value. Because students showed a higher success rate at evaluating numeric expression, it is suggested that teachers skip the algebraic simplification step, at least until simplification has become a mastered skill. Students are more likely to make an error in the simplification process when a variable is present. Eliminating the variable earlier on will likely make students less prone to error on these types of problems.

Implications for Further Research

While this research did provide very interesting results as well as classroom implications, they did not ultimately agree with the hypothesis statement. That being said, the research process could have been tweaked so as to more accurately provide a better window into the heart of the hypothesis statement. Such changes could possibly include:

- Choose participants from a high school setting
- Use pairs of *isomorphic* expressions rather than *comparable* expressions
- Separate the assessment into two parts
- Provide more explicit instructions on the assessment

The following sections elaborate on these changes in an attempt to provide details for providing a more successful and accurate research project.

- ***Choose participants from a high school setting***

The intent of this research was meant to be most impactful on the middle and high school curriculum. Although the researcher tested students who had all completed such a curriculum, the data might be more meaningful had it come from students who are currently participating in them. It would also eliminate factors that were addressed on the pre-survey such as mathematics background and how long ago that training was. Selecting participants from a high school setting would provide less inhibiting factors in the research by providing more consistency amongst their mathematical backgrounds.

- ***Use pairs of isomorphic expressions rather than comparable expressions***

The assessment that was used for this research was comprised of structurally "comparable" expressions. That is, each pair consisted of the same separation of groupings by use of both parentheses and operations. The constants and coefficients that made up each term, however, were not consistent. Similarly, the numbers that replaced the variable in the numeric expressions were not consistent. In a further study, the researcher suggests using isomorphic expressions. For example:

$$2(3+2)-4(3)+10$$

is isomorphic to

$$2(x+2)-4(x)+10.$$

Note that aside from the variable taking the place of the 3, these expressions are exactly the same. Such isomorphic expressions would help to provide more consistency as well as to help pinpoint the relationship between any errors made in both examples.

- ***Separate the assessment into two parts***

Separating the assessment into two parts is a direct result of the previous implication. By making each pair isomorphic, students may catch on to the fact that they can first simplify the algebraic expression and then plug in the particular value from the numeric expression after they have simplified it. To avoid any confusion that might result from this, the assessment would work best in two parts. The first part has all three numeric expressions on it. This gets administered and then collected. Once collected, the second part is administered. The second part consists of all three algebraic expressions.

- ***Provide more explicit instructions on the assessment***

One prevalent issue, which appeared numerous times, was that students did not always seem to fully understand what they were being asked to do. The most glaring example of this would be attempting to solve for the variable on the algebraic expressions. By placing more direct and explicit directions on the assessment, student may be more capable of achieving the desired result, which would provide more accurate data. Any results that are skewed by a lack of specificity are unnecessary, and so providing clearer expectations may help to ensure clearer results. Similarly, it might be helpful to request that students show each step of the simplification process. This would help to identify errors that occurred and possibly help clarify the reasoning behind those errors.

The research conducted did not definitively agree nor disagree with the hypothesis statement. The analysis of the data tends to suggest, however, that students are more successful at evaluating numeric expressions over simplifying algebraic expressions. The preceding implications are meant to provide a basis to more accurately answer the research question, whether it agrees or disagrees with the hypothesis statement. The correlation of student understanding of the relationship between arithmetic and algebra can be improved upon and refined to suggest greater implications for middle and high school mathematics curricula.

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Pre-Assessment Survey

1. What is the highest level of mathematics training you have received prior to taking this course?

   a. Integrated Algebra (or equivalent)

   b. Geometry (or equivalent)

   c. Algebra II/Trigonometry (or equivalent)

   d. Other (Please Specify): ___________________

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2. How many years has it been since you have last taken a mathematics course?

*Please reply with 0 if you have taken a math class within the past year.

______ Year(s)

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3. Provide a numeric rating of your skills in basic number sense. This skill set encompasses topics such as *order of operations, basic arithmetic, and properties of real numbers*.

0 **1** **2** **3** **4** **5** **6** **7** **8** **9** **10**
(Weakest) (Strongest)

---

4. Provide a numeric rating of your skills in basic algebra. This skill set encompasses topics such as *algebraic form, structure,* and *simplification*.

0 **1** **2** **3** **4** **5** **6** **7** **8** **9** **10**
(Weakest) (Strongest)

Assessment

Note: Brackets serve the same purpose as parentheses.

Show **as much work as possible** in the simplification process and **circle your final answers**.

1) Evaluate the following expression for a single value:

$$3(6) + 4[8 - (5 - 2)] - [3(7) + 11]$$

2) Simplify the following algebraic expression, completely:

$$6x + 3[2x - (1 - x)] - (3x - 18)$$

3) Evaluate the following expression for a single value:

$$3(16 - 4) - [4(7) + 6] + [20 - 2(9)]$$

Assessment (p. 2)

Show **as much work as possible** in the simplification process and **circle your final answers**.

4) Simplify the following algebraic expression, completely:

$$3(9a - 7) - (5a + 7) + (13 - 3a)$$

5) Evaluate the following expression for a single value:

$$3[4^2 + 3(-3) + 2] - 2[2(3^2) - 4]$$

6) Simplify the following algebraic expression, completely:

$$3(4y^2 + 11y + 5) - 2(3y^2 - 3)$$

Post-Assessment Survey

After taking the assessment, please re-evaluate your original rating on the following questions and make any changes you feel necessary.

1. Provide a numeric rating of your skills in basic number sense. This skill set encompasses topics such as *order of operations, basic arithmetic, and properties of real numbers*.

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
(Weakest)										(Strongest)

2. Provide a numeric rating of your skills in basic algebra. This skill set encompasses topics such as *algebraic form, structure,* and *algebraic simplification*.

0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
(Weakest)										(Strongest)

To: Students of MATH 110
From: Michael Humbert
Re: Consent Form

- You are being asked to participate in a research study searching for trends in perception of structure, as it relates to both algebra and number sense.
- The participant materials consist of a pre-assessment survey, a six question formal assessment, and a post-assessment survey. The surveys will gather information about participants' history with mathematics as well as competency in the aforementioned content areas. The assessment addresses testing those skills directly.
- The attached consent form is labeled with a number unique to you, as are your other participant materials. The consent forms will be collected and sealed in an envelope. No other materials will be labeled with names, as to keep the data anonymous.
- If at any point you wish to withdraw, you may contact me to locate your numbered consent form and remove the data associated with the matching number.
- By signing the consent form, you are allowing the researcher to use the materials he receives as necessary, and to truthfully report the anonymous data he is provided.
- There is no risk involved. Your participation has no effect on your class grade. There is no penalty for choosing not to participate/withdrawing.
- Calculators are not allowed on the assessment, but you will be given a multiplication table.

If you have any questions, you may contact the following persons by means of email or telephone.

Michael Humbert Researcher	humbert@fredonia.edu	(716) 673-4811
Dr. Keary Howard Faculty Advisor	keary.howard@fredonia.edu	(716) 673-3873
Maggie Bryan Peterson Human Subjects Administrator	petersmb@fredonia.edu	(716) 673-3528

Student Consent Form

SUNY Fredonia

Thank you for being a part of this anonymous study. Please print and sign your name in the space provided to show that you agree to participate. Remember that signing the form allows Mr. Humbert to use your data (anonymously) for his research project.

Voluntary Consent: I have read this memo and I am fully aware of all that this study involves. My signature below shows that I freely agree to participate in this study. I understand that there will be no penalty for not participating. I understand that I may withdraw from the study at any time, also without penalty. I understand that my name and any other personal information will be kept out of the study.
I understand that if I have any questions about the study, I may contact Mr. Humbert, or the others listed on the attached memo, by means of email and telephone.

Please return this original, completed consent form as soon as possible. Thank you for your cooperation.

Student Name (Please Print): ____

Student Signature:____