ON THE TRACE OF HECKE OPERATORS FOR MAASS FORMS

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Abstract. The trace of the Hecke operator $T(n)$ acting on a Hilbert space of functions spanned by the eigenfunctions of the Laplace-Beltrami operator with a positive eigenvalue is computed, which can be considered as the analogue of Eichler-Selberg's trace formula for non-holomorphic cusp forms of weight zero.

1. Introduction

Denote by Γ the group $PSL_2(\mathbb{Z})$. The Laplace-Beltrami operator Δ on the upper half-plane $\mathcal{H}$ is given by

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$

Define D to be a fundamental domain of Γ, which contains the points $z = x + iy$ with $0 < x < 1$ and $|z-\frac{1}{2}| > 1$. Eigenfunctions of the discrete spectrum of Δ are nonzero real-analytic solutions of the equation

$$\Delta\psi = \lambda\psi$$

such that $\psi(\gamma z) = \psi(z)$ for all γ in Γ and such that

$$\int_D |\psi(z)|^2 dz < \infty$$

where dz represents the Poincaré measure of the upper half-plane.

The Hecke operators $T(n)$, $n = 1, 2, \cdots$, which act in the space of automorphic functions with respect to Γ, are defined by

$$(T(n)f)(z) = \frac{1}{\sqrt{n}}\sum_{ad=n,\, 0\le b<d} f\left(\frac{az+b}{d}\right).$$

An orthogonal system of eigenfunctions of Δ exists [6] such that each of them is an eigenfunction of all the Hecke operators. Let λ be a positive discrete eigenvalue of

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Δ. Then $\lambda > \frac{1}{4}$. If $\psi(z)$ is such an eigenfunction of Δ with a positive eigenvalue λ, then

$$\psi(z) = \sqrt{y} \sum_{m \neq 0} \rho(m) K_{i\kappa}(2\pi|m|y) e(mx)$$

where $\kappa = \sqrt{\lambda - \frac{1}{4}}$ and where $K_\nu(y)$ is given by the formula §6.32, [13]

$$K_\nu(y) = \frac{2^\nu \Gamma(\nu + \frac{1}{2})}{y^\nu \sqrt{\pi}} \int_0^\infty \frac{\cos(yt)}{(1+t^2)^{\nu+\frac{1}{2}}} dt. \tag{1.1}$$

If $\psi(z)$ is normalized so that $\rho(1) = 1$, then the identity [6]

$$\left(T(n)\psi\right)(z) = \rho(n)\psi(z)$$

holds for all positive integers n. The Petersson-Ramanujan conjecture for non-holomorphic cusp forms of weight zero says that the inequality

$$|\rho(n)| \leq d(n)$$

holds for all positive integers n, where $d(n)$ denotes the number of divisors of n. Let $\mathcal{E}_\lambda$ be a Hilbert space of functions spanned by the eigenfunctions of Δ with a positive eigenvalue λ. The inner product of the space is given by

$$\langle F(z), G(z) \rangle = \int_D F(z) \bar{G}(z) dz. \tag{1.2}$$

The Eichler-Selberg trace formula [8], p.85 is a useful formula for studying holomorphic modular forms of integral weights (cf. Deligne [1] and Ihara [4]). The analogue of Eichler-Selberg's trace formula for non-holomorphic cusp forms of weight zero is obtained in the Main Theorem, whose proof is given in section 4. In particular, the trace $\mathrm{tr}T(n)$ of Hecke operators acting on the space $\mathcal{E}_\lambda$ is computed, and the computation is already implicit in Hejhal [2].

Write $\tau = \frac{1}{2} + i\kappa$. Denote by h_d the class number of indefinite rational quadratic forms with discriminant d. Define

$$\epsilon_d = \frac{v_0 + u_0\sqrt{d}}{2} \tag{1.3}$$

where the pair (v_0, u_0) is the fundamental solution [11] of Pell's equation $v^2 - du^2 = 4$. Denote by Ω the set of all the positive integers d such that $d \equiv 0$ or $1 \pmod 4$ and such that d is not a square of an integer.

Main Theorem. *Define*

$$L_n(\sigma) = \sum_{d\in\Omega,\, u} \frac{h_d \ln \epsilon_d}{(du^2)^\sigma}$$

for $Re\,\sigma > 1$, where the summation on u is taken over all the positive integers u which together with t are the integral solutions of the equation $t^2 - du^2 = 4n$. Then

$$trT(n) = 2n^{i\kappa} Res_{\sigma=\tau} L_n(\sigma)$$

for every positive integer n, where $L_n(\sigma)$ is an analytic function of σ for $Re\,\sigma > 1$ and can be extended by analytic continuation to the half-plane $Re\,\sigma > 0$ except for possible simple poles at $\sigma = 1, \frac{1}{2}, \frac{1}{2} \pm i\kappa$ with $\frac{1}{4} + \kappa^2$ being taken over all the positive discrete eigenvalues of the Laplace-Beltrami operator for the modular group.

2. Trace formula

Let σ be a complex number with $Re\,\sigma > 1$. Define

$$k(t) = (1 + \frac{t}{4})^{-\sigma}$$

and

$$k(z, z') = k\left(\frac{|z - z'|^2}{yy'}\right),$$

for $z = x + iy$ and $z' = x' + iy'$ in the upper half-plane. Then $k(mz, mz') = k(z, z')$ for every 2×2 matrix m of determinant one with real entries. The kernel $k(z, z')$ is of (a)-(b) type in the sense of Selberg [8], p.60. Let

$$g(u) = \int_w^\infty k(t) \frac{dt}{\sqrt{t - w}}$$

with $w = e^u + e^{-u} - 2$. Write

$$h(r) = \int_{-\infty}^\infty g(u) e^{iru} du.$$

Then

$$g(u) = \sqrt{w} \int_0^1 (t + \frac{w}{4})^{-\sigma} t^{\sigma - \frac{3}{2}} \frac{dt}{\sqrt{1 - t}} = c(1 + \frac{w}{4})^{\frac{1}{2} - \sigma} \tag{2.1}$$

where $c = 2\sqrt{\pi}\Gamma(\sigma - \frac{1}{2})\Gamma^{-1}(\sigma)$. Since

$$h(r) = c4^{\sigma - \frac{1}{2}} \int_0^\infty (u + \frac{1}{u} + 2)^{\frac{1}{2} - \sigma} u^{ir - 1} du$$

for $Re\,\sigma > \frac{1}{2}$, by computation we find that

$$\lim_{\sigma\to\tau}(\sigma-\tau)h(r) = \begin{cases} 4^\tau\sqrt{\pi}\frac{\Gamma(i\kappa)}{\Gamma(\tau)}, & \text{for } r=\pm\kappa; \\ 0, & \text{for } r\neq\pm\kappa. \end{cases} \tag{2.2}$$

Define

$$\Gamma^* = \cup_{\substack{ad=n \\ 0\le b<d}} \frac{1}{\sqrt{n}} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \Gamma.$$

Then γT belongs to Γ^* whenever $\gamma \in \Gamma$ and $T \in \Gamma^*$. Every element of Γ^* is represented uniquely in the form

$$\frac{1}{\sqrt{n}} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix} \gamma$$

with $ad = n$, $0 \le b < d$ and $\gamma \in \Gamma$. It follows that Γ^* satisfies all the requirements given in [8], p.69. The Eisenstein series is given by

$$E(z,s) = \frac{1}{2}\sum_{(c,d)=1} \frac{y^s}{|cz+d|^{2s}}$$

for z in the upper half-plane when $Re\, s > 1$. Define

$$K(z,z') = \sum_{T\in\Gamma^*} k(z, Tz')$$

and

$$H(z,z') = \sum_{ad=n,\,0\le b<d} \frac{1}{4\pi}\int_{-\infty}^{\infty} h(r)E(\frac{az+b}{d}, \frac{1}{2}+ir)E(z', \frac{1}{2}-ir)dr.$$

Let ℓ be a positive number such that $\frac{1}{4}+\ell^2$ is a discrete eigenvalue of Δ distinct from λ. Denote by t_ℓ the trace of the Hecke operator $T(n)$ acting on the space $\mathcal{E}_{\frac{1}{4}+\ell^2}$. It follows from (2.14) of [8], the argument of [5], pp.96-98, Theorem 5.3.3 of [5], and the spectral decomposition formula (5.3.12) of [5] that

$$\sum_{d|n} dh(-\frac{i}{2}) + \sqrt{n}h(\kappa)\mathrm{tr}T(n) + \sqrt{n}\sum_{\ell} h(\ell)t_\ell = \int_D \{K(z,z) - H(z,z)\}dz \tag{2.3}$$

for $Re\,\sigma > 1$, where the summation is taken over all distinct positive numbers ℓ not equal to κ such that $\frac{1}{4}+\ell^2$ is a discrete eigenvalue of Δ.

3. Evaluation of components of the trace

For every element T of Γ^*, denote by Γ_T the set of all the elements of Γ which commute with T. Put $D_T = \Gamma_T\backslash\mathcal{H}$. The elements of Γ^* can be divided into four types, of which the first consists of the identity element, while the others are respectively the hyperbolic, the elliptic and the parabolic elements. If T is not a parabolic element, put

$$c(T) = \int_{D_T} k(z, Tz)dz.$$

3.1. The identity component.

If Γ^* contains the identity element I, then

$$c(I) = \frac{\pi}{3}.$$

3.2. Elliptic components.

There are only a finite number of elliptic conjugacy classes.

Lemma 3.1. *Let R be an elliptic element of Γ^*. Then*

$$c(R) = \frac{\pi}{2m\sin\theta}\int_0^\infty \frac{k(t)}{\sqrt{t + 4\sin^2\theta}}dt,$$

where m represents the order of a primitive element of Γ_R and where θ is defined by the formula trace$(R) = 2\cos\theta$.

Proof. Since R is an elliptic element of Γ^*, an element η exists in $SL_2(\mathbb{R})$ such that

$$\eta R\eta^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \widetilde{R}$$

for some real number $0 < \theta < \pi$. Denote by $(\eta\Gamma\eta^{-1})_{\widetilde{R}}$ the set of all the elements of $\eta\Gamma\eta^{-1}$ which commute with $\widetilde{R}$. We have

$$c(R) = \int_{D_{\widetilde{R}}} k(z, \widetilde{R}z)dz$$

where $D_{\widetilde{R}} = (\eta\Gamma\eta^{-1})_{\widetilde{R}}\backslash\mathcal{H}$.

Let $\gamma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ be an element of Γ which has the same fixed points as $R = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $(\alpha - \delta)c = \gamma(a - d)$ and $\beta c = \gamma b$. It follows that γ commutes with R. By Proposition 1.13 of [9], a primitive elliptic element γ_0 of Γ exists such that $(\eta\Gamma\eta^{-1})_{\widetilde{R}}$ is generated by $\eta\gamma_0\eta^{-1}$. Since $\eta\gamma_0\eta^{-1}$ commutes with $\widetilde{R}$, it is of the form

$$\begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix}$$

for some real number θ_0. By Proposition 1.16 of [9], $\theta_0 = \pi/m$ for some positive integer m. In fact, $m = 2$ or 3. It follows from the argument of [5], p.99 that

$$c(R) = \frac{1}{m} \int_0^\infty \int_{-\infty}^\infty k\left(\frac{|z^2+1|^2}{y^2}\sin^2\theta\right) dz.$$

By the argument of [5], p.100 we have

$$c(R) = \frac{\pi}{2m\sin\theta} \int_0^\infty \frac{k(t)}{\sqrt{t+4\sin^2\theta}} dt. \quad \square$$

3.3. Hyperbolic components.

Let P be a hyperbolic element of Γ^*. Then an element ρ exists in $SL_2(\mathbb{R})$ such that

$$\rho P \rho^{-1} = \begin{pmatrix} \lambda_P & 0 \\ 0 & \lambda_P^{-1} \end{pmatrix} = \widetilde{P}$$

with $\lambda_P > 1$. The number λ_P^2 is called the norm of P and will be denoted by NP. It follows that

$$c(P) = \int_{D_{\widetilde{P}}} k(z, NPz) dz$$

where $D_{\widetilde{P}} = (\rho\Gamma\rho^{-1})_{\widetilde{P}} \backslash \mathcal{H}$.

Lemma 3.2. *Let P be a hyperbolic element of Γ^* such that $\Gamma_P \neq \{1_2\}$. If P_0 is a primitive hyperbolic element of Γ which generates the group Γ_P, then*

$$c(P) = \frac{\ln NP_0}{(NP)^{1/2} - (NP)^{-1/2}} g(\ln NP).$$

Proof. An argument similar to that made for the elliptic elements shows that every element of Γ, which has the same fixed points as P, commutes with P. Because $\rho P_0 \rho^{-1}$ commutes with $\widetilde{P}$, it is of the form

$$\begin{pmatrix} \lambda_{P_0} & 0 \\ 0 & \lambda_{P_0}^{-1} \end{pmatrix}$$

for some real number $\lambda_{P_0} > 1$. Then

$$c(P) = \int_1^{NP_0} \frac{dy}{y^2} \int_{-\infty}^\infty k\left(\frac{(NP-1)^2}{NP}\frac{|z|^2}{y^2}\right) dx.$$

The stated identity follows. $\square$

Let Y be a large positive number. Define

$$D_Y = \{z \in D : Im\, z < Y\}.$$

Denote by $(D_Y)_P$ the set

$$\sum \gamma D_Y$$

where the summation is taken over all elements γ of Γ. Write

$$c(P)_Y = \int_{(D_Y)_P} k(z, Pz) dz.$$

Lemma 3.3. *Let P be a hyperbolic element of Γ^* such that $\Gamma_P = \{1_2\}$. Then*

$$c(P)_Y = \frac{\ln \frac{Y}{2\rho}}{(NP)^{1/2} - (NP)^{-1/2}} g(\ln NP) + \int_1^\infty k\left((NP + \frac{1}{NP} - 2)t\right) \frac{\ln t}{\sqrt{t-1}} dt + o(1)$$

where the term $o(1)$ has a limit zero as $Y \to \infty$ and where ρ is defined in the proof.

Proof. Since $\Gamma_P = \{1_2\}$, the fixed points of P are cusps of Γ by Proposition 1.13 of [9]. Since cusps of Γ are exactly the points in $\mathbb{Q} \cup \{\infty\}$, an element γ of Γ exists such that $\gamma(\infty)$ is one of the fixed points of P. Since $c(P)$ depends only on the conjugacy class $\{P\}$ represented by P, the element P can be replaced by $\gamma^{-1}P\gamma$ without changing the value of $c(P)$. Thus, P can be chosen in its conjugacy class to be of the form

$$\frac{1}{\sqrt{n}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

with $1 \leq b \leq |a-d|$. Let a, b and d be positive integers. Define

$$\alpha = \frac{b}{(b, |a-d|)}$$

and

$$\gamma = \frac{d-a}{(b, |a-d|)}.$$

Then integers β and δ exist such that

$$\alpha\delta - \beta\gamma = 1.$$

Let $b_1 = \delta(b, |a-d|)$. Write

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

We have

$$A^{-1} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} A = \begin{pmatrix} d & b_1 \\ 0 & a \end{pmatrix}.$$

It follows that elements of the form

$$\frac{1}{\sqrt{n}} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \ ad = n,\ 0 < d < a,\ 1 \leq b \leq a-d \tag{3.1}$$

constitute a complete set of representatives for the conjugacy classes of hyperbolic elements of Γ^* whose fixed points are cusps of Γ.

Let

$$\gamma = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$$

be an element of Γ. The linear fractional transformation, which takes every complex z in the upper half-plane into $\gamma(z)$, maps the horizontal line $Im\, z = Y$ into a circle of radius $\frac{1}{2q^2Y}$ with center at $\frac{p}{q} + \frac{i}{2q^2Y}$. Let

$$\mu = \begin{pmatrix} 1 & \frac{b}{a-d} \\ 0 & 1 \end{pmatrix}.$$

Then

$$c(P)_Y = \int_{\mu\{(D_Y)_P\}} k(z, \frac{a}{d} z) dz.$$

Let γ is an element of Γ such that $(\mu\gamma)(\infty) = 0$. Then the linear fractional transformation, which takes every complex z in the upper half-plane into $(\mu\gamma)(z)$, maps the horizontal line $Im\, z = Y$ into a circle of radius ρ with center at ρi, where

$$\rho = \frac{(b, a-d)^2}{2Y(a-d)^2}.$$

It follows that

$$\begin{aligned} c(P)_Y &= \int_0^\pi d\theta \int_{2\rho \sin\theta}^{Y/\sin\theta} k\left(\frac{(a-d)^2}{n \sin^2\theta}\right) \frac{dr}{r \sin^2\theta} + o(1) \\ &= \int_1^\infty k\left(\frac{(a-d)^2}{n} t\right) \frac{\ln(\frac{Y}{2\rho} t)}{\sqrt{t-1}} dt + o(1) \end{aligned}$$

where $o(1)$ has a limit zero when $Y \to \infty$. The identity

$$c(P)_Y = \frac{\sqrt{n} \ln \frac{Y}{2\rho}}{a-d} g(\ln \frac{a}{d}) + \int_1^\infty k\left(\frac{(a-d)^2}{n} t\right) \frac{\ln t}{\sqrt{t-1}} dt + o(1)$$

holds when $ad = n$, $0 < d < a$ and $1 \le b \le a - d$. $\square$

3.4. Parabolic components.

Let S be a parabolic element of Γ^*. An argument similar to that made for the elliptic elements shows that every element of Γ which has the same fixed point as S commutes with S. Since the cusps of Γ are exactly the points in $\mathbb{Q} \cup \{\infty\}$, it follows from Proposition 1.17 of [9] that an element ν of Γ exists such that

$$\nu \Gamma_S \nu^{-1} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

Since $\nu S \nu^{-1}$ commutes with every element of $\nu \Gamma_S \nu^{-1}$, it is of the form

$$\frac{1}{\sqrt{n}} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

for some integers a and b. It follows that Γ^* has parabolic elements only if n is the square of an integer. Furthermore, elements of the form

$$\begin{pmatrix} 1 & b/\sqrt{n} \\ 0 & 1 \end{pmatrix}, \qquad 0 \neq b \in \mathbb{Z}$$

constitute a complete set of representatives for the conjugacy classes of parabolic elements of Γ^*. If n is the square of an integer, then ∞ is the only cusp of Γ^* up to Γ-equivalence. Define δ_n to be one if n is the square of an integer and to be zero otherwise.

Lemma 3.4. *Put*

$$c(\infty)_Y = \delta_n \int_0^Y \int_0^1 \sum_{0 \neq b \in \mathbb{Z}} k(z, z + \frac{b}{\sqrt{n}}) dz - \int_{D_Y} H(z,z) dz.$$

Then

$$\begin{aligned}\frac{c(\infty)_Y}{\sqrt{n}} =& g(0)\delta_n \ln \frac{\sqrt{n}}{2} + \frac{\delta_n + d(n)}{4} h(0) \\ & - \ln Y \sum_{ad=n, a\neq d>0} g(\ln \frac{a}{d}) - \frac{\delta_n}{2\pi} \int_{-\infty}^{\infty} h(r) \frac{\Gamma'}{\Gamma}(1+ir) dr \\ & + \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'}{\varphi}(\frac{1}{2}+ir) \sum_{ad=n,\, d>0} a^{ir} d^{-ir} dr + o(1).\end{aligned}$$

Proof. By the argument of [5], pp.102–106 we have

$$\begin{aligned}&\frac{1}{\sqrt{n}} \int_0^Y \int_0^1 \sum_{0\neq b\in\mathbb{Z}} k(z, z+\frac{b}{\sqrt{n}}) dz \\ &= g(0)\ln(\sqrt{n}Y) - \frac{1}{2\pi}\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr - g(0)\ln 2 + \frac{1}{4}h(0) + o(1).\end{aligned}$$

If

$$\varphi(s) = \sqrt{\pi}\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)},$$

then

$$E(z,s) = y^s + \varphi(s) y^{1-s} + r(z,s)$$

for $z = x + iy$ with $y > 0$, where

$$r(z,s) = 2\sqrt{y}\pi^s \Gamma(s)^{-1} \sum_{m\neq 0} |m|^{s-\frac{1}{2}} \varphi_m(s) K_{s-\frac{1}{2}}(2|m|\pi y) e(mx)$$

and

$$\varphi_m(s) = \sum_{d|m} \frac{d^{1-2s}}{\zeta(2s)}.$$

By the functional identity of the Riemann zeta function $\zeta(s)$, we have $|\varphi(s)| = 1$ for $Re\, s = 1/2$. It follows from Theorem 2.3.3 of [5] that

$$\int_{D_Y} \left(\sum_{ad=n,\, 0\le b<d} E(\frac{az+b}{d}, s) \right) E(z,\bar{s})dz$$
$$= \sum_{ad=n, d>0} a^s d^{1-s} \left(\frac{Y^{s+\bar{s}-1} - |\varphi(s)|^2 Y^{1-s-\bar{s}}}{s+\bar{s}-1} + \frac{\varphi(\bar{s})Y^{s-\bar{s}} - \varphi(s)Y^{\bar{s}-s}}{s-\bar{s}} \right) + \omega_Y(s)$$

for $Re\, s > \frac{1}{2}$ with s not equal to one, where

$$\omega_Y(s) = - \sum_{\substack{ad=n \\ 0\le b<d}} \int_0^1 \int_Y^\infty [\varphi(\bar{s})y^{1-\bar{s}} + r(z,\bar{s})] r(\frac{az+b}{d}, s) \frac{dxdy}{y^2}.$$

The argument of [5], p.107 shows that

$$\frac{1}{\sqrt{n}} \int_{D_Y} H(z,z)dz$$
$$= \frac{\ln Y}{2\pi} \int_{-\infty}^{\infty} h(r) \sum_{\substack{ad=n \\ d>0}} a^{ir} d^{-ir} dr - \frac{1}{4\pi} \int_{-\infty}^{\infty} h(r) \frac{\varphi'}{\varphi}(\frac{1}{2}+ir) \sum_{\substack{ad=n \\ d>0}} a^{ir} d^{-ir} dr$$
$$+ \frac{1}{4\sqrt{n}\pi} \int_{-\infty}^{\infty} h(r)\omega_Y(\frac{1}{2}+ir)dr - \frac{d(n)}{4} h(0) + o(1).$$

By partial integration, we obtain

$$h(r) = \frac{1}{r^4} \int_0^\infty g^{(4)}(\ln u) u^{ir-1} du \tag{3.2}$$

for nonzero r. It follows from (3.2) and partial integration of (1.1) that

$$\frac{1}{4\sqrt{n}\pi} \int_{-\infty}^{\infty} h(r)\omega_Y(\frac{1}{2}+ir)dr = o(1)$$

as $Y \to \infty$. Then the stated identity holds. $\square$

It follows from Lemma 3.3, Lemma 3.4 and the statement concerning (3.1) that

$$\begin{aligned}
&\lim_{Y\to\infty}\left(c(\infty)_Y+\sum_{\{P\},\Gamma_P=\{1_2\}}c(P)_Y\right)\\
&=\frac{1}{2}\sum_{\substack{ad=n,\,d>0\\1\le b\le|a-d|}}\left(\frac{\ln\frac{(a-d)^2}{(b,a-d)^2}}{|a-d|}\sqrt{n}g(\ln\frac{a}{d})+\int_1^\infty k\left(\frac{(a-d)^2}{n}t\right)\frac{\ln t}{\sqrt{t-1}}dt\right)\\
&+\delta_n g(0)\sqrt{n}\ln\frac{\sqrt{n}}{2}+\frac{\sqrt{n}}{4}\{\delta_n+d(n)\}h(0)-\delta_n\frac{\sqrt{n}}{2\pi}\int_{-\infty}^{\infty}h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr\\
&+\frac{\sqrt{n}}{4\pi}\int_{-\infty}^{\infty}h(r)\frac{\varphi'}{\varphi}(\frac{1}{2}+ir)\sum_{ad=n,\,d>0}a^{ir}d^{-ir}dr.
\end{aligned}\tag{3.3}$$

Denote by $c(\infty)$ the right side of the identity (3.3). We conclude that the formula (2.3) can be written as

$$\begin{aligned}
&\sum_{d|n}dh(-\frac{i}{2})+\sqrt{n}h(\kappa)\mathrm{tr}T(n)+\sqrt{n}\sum_{\ell}h(\ell)t_\ell\\
&=c(I)+\sum_{\{R\}}c(R)+\sum_{\{P\},\,\Gamma_P\neq\{1_2\}}c(P)+c(\infty)
\end{aligned}\tag{3.4}$$

for $Re\,\sigma>1$, where the summations on the right side of the identity are taken over the conjugacy classes.

Lemma 3.5. *The series*

$$\sum_{\{P\},\,\Gamma_P\neq\{1_2\}}c(P)$$

represents an analytic function in the half-plane $Re\,\sigma>0$ *except for possible simple poles at* $\sigma=1,\frac{1}{2},\frac{1}{2}\pm i\kappa$ *with* $\frac{1}{4}+\kappa^2$ *being taken over all the positive discrete eigenvalues of* Δ.

Proof. Write

$$g^{(4)}(\log u)=A(\sigma)u^{\frac{1}{2}-\sigma}+O_\sigma(u^{-\frac{1}{2}}),$$

where $A(\sigma)$ is an analytic function of σ for $Re\,\sigma>0$ and where $O_\sigma(u^{-\frac{1}{2}})$ means that, for every complex number σ with $Re\,\sigma>0$, there exists a finite constant $B(\sigma)$ depending only on σ such that

$$|O_\sigma(u^{-\frac{1}{2}})|\leqslant B(\sigma)u^{-\frac{1}{2}}.$$

Moreover, for every fixed value of u, the term $O_\sigma(u^{-\frac{1}{2}})$ also represents an analytic function of σ for $Re\,\sigma > 0$. By (3.2), we have

$$\begin{aligned} h(r) &= \frac{1}{r^4}\int_1^\infty g^{(4)}(\ln u)(u^{ir}+u^{-ir})\frac{du}{u} \\ &= \frac{A(\sigma)}{r^4}\int_1^\infty u^{-\frac{1}{2}-\sigma}(u^{ir}+u^{-ir})du + O_\sigma\left(\frac{1}{r^4}\int_1^\infty u^{-\frac{3}{2}}du\right) \\ &= \frac{A(\sigma)}{r^4}\left(\frac{1}{\sigma-\frac{1}{2}-ir}+\frac{1}{\sigma-\frac{1}{2}+ir}\right) + O_\sigma(r^{-4}) \end{aligned}$$

for $Re\,\sigma > \frac{1}{2}$. By analytic continuation, we obtain that

$$h(r) = \frac{2A(\sigma)(\sigma-\frac{1}{2})}{r^4[(\sigma-\frac{1}{2})^2+r^2]} + O_\sigma(r^{-4}) \tag{3.5}$$

for $Re\,\sigma > 0$. It follows from (2.1) and the results of [12] that the left side of (3.4) is an analytic function of σ for $Re\,\sigma > 0$ except for simple poles at $\sigma = 1, \frac{1}{2}, \frac{1}{2} \pm i\kappa$ with $\frac{1}{4} + \kappa^2$ being taken over all the positive discrete eigenvalues of Δ. Then the right side of (3.4) can be interpreted as an analytic function of σ in the same region by analytic continuation.

It follows from the definition of $k(t)$ and Lemma 3.1 that $c(R)$ is analytic for $Re\,\sigma > 0$ except for simple poles at $\sigma = \frac{1}{2}$. There are only a finite number of elliptic conjugacy classes. The term $c(I)$ is a constant. We have

$$\frac{\varphi'}{\varphi}(s) = 2\ln\pi - \frac{\Gamma'}{\Gamma}(s) - \frac{\Gamma'}{\Gamma}(1-s) - 2\frac{\zeta'}{\zeta}(2s) - 2\frac{\zeta'}{\zeta}(2-2s) \tag{3.6}$$

when $Re\,s = \frac{1}{2}$. By Stirling's formula the identity

$$\frac{\Gamma'}{\Gamma}(z) = \ln z + O(1) \tag{3.7}$$

holds uniformly when $|\arg z| \le \pi - \delta$ for a small positive number δ. The expression (3.3) together with (2.1), (3.5), (3.6) and (3.7) implies that $c(\infty)$ is an analytic function of σ in the half-plane $Re\,\sigma > 0$ except for possible simple poles at $\sigma = 1, \frac{1}{2}, \frac{1}{2} \pm i\kappa$ with $\frac{1}{4} + \kappa^2$ being taken over all the positive discrete eigenvalues of Δ. Therefore the series

$$\sum_{\{P\},\,\Gamma_P \neq \{1_2\}} c(P)$$

represents an analytic function of σ in the half-plane $Re\,\sigma > 0$ except for possible simple poles at $\sigma = 1, \frac{1}{2}, \frac{1}{2} \pm i\kappa$ with $\frac{1}{4} + \kappa^2$ being taken over all the positive discrete eigenvalues of Δ.

4. Proof of the Main Theorem

Lemma 4.1. *We have*

$$4^{\tau}\sqrt{\pi n}\frac{\Gamma(i\kappa)}{\Gamma(\tau)} trT(n) = \lim_{\sigma\to\tau}(\sigma-\tau)\sum_{\{P\},\,\Gamma_P\neq\{1_2\}} c(P)$$

where the right side is defined as in Lemma 3.5.

Proof. It follows from (3.5) and the results of [12] that

$$\lim_{\sigma\to\tau}(\sigma-\tau)\sum_{\ell} h(\ell)t_{\ell} = 0.$$

By (3.5), (3.6) and (3.7), we have

$$\lim_{\sigma\to\tau}(\sigma-\tau)\int_{-\infty}^{\infty} h(r)\frac{\Gamma'}{\Gamma}(1+ir)dr = 0$$

and

$$\lim_{\sigma\to\tau}(\sigma-\tau)\int_{-\infty}^{\infty} h(r)\frac{\varphi'}{\varphi}(\frac{1}{2}+ir)a^{ir}d^{-ir}dr = 0,$$

where $ad = n$ with $d > 0$. The stated identity then follows from (2.2), (3.3), (3.4) and Lemma 3.1. □

A quadratic form $ax^2 + bxy + cy^2$, which is denoted by $[a, b, c]$, is said to be primitive if $(a, b, c) = 1$ and $b^2 - 4ac = d \in \Omega$. Two quadratic forms $[a, b, c]$ and $[a', b', c']$ are equivalent if an element γ of Γ exists such that

$$\begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix} = \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\gamma,$$

where γ^t is the transpose of γ. This relation partitions the quadratic forms into equivalence classes, and two such forms from the same class have the same discriminant. The number of classes h_d of a given discriminant d is finite, and is called the class number of indefinite quadratic forms.

Remark. Siegel [10] proved that

$$\lim_{d\to\infty}\frac{\ln(h_d \ln\epsilon_d)}{\ln d} = \frac{1}{2}. \tag{4.1}$$

Lemma 4.2. *We have*

$$\sum_{\{P\},\,\Gamma_P\neq\{1_2\}} c(P) = c\sqrt{n}\sum_{d\in\Omega,u}\frac{2h_d\ln\epsilon_d}{\sqrt{d}u}(1+\frac{du^2}{4n})^{\frac{1}{2}-\sigma}$$

for $\mathrm{Re}\,\sigma > 1$*, where the summation on* u *is taken over all the positive integers* u *which together with* t *are integral solutions of the equation* $t^2 - du^2 = 4n$*. The series on the right side of the identity converges absolutely for* $\mathrm{Re}\,\sigma > 1$*.*

Proof. Let

$$P = \frac{1}{\sqrt{n}}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a hyperbolic element of Γ^* such that $\Gamma_P \neq \{1_2\}$. Then fixed points r_1, r_2 of P are not cusps of Γ. This implies that Γ_P is the subgroup of Γ consisting of hyperbolic transformations with r_1, r_2 as fixed points. Define $[a, b, c]$ to be the primitive quadratic form such that r_1, r_2 are the roots of the equation $ar^2 + br + c = 0$. By Sarnak [7], the subgroup Γ_P consists of matrices of the form

$$\begin{pmatrix} \frac{v-bu}{2} & -cu \\ au & \frac{v+bu}{2} \end{pmatrix}$$

with $v^2 - du^2 = 4$ and is generated by the primitive hyperbolic element

$$P_0 = \begin{pmatrix} \frac{v_0-bu_0}{2} & -cu_0 \\ au_0 & \frac{v_0+bu_0}{2} \end{pmatrix}$$

where the pair (v_0, u_0) is the fundamental solution of Pell's equation $v^2 - du^2 = 4$. Since P and P_0 have the same fixed points, we have $A = D - bC/a$ and $B = -cC/a$. Since P belongs to Γ^* and $AD - BC = n$, C satisfies the equation

$$\begin{cases} aD^2 - bDC + cC^2 = na \\ a|C. \end{cases} \tag{4.2}$$

Let λ_P be an eigenvalue of P. Then

$$\lambda_P - \frac{1}{\lambda_P} = \pm\frac{C\sqrt{d}}{a\sqrt{n}} \tag{4.3}$$

and

$$\lambda_P + \frac{1}{\lambda_P} = \frac{1}{\sqrt{n}}(2D - \frac{b}{a}C). \tag{4.4}$$

Conversely, let a pair (C, D) be a solution of the equation (4.2). Define

$$A = D - \frac{b}{a}C$$

and

$$B = -\frac{c}{a}C.$$

Then the matrix

$$P = \frac{1}{\sqrt{n}} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has the same fixed points as P_0, and eigenvalues of P satisfies the identity (4.3). We have the decomposition

$$P = \frac{1}{\sqrt{n}} \begin{pmatrix} \frac{n}{(D,C)} & -\frac{c}{a}\alpha C - (D - \frac{b}{a}C)\beta \\ 0 & (D,C) \end{pmatrix} \sigma \cdot \sigma^{-1} \begin{pmatrix} \alpha & \beta \\ C/(D,C) & D/(D,C) \end{pmatrix},$$

where α and β are integers such that

$$\alpha D - \beta C = (D, C)$$

and where $\sigma = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ is chosen so that

$$\begin{pmatrix} n/(D,C) & * \\ 0 & (D,C) \end{pmatrix} \sigma = \begin{pmatrix} n/(D,C) & -s \\ 0 & (D,C) \end{pmatrix}$$

with $0 \leq s < n/(D,C)$. Therefore, P belongs to Γ^*.

Let P_0' be the primitive hyperbolic element of Γ corresponding to $[a', b', c']$. Since the identity

$$\begin{pmatrix} v_0/2 & 0 \\ 0 & v_0/2 \end{pmatrix} + u_0 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma^t \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \gamma = \gamma^{-1} P_0 \gamma$$

holds for every element γ of Γ, two forms $[a, b, c]$ and $[a', b', c']$ of the same discriminant are equivalent if, and only if, an element γ of Γ exists such that $\gamma^{-1} P_0 \gamma = P_0'$. For a given discriminant d, let C be a solution of the equation (4.2). Then the quotient $C/|a|$ depends only on the equivalence classes $\{[a, b, c]\}$ by the identity (4.3). Let $\mathcal{F}$ be the set of all the pairs $(\{P_0\}, C)$, where d belongs to Ω and where C satisfies the equation (4.2) with $[a, b, c]$ being a representative of its equivalence class. Then an one-to-one correspondence exists between $\mathcal{F}$ and the set of all conjugacy classes of hyperbolic elements of Γ^* whose fixed points are not cusps of Γ.

It follows from (2.1) and Lemma 3.2 that

$$\sum_{\{P\},\, \Gamma_P \neq \{1_2\}} c(P) = c\sqrt{n} \sum \frac{2 \ln \epsilon_d}{\sqrt{d} \frac{|C|}{|a|}} \left(1 + \frac{dC^2}{4na^2}\right)^{\frac{1}{2} - \sigma}$$

where the summation is taken over the set of all the pairs $(\{P_0\}, C)$ in $\mathcal{F}$. Define $v = 2D - b\frac{C}{a}$ and $u = \frac{C}{a}$. Then the equation (4.2) becomes

$$v^2 - du^2 = 4n. \tag{4.5}$$

Moreover, we have

$$\frac{1}{\sqrt{n}}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{\sqrt{n}}\begin{pmatrix} \frac{v-bu}{2} & -cu \\ au & \frac{v+bu}{2} \end{pmatrix}. \tag{4.6}$$

It follows that the stated identity holds.

Two solutions (v,u) and (v',u') of the equation (4.5) are said to be equivalent if

$$v' + u'\sqrt{d} = (v + u\sqrt{d})\epsilon_d^q$$

for some integer q. This relation partitions all the solutions of (4.5) into equivalence classes. Denote by J_d the number of such equivalence classes. Assume that $T_1 = \frac{1}{\sqrt{n}}\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\gamma_1$ and $T_2 = \frac{1}{\sqrt{n}}\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\gamma_2$ with γ_i in Γ are of the form (4.6). Since T_1 and T_2 have the same two distinct fixed points, they commute with each other. This implies that $\gamma_1^{-1}\gamma_2$ commutes with T_1. It follows that $\gamma_1^{-1}\gamma_2$ is a power of the primitive hyperbolic element P_0. Since T_1, T_2 and P_0 can be diagonalized simultaneously, by using (4.3) and (4.4) we see that the eigenvalue of T_1 differs from that of T_2 by a power factor of ϵ_d. Therefore, for fixed integers a, b, d with $ad = n$, $0 \leqslant b < d$, the eigenvalues of elements in Γ^* of the form $\frac{1}{\sqrt{n}}\left(\begin{smallmatrix} d & -b \\ 0 & a \end{smallmatrix}\right)\gamma$, $\gamma \in \Gamma$, and of the form (4.6) corresponds to one equivalence class of solutions of (4.5). It follows from the statement given at the beginning of the second paragraph in section 2 that $J_d \le \sum_{c|n} c$. An unique solution (v_j, u_j) of the equation (4.5) with $v_j, u_j > 0$ exists in each equivalence class such that $2\sqrt{n}\lambda_j = v_j + u_j\sqrt{d}$ is the smallest among all the positive solutions of the equivalence class for $j = 1, \cdots, J_d$. Then all the solutions of the equation (4.5) with $v, u > 0$ are given by

$$\begin{cases} \frac{u\sqrt{d}}{\sqrt{n}} = \lambda_j\epsilon_d^q - \lambda_j^{-1}\epsilon_d^{-q} \\ \frac{v}{\sqrt{n}} = \lambda_j\epsilon_d^q + \lambda_j^{-1}\epsilon_d^{-q} \end{cases} \tag{4.7}$$

for all nonnegative integers q and for $j = 1, \cdots, J_d$. It follows from (4.1) that

$$\sum_{d\in\Omega,u} \frac{h_d \ln \epsilon_d}{\sqrt{d}u}(1 + \frac{du^2}{4n})^{\frac{1}{2}-\sigma} \ll \sum_{d\in\Omega,u} d^{\frac{1}{2}+\epsilon-\sigma}u^{-2\sigma} = S_1 + S_2$$

for a small positive number ϵ when $\sigma > 1$, where

$$S_1 = \sum_{d\in\Omega,u<\sqrt{d}} (du^2)^{\frac{1}{2}+\epsilon-\sigma}u^{-1-2\epsilon}$$

and

$$S_2 = \sum_{d\in\Omega,u>\sqrt{d}} d^{\frac{1}{2}+\epsilon-\sigma}u^{-2\sigma}.$$

If two distinct integers d and d' of Ω are square free, then the equation (4.5) implies that

$$\min_{1\le j\le J_d} v_j \neq \min_{1\le j\le J_{d'}} v'_j$$

where v'_j, $j = 1, \cdots, J_{d'}$, correspond the equation ${v'}^2 - d'{u'}^2 = 4n$ similarly as the above. Then the inequality

$$S_1 \ll \sum_{\substack{d\in\Omega \\ d\,\text{square free}}} (\min_{1\le j\le J_d} v_j)^{1+2\epsilon-2\sigma} \sum_{j=1}^{J_d}\sum_{l=1}^{\infty}(l^2u_j)^{-1-2\epsilon} \ll \sum_{v=1}^{\infty} v^{1+2\epsilon-2\sigma} < \infty$$

holds for $\sigma > 1$. Since $J_d \le \sum_{d|n} d$, it follows from (4.7) that

$$S_2 \ll \sum_{d=1}^{\infty} d^{\frac{1}{2}+\epsilon-2\sigma} < \infty$$

for $\sigma > 1$. Therefore the series on the right side of the stated identity converges absolutely for $Re\,\sigma > 1$. □

Proof of the Main Theorem. An argument similar to the estimation of terms S_1 and S_2 in the proof of Lemma 4.2 shows that

$$\sum_{d\in\Omega,u} \frac{h_d \ln \epsilon_d}{\sqrt{d}u}(1+\frac{du^2}{4n})^{\frac{1}{2}-\sigma} - \sum_{d\in\Omega,u}\frac{h_d\ln\epsilon_d}{(du^2)^\sigma} \ll \sum_{d\in\Omega,u}\frac{(du^2)^{\frac{1}{2}+\epsilon-1-\sigma}}{u^{1+2\epsilon}} < \infty$$

for $\sigma > 0$. It follows from Lemma 4.2 that

$$\lim_{\sigma\to\tau}(\sigma-\tau)\sum_{\{P\},\,\Gamma_P\neq\{1_2\}} c(P) = \frac{(4\pi)^{\frac{1}{2}}\Gamma(i\kappa)}{(4n)^{-\tau}\Gamma(\tau)}\lim_{\sigma\to\tau}(\sigma-\tau)\sum_{d\in\Omega}\frac{h_d\ln\epsilon_d}{d^\sigma}\sum_u u^{-2\sigma}. \quad (4.8)$$

Lemma 3.5 shows that the function on the right side of (4.8) represents an analytic function of σ in the half-plane $Re\,\sigma > 0$ except for possible simple poles at $\sigma = 1, \frac{1}{2}, \frac{1}{2} \pm i\kappa$ with $\frac{1}{4}+\kappa^2$ being taken over all the positive discrete eigenvalues of Δ. Then the stated identity follows from Lemma 4.1.

This completes the proof of the theorem.

Remark. It is conjectured that $|\mathrm{tr}T(p)| \leqslant 2$ for every prime number p.

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