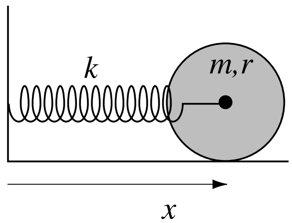


Physics 24

PS20, 14 April 2003

Problem 1 – Harrison's Clocks To win the Longitude Prize required an accuracy of better than 30 miles on a trans-Atlantic journey from London to the Caribbean. London is on the Prime Meridian (longitude of 0°), and at 51° north latitude; the latitude and longitude of Barbados, not far from the South American coast in the Caribbean, are about 13° and 60°, respectively. Such a journey typically took about 90 days.

(a) What relative accuracy must a clock have to qualify for the Prize? That is, when the clock records an interval of time t, what is the largest permissible discrepancy δt between the clock reading and the actual interval of elapsed time? Note that 360° of longitude correspond to a day of 24 hours.

(b) To illustrate why temperature compensation is important, suppose that the timekeeper is based on a gravitational pendulum that is 1 m in length and made of brass. The thermal expansion coefficient of brass is $\eta = 1.89 \times 10^{-5}\ {}^\circ\text{C}^{-1}$. This means that for a temperature rise of 1°C the length of a 1-m rod will increase by 1.89×10^{-5} m = 18.9 μm. Suppose the ship leaves London in the spring with its pendulum perfectly adjusted to beat time in London. *Estimate* the magnitude of the error in the reading of the clock caused by temperature variation over the length of the voyage to Barbados. Please explain the approximations you make, and neglect all sources of error besides temperature.

Problem 2 – Amplitude dependence of a pendulum (15 points)

The period of a gravitational pendulum depends on the amplitude of oscillation (for finite amplitude of oscillation). In this problem we will derive an integral for computing this amplitude dependence $T(\theta_0)$, and perform the integration numerically.[1]

[1] It is possible to use analytic methods to find a series that gives the amplitude dependence of the period. The result is

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \left(\frac{1}{2}\right)^2 \sin^2\theta_0/2 + \left(\frac{3\times 1}{4\times 2}\right)^2 \sin^4\theta_0/2 + \cdots\right]$$

You will be modifying a short Java program for this assignment. There is a CodeWarrior project file prepared for downloading at

http://saeta.physics.hmc.edu/courses/p24/Pendulum.zip

Inside the project you will find a working program that has the necessary structure. You need to edit a few functions in the file to make the program calculate the amplitude dependence (as described below).

(a) Starting from energy considerations, show that the angular velocity of the pendulum may be expressed

$$\frac{d\theta}{dt} = \sqrt{\frac{2g}{L}}\sqrt{\cos\theta - \cos\theta_0} \tag{1}$$

where L is the length of the (simple) pendulum.

(b) Separating variables we obtain an integral to give the period:

$$\int dt = \sqrt{\frac{L}{2g}}\int \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{2}$$

One quarter of the period is given by integrating from 0 to θ_0, so

$$T = 4\sqrt{\frac{L}{2g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \tag{3}$$

All we have to do is evaluate this integral for different values of amplitude and prepare a plot of $T(\theta_0)$. However, there is a problem. Although Eq. (3) is perfectly correct and well-behaved analytically, the integrand diverges (in an integrable way) at one limit of integration. Following the great mathematician Legendre, we will convert this integral to a standard form using a trigonometric identity,

$$\cos\theta - \cos\theta_0 = 2\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right) \tag{4}$$

and a substitution,

$$\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin\phi \tag{5}$$

where ϕ is our new integration variable, to re-express Eq. (3) in the form

$$\boxed{T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2\frac{\theta_0}{2}\sin^2\phi}}} \tag{6}$$

The great advantage of Legendre's way of writing the integral is that the integrand is perfectly well behaved everywhere. We still have to do it numerically, but at least we won't have any divergences to worry about (except at $\theta_0 = \pi$).

Your task is to write a computer program in Java, C, C++, or similar programming language to evaluate the period for various values of the amplitude θ_0. The most convenient way to study the results is to compare the finite amplitude period to its limiting value T_0 as $\theta_0 \to 0$. If you wish, you may write your program in a numerical package such as MatLab, but your program should not use a built-in integrator.

What you should calculate:

- Prepare a table of values of $T(\theta_0)/T_0$ that are accurate to 0.01% for the values

$$\theta_0 = \{1^\circ, 10^\circ, 20^\circ, 30^\circ, 40^\circ, 60^\circ, 90^\circ, 170^\circ\}$$

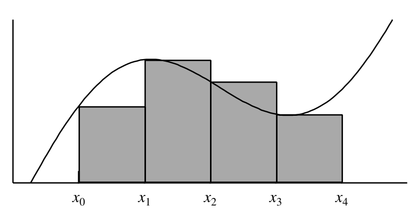
Include in your table as well the number of steps N required to obtain an accuracy of 0.01%.

- Also prepare a plot of $T(\theta_0)/T_0$ for $0 \leq \theta_0 \leq 170^\circ$. You may use Kaleidagraph, Origin, gnuplot, or other plotting package to prepare your plot.
- Include a listing of your computer code with your solution to this problem.

(c) Suppose that Harrison had wanted to use a pendulum clock to solve the Longitude problem. The pendulum of such a clock would have to swing with a finite amplitude, which we will assume is 10°. Using your plot or your computer program, estimate the largest (systematic) variation in the amplitude of the motion of the pendulum allowed by the terms of the Longitude Act. That is, by how much must the amplitude change from 10° before the change in period is too large to win the Prize. (See the previous problem for details of the Longitude Act.)

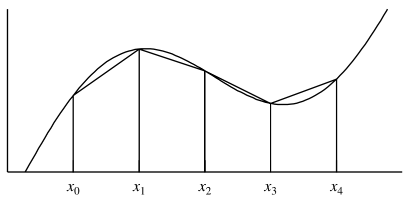
Numerical Integration

A simple way to conceive of an integral is as a Riemann sum, as suggested in the adjacent figure. For simplicity, we divide the interval between $a = x_0$ and $b = x_4$ into N equal subranges, and sum the area of each rectangle of width $h = (b - a)/N$ and height $f(x_n)$, where $n = 0 \ldots N - 1$. That is, we use the expression



$$\int_a^b f(x)\,dx \approx h \sum_{n=0}^{N-1} f(x_n) \qquad (7)$$

where the positions x_n are given by $x_n = a + nh$. This method is somewhat similar to the Euler method for integrating differential equations that was discussed in Physics 23. In principle, as $N \to \infty$ Eq. (7) gives the exact value of the definite integral. (Unfortunately, infinity is a long way away.)



A somewhat better approach is to use the trapezoid rule, which represents the integrand with a bunch of line segments, as illustrated in the figure. The trapezoids do a better job of following the smooth curve, and lead to a more accurate evaluation of the integral for given step size h. The trapezoid rule corresponds to the sum

$$\int_a^b f(x)\,dx \approx h\left(\tfrac{1}{2}f_0 + f_1 + f_2 + \cdots + f_{N-2} + f_{N-1} + \tfrac{1}{2}f_N\right) \tag{8}$$

where the step size h is $(b-a)/N$ and $f_n = f(a+nh)$.

One step better is to use parabolas instead of lines, which gives Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{N-2} + 4f_{N-1} + f_N\right) \tag{9}$$

where the 4's and 2's alternate on the interior points. These ideas are discussed at greater length in *Motion I* (pp. 11-30, -32, and -33).

For a given step size, Simpson's method is more accurate than either of the other two methods. Regardless of the method you choose, you can estimate the accuracy of your result by running the same integration a second time with a step size half as big. The amount by which the result changes gives an estimate of the error. You should obtain results that are accurate to 0.01%.

A variety of more sophisticated formulas are possible, but unnecessary for this problem. Curious students may consult a numerical analysis text, such as *A First Course in Computational Physics*, by Paul L. DeVries.