# Math 362: Real and Abstract Analysis
# College of the Holy Cross, Spring 2005
# Notes on Taylor's Theorem

Here are two versions of Taylor's Theorem for multivariable functions.

**Theorem 1.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be $k$-times differentiable. Then*

$$f(x) = f(p) + Df_p(x-p) + \frac{1}{2}D^2 f_p(x-p, x-p) + \cdots + \frac{1}{k!}D^k f_p(x-p, x-p, \ldots, x-p) + R(x),$$

*where*

$$\lim_{x \to p} \frac{R(x)}{|x-p|^k} = 0.$$

*Proof.* We'll just prove the case $k = 2$. So write

$$f(x) = f(p) + Df_p(x-p) + \frac{1}{2}D^2 f_p(x-p, x-p) + R(x)$$

Then we have $R(p) = 0$, $DR(p) = 0$ and $D^2R(p) = 0$. By the Mean Value Theorem,

$$|R(x)| = |R(x) - R(p)| \leq M|x-p|$$

where $M = \sup\{\|DR(y)\| : y = p + t(x-p), 0 \leq t \leq 1\}$. Now

$$DR(y) = DR(p) + D^2R(p)(y-p) + R_2(y) \quad \text{where} \quad \lim_{y \to p} \frac{R_2(y)}{|y-p|} = 0.$$

Given $\epsilon > 0$, there is some $\delta > 0$ such that

$$|y-p| < \delta \quad \Longrightarrow \quad \frac{R_2(y)}{|y-p|} < \epsilon.$$

Since $DR(p) = 0$ and $D^2R(p) = 0$, we have $DR(y) = R_2(y)$. Thus $|x-p| < \delta$ implies $|y-p| < \delta$, which implies

$$\frac{\|DR(y)\|}{|y-p|} < \epsilon$$

so $\|DR(y)\| \leq \epsilon|y-p| \leq \epsilon|x-p|$ and thus $M \leq \epsilon|x-p|$. It then follows that $|R(x)| \leq \epsilon|x-p|^2$ whenever $|x-p| < \delta$. Hence

$$\lim_{x \to p} \frac{R(x)}{|x-p|^2} = 0$$

□

Another version of Taylor's Theorem which gives a precise form of the remainder follows from the $C^1$ Mean Value Theorem.

**Theorem 2.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be twice differentiable. Then*

$$f(x) = f(p) + Df(p)(x-p) + \beta(x-p, x-p)$$

*where*

$$\beta = \int_0^1 \int_0^1 D^2 f(p + st(x-p))\, dst\, dt.$$

*Proof.* By the $C^1$ Mean Value Theorem

$$f(x) = f(p) + \int_0^1 Df(p + t(x-p))\, dt \cdot (x-p).$$

Now apply the $C^1$ Mean Value Theorem to $Df$ to get

$$Df(y) = Df(p) + \int_0^1 D^2 f(p + s(y-p))\, ds(y-p)$$

Applying this with $y = p + t(x-p)$ gives

$$Df(p + t(x-p)) = Df(p) + \int_0^1 D^2 f(p + st(x-p))\, ds \cdot t(y-p)$$

so substituting for the integrand in the expression for $f(x)$ above proves the theorem. □