Some basic statistical concepts

- Describing sample data
  - Random samples
  - Sample mean, variance, standard deviation
  - Populations versus samples
  - Population mean, variance, standard deviation
  - Estimating parameters
- Simple *comparative* experiments
  - The hypothesis testing framework
  - The two-sample *t*-test
  - Checking assumptions, validity

An Example:

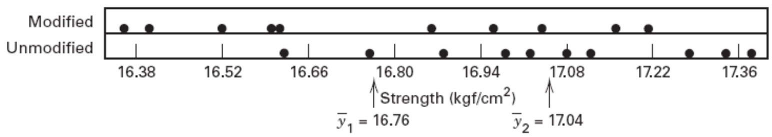
- Portland cement formulation
  - See data, page 24

TABLE 2.1
Tension Bond Strength Data for the Portland Cement Formulation Experiment

j	Modified Mortar y_{1j}	Unmodified Mortar y_{2j}
1	16.85	16.62
2	16.40	16.75
3	17.21	17.37
4	16.35	17.12
5	16.52	16.98
6	17.04	16.87
7	16.96	17.34
8	17.15	17.02
9	16.59	17.08
10	16.57	17.27

Graphical view of the data

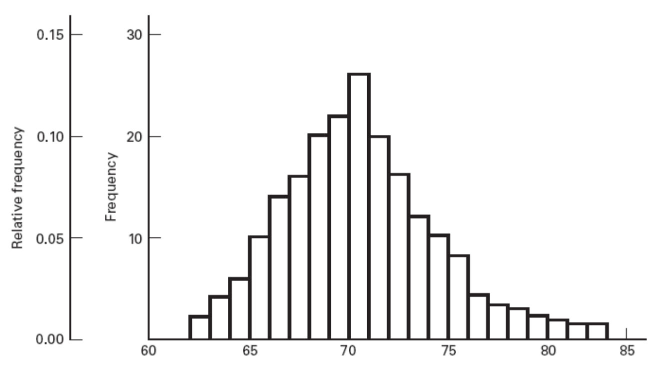
- Dot diagram
  - fig. 2.1, pp. 24



■ **FIGURE 2.1** Dot diagram for the tension bond strength data in Table 2.1

- What do you see?

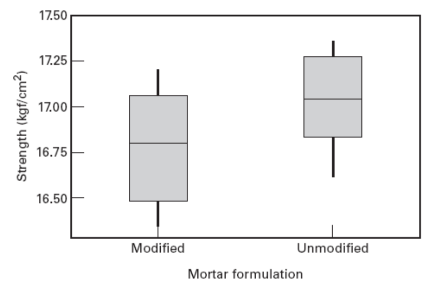
If you have a large sample, a histogram may be useful



■ **FIGURE 2.2** Histogram for 200 observations on metal recovery (yield) from a smelting process

Another graphical comparison

- Box plots
  - Fig. 2.3, pp. 26



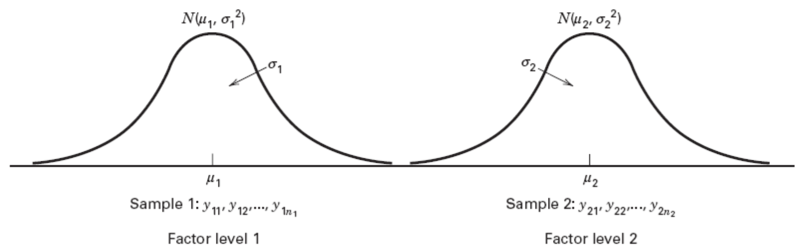
■ **FIGURE 2.3** Box plots for the Portland cement tension bond strength experiment

- What do you see here?

The hypothesis testing framework

- *Statistical hypothesis testing* is a useful framework for many experimental situations
- Origins of the methodology date from the early 1900s
- For the Portland cement example, we will use the *two-sample t-test*

The hypothesis testing framework



■ **FIGURE 2.9** The sampling situation for the two-sample *t*-test

- Sampling from a normal distribution
- Statistical hypotheses: $H_0 : \mu_1 = \mu_2$

  $H_1 : \mu_1 \neq \mu_2$

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Estimation of parameters

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \text{ estimates the population mean } \mu$$

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2 \text{ estimates the variance } \sigma^2$$

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Portland cement example …

- Summary statistics (see pg. 36)

Modified Mortar	Unmodified Mortar
$\bar{y}_1 = 16.76$ $S_1^2 = 0.100$ $S_1 = 0.316$ $n_1 = 10$	$\bar{y}_2 = 17.04$ $S_2^2 = 0.061$ $S_2 = 0.248$ $n_2 = 10$

The two-sample *t*-test:

- Use the sample mean to draw inferences about the population means …

  $\bar{y}_1 - \bar{y}_2 =$ ______________ ______________ _

- Comparison is …

  $$\frac{\text{differences in means}}{\text{standard deviation of the difference in the means}},$$

  where $\sigma_{\bar{y}}^2 = \dfrac{\sigma^2}{n}$

- This suggests a statistic, $Z_0 = \dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$

The two-sample *t*-test:

Use S_1^2 and S_2^2 to estimate σ_1^2 and σ_2^2

The previous ratio becomes $\dfrac{\bar{y}_1 - \bar{y}_2}{\sqrt{\dfrac{S_1^2}{n_1} + \dfrac{S_2^2}{n_2}}}$

However, we have the case where $\sigma_1^2 = \sigma_2^2 = \sigma^2$

Pool the individual sample variances:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

The two-sample *t*-test:

The test statistic is

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$$

- Values of t_0 that are near zero are consistent with the null hypothesis
- Values of t_0 that are very different from zero are consistent with the alternative hypothesis
- t_0 is a “distance” measure-how far apart the averages are expressed in standard deviation units
- Note: this is an interpretation of t_0 as a *signal-to-noise* ratio

For the Portland cement example

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2} = \frac{9(0.100) + 9(0.061)}{10 + 10 - 2} = 0.081$$

$$S_p = 0.284$$

$$t_0 = \frac{\bar{y}_1 - \bar{y}_2}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{16.76 - 17.04}{0.284\sqrt{\frac{1}{10} + \frac{1}{10}}} = -2.20$$

The two sample means are a little over two standard deviations apart
Is this a "large" difference?

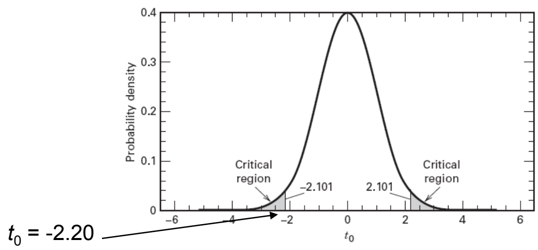
The two-sample (pooled) t-test

- So far, we haven't really done any "statistics"
- We need an objective basis for deciding how large the test statistic t_0 really is
  - In 1908, W. S. Gosset derived the *reference distribution* for t_0 … called the *t* distribution
  - Tables of the *t* distribution – see Appendix II, pg. 614
- For our example, with $v = n_1 + n_2 - 2 = 18$ degrees of freedom and α level of 0.05 ($\alpha/2 = 0.025$),

t_{crit} = ____________________

The two-sample (pooled) *t*-test

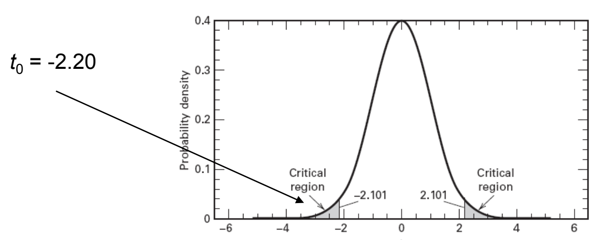
- A value of t_0 between –2.101 and 2.101 is consistent with equality of means
  - It is possible for the means to be equal and t_0 to exceed either 2.101 or –2.101. but it would be a "rare event"



■ **FIGURE 2.10** The *t* distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

  - leads to the conclusion that the means are different
- Note: we could also use the *P*-value approach

The two-sample (pooled) *t*-test



■ **FIGURE 2.10** The *t* distribution with 18 degrees of freedom with the critical region $\pm t_{0.025,18} = \pm 2.101$

- The *P*-value is the area (probability) in the tails of the *t*-distribution beyond -2.20 + the probability beyond +2.20 (it's a two-sided test)
- The *P*-value is a measure of how unusual the value of the test statistic is given that the null hypothesis is true
- The *P*-value the risk of *wrongly rejecting* the null hypothesis of equal means (it measures rareness of the event)
- The *P*-value in our problem is $P = 0.042$

Computer two-sample *t*-test results

t-Test: Two-Sample Assuming Equal Variances

	Modified	*Unmodified*
Mean	16.764	17.042
Variance	0.100138	0.061462222
Observations	10	10
Pooled Variance	0.0808	
Hypothesized Mean Difference	0	
df	18	
t Stat	-2.18688	
P(T<=t) one-tail	0.021098	
t Critical one-tail	1.734064	
P(T<=t) two-tail	0.042197	
t Critical two-tail	2.100922	

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Welcome to Minitab, press F1 for help.
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Two-Sample T-Test and CI: Modified, Unmodified

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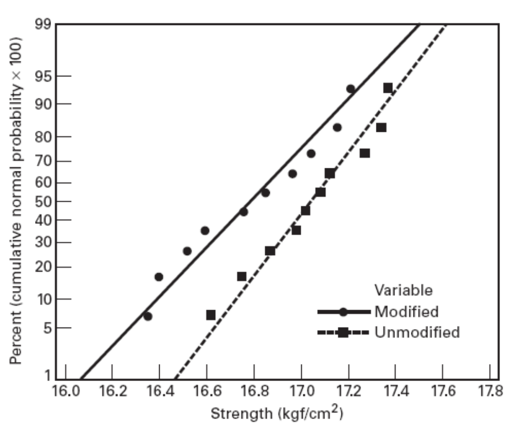
Two-sample T for Modified vs Unmodified

             N    Mean  StDev  SE Mean
Modified    10  16.764  0.316     0.10
Unmodified  10  17.042  0.248    0.078

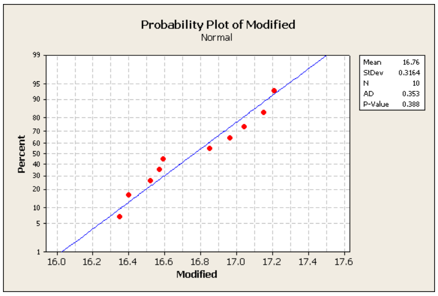

Difference = mu (Modified) - mu (Unmodified)
Estimate for difference:  -0.278
95% CI for difference:  (-0.545, -0.011)
T-Test of difference = 0 (vs not =): T-Value = -2.19  P-Value = 0.042  DF = 18
Both use Pooled StDev = 0.2843
```

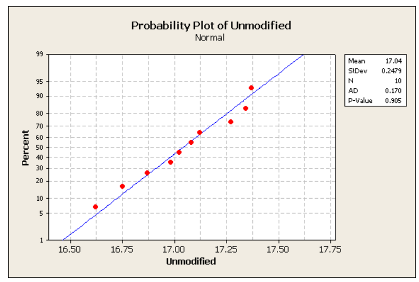

Checking assumptions – the normal probability plot



■ **FIGURE 2.11** Normal probability plots of tension bond strength in the Portland cement experiment

Normality testing in Minitab





Importance of the *t*-Test

- Provides an *objective* framework for simple comparative experiments
- Could be used to test all relevant hypotheses in a two-level factorial design, because all of these hypotheses involve the mean response at one “side” of the cube versus the mean response at the opposite “side” of the cube

Confidence intervals (See pg. 44)

- Hypothesis testing gives an objective statement concerning the difference in means, but it doesn't specify "how different" they are
- *General form* of a confidence interval

$$L \le \theta \le U \text{ where } P(L \le \theta \le U) = 1 - \alpha$$

- The 100(1- α)% *confidence interval* on the difference in two means:

$$\bar{y}_1 - \bar{y}_2 - t_{\alpha/2, n_1+n_2-2} S_p \sqrt{(1/n_1) + (1/n_2)} \le \mu_1 - \mu_2 \le \bar{y}_1 - \bar{y}_2 + t_{\alpha/2, n_1+n_2-2} S_p \sqrt{(1/n_1) + (1/n_2)}$$

The Portland cement example

(see pg. 45)

The actual 95 percent confidence interval estimate for the difference in mean tension bond strength for the formulations of Portland cement mortar is found by substituting in Equation 2.30 as follows:

$$16.76 - 17.04 - (2.101)0.284\sqrt{\tfrac{1}{10} + \tfrac{1}{10}} \le \mu_1 - \mu_2 \le 16.76 - 17.04 + (2.101)0.284\sqrt{\tfrac{1}{10} + \tfrac{1}{10}}$$

$$-0.28 - 0.27 \le \mu_1 - \mu_2 \le -0.28 + 0.27$$

$$-0.55 \le \mu_1 - \mu_2 \le -0.01$$

Thus, the 95 percent confidence interval estimate on the difference in means extends from -0.55 to -0.01 kgf/cm^2. Put another way, the confidence interval is $\mu_1 - \mu_2 = -0.28 \pm 0.27$ kgf/cm^2, or the difference in mean strengths is -0.28 kgf/cm^2, and the accuracy of this estimate is ± 0.27 kgf/cm^2. Note that because $\mu_1 - \mu_2 = 0$ is *not* included in this interval, the data do not support the hypothesis that $\mu_1 = \mu_2$ at the 5 percent level of significance (recall that the *P*-value for the two-sample *t*-test was 0.042, just slightly less than 0.05). It is likely that the mean strength of the unmodified formulation exceeds the mean strength of the modified formulation. Notice from Table 2.2 that both Minitab and JMP reported this confidence interval when the hypothesis testing procedure was conducted.

Other chapter topics

- Hypothesis testing when the variances are known
- One sample inference
- Hypothesis tests on variances
- Paired experiments

Homework (due Tuesday, Sept. 1)

- Use Minitab or Excel to solve the following:
  - 2.8
  - 2.15
  - 2.19
    - include a test of normality for the data
- Helpful hints:
  - An Excel data set is available through the textbook website (www.wiley.com/college/montgomery)
  - Turn in a printout of your results, appropriately formatted and annotated.