Nonlinearity Index of the Cayley Form¹

Andrew J. Sinclair,² John E. Hurtado,³ and John L. Junkins⁴

Abstract

The nonlinearity index is a measure of the nonlinearity of dynamical systems based on computing the initial-condition sensitivity of the state-transition matrix. The Cayley form is a representation for dynamical systems that relates their motion to *N*-dimensional rotations. The generalized coordinates of the system are used to define an *N*-dimensional orientation, and a set of quasi velocities is defined equal to the corresponding angular velocity. The non-linearity index of the Cayley-form representation is computed for an elastic spherical pendulum and a planar satellite example. These results are compared to values for alternative dynamical representations. Additionally, the nonlinearity is evaluated by analyzing how well the linearized equations of each representation capture certain properties of the motion. These results show that the Cayley form can have lower nonlinearity than traditional representations, in particular those representations that suffer from kinematic singularities.

Introduction

Much of the development in dynamics and control has focused on varying representations for physical systems. An example of this is the Cayley form, which describes dynamical systems using the kinematic and dynamic equations of *N*-dimensional rotations [1-3]. This representation defines a new set of quasi velocities, the Cayley quasi velocities, which are equivalent to the angular velocity of some associated rotational motion, and the generalized coordinates are equated to the extended Rodrigues parameters [4, 5] of that rotational motion.

Previous work has presented applications of the Cayley form for developing representations of system dynamics and for designing feedback controllers [3, 6]. In that work it was significant that the Cayley form produces coupled system representations. In reference [3] it was noted that the resulting equations of motion from the Cayley form can be more complicated than alternative methods that produce

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²Assistant Professor, Department of Aerospace Engineering, Auburn University, 211 Aerospace Engineering Building, Auburn University, AL 36849-5338. E-mail: sinclair@auburn.edu.

³Assistant Professor, Department of Aerospace Engineering, Texas A&M University, TAMU-3141, College Station, TX 77843-3141. E-mail: jehurtado@tamu.edu.

⁴Distinguished Professor, Department of Aerospace Engineering, Texas A&M University, TAMU-3141, College Station, TX 77843-3141. E-mail: junkins@tamu.edu.

decoupled equations of motion, and in reference [6] numerical simulation results indicated that designing controllers based on coupled representations can in some cases result in superior performance. These issues motivate a desire to analyze the Cayley form to quantitatively measure these properties related to complexity and coupling.

The Cayley form, of course, is just one example of the generally infinite possibilities for representing dynamic systems. Due to the broad variety of system representations, the idea of comparing different representations of a physical system (or indeed, representations of different physical systems) is not a new one. Methods have been developed to analyze and compare system representations. One of these is the nonlinearity index developed by Junkins [7] and Junkins and Singla [8]. This index applies to a particular initial condition of a particular problem. The results, however, provide illustrative insight into the possible behaviors of the Cayley form. In this paper the nonlinearity index of the Cayley form and alternative representations are computed for two sample problems. First, however, the definitions of the Cayley form and the nonlinearity index are reviewed.

Cayley Form

Rotations in N dimensions are described by a proper orthogonal matrix $C \in SO(N)$ called a rotation matrix. These rotation matrices can be related to a skew-symmetric representation, Q, by the Cayley transform [9]

Forward
$$C = (I - Q)(I + Q)^{-1} = (I + Q)^{-1}(I - Q)$$
 (1)

Inverse
$$Q = (I - C)(I + C)^{-1} = (I + C)^{-1}(I - C)$$
 (2)

Here, *Q* is an $N \times N$ skew-symmetric matrix, and *I* is the identity matrix. The matrix *Q* comprises a set of M = N(N - 1)/2 distinct parameters whose values vary from $+\infty$ to $-\infty$. These parameters represent the orientation of an *N*-dimensional reference frame and are called the extended Rodrigues parameters (ERPs) [4, 5]. The kinematics of these ERPs are related to the *N*-dimensional angular velocity through the Cayley-transform kinematic relationships [10].

$$\Omega = 2(I+Q)^{-1}\dot{Q}(I-Q)^{-1}$$
(3)

$$\dot{Q} = \frac{1}{2}(I+Q)\Omega(I-Q) \tag{4}$$

In the Cayley form, these equations are used as definitions for the Cayley quasivelocities, Ω , related to the generalized coordinates and velocities, Q and \dot{Q} , of a system. Of course, traditionally these motion variables are represented in vector forms: $\boldsymbol{\omega}$, \mathbf{q} , and $\dot{\mathbf{q}}$. The mapping between the skew-symmetric matrix form and vector forms is performed by the relative numerical tensor χ [1]. For N = 3, χ simplifies to the Levi-Civita permutation symbol.

The equations of motion for the Cayley quasi velocities are given by the *N*-dimensional rotational dynamics [1, 2]. These are shown below in index notation for the vector form of the motion variables.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial\omega_k}\right) + \frac{1}{2}\chi_{ij}^r(\chi_{ic}^k\Omega_{cj} - \chi_{cj}^k\Omega_{ic})\left(\frac{\partial T}{\partial\omega_r}\right) = A_{rk}\left(f_r + \frac{\partial T}{\partial q_r}\right) \tag{5}$$

Here, T is the kinetic energy as a function of the generalized coordinates and angular velocity, and **f** are the generalized forces and include potential and nonpotential

forces. Also, A is the linear mapping from the angular-velocity vector $\boldsymbol{\omega}$ to the generalized velocities $\dot{\mathbf{q}}$ and is derived from equation (4) [2].

$$A_{ij} = \frac{1}{2} \left(\delta_{ij} - \chi^i_{\nu p} \chi^j_{\nu l} Q_{lp} - \frac{1}{2} \chi^i_{\nu p} \chi^j_{kl} Q_{\nu k} Q_{lp} \right)$$
(6)

Nonlinearity Index

Generally, infinite possibilities exist for coordinate choices to represent any given physical system. Much of the history of analytical mechanics has spawned from the development of new coordinate choices. A typical approach defines one set of *position-level coordinates* to describe the configuration of the system and a second set of *velocity-level coordinates* to describe the evolution of that configuration. One issue that affects the choice of position-level coordinates is the presence of singularities, e.g., configurations that can not be described by a particular set of coordinates or configurations for which the coordinates are undefined. A classic example of this is the variety of popular choices for representing the orientation of a rigid body. Choices for velocity-level coordinates (such as the Cayley form) generally provide canonical representations for the dynamics of broad classes of problems. Examples of this are the conjugate momenta and quasi velocities. Of course, alternatives to the split position- and velocity-level coordinates also exist, such as the classic orbital elements that describe both the position and velocity of a space-craft in a single set of variables.

Along with the issues of singularities and canonical representation, another issue related to coordinate choice is the linearity or nonlinearity of the resulting dynamical system. Of course, linear equations are desirable, and when working with non-linear equations it can be useful to define exactly how nonlinear the system is. Lower nonlinearity can result in improved performance in the application of linear control and estimation methods. One approach to determine the amount of nonlinearity is the nonlinearity index developed by Junkins, which provides a measure for the nonlinearity of a dynamical system and a particular initial condition. Consider the dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}); \qquad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{7}$$

The state vector \mathbf{x} consists of both position-level and velocity-level coordinates. The first-order sensitivity of the trajectory to the initial conditions is described by the state-transition matrix

$$\Phi(t, t_0) = \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}(t_0)}$$
(8)

The state-transition matrix satisfies the differential equation

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$$\dot{\Phi}(t, t_0) = F\Phi(t, t_0); \qquad F = \frac{\partial \mathbf{f}(t, \mathbf{x})}{\partial \mathbf{x}(t)}; \qquad \Phi(t_0, t_0) = I$$
(9)

For a general linear system the Jacobian matrix *F* can be a function of time. For a linear autonomous system the Jacobian matrix is constant. Therefore, for this type of system $\Phi(t, t_0)$ is independent of the initial condition $\mathbf{x}(t_0)$. In other words, the state-transition matrix $\Phi(t, t_0)$ evaluated along a nominal trajectory with initial condition $\mathbf{x}(t_0)$ will be exactly equal to the state-transition matrix $\Phi(t, t_0)$ evaluated along any neighboring trajectory with initial condition $\mathbf{x}(t_0)$. This suggests using the

magnitude of the difference between state-transition matrices evaluated along neighboring trajectories as a measure of nonlinearity. In particular the following nonlinearity index was suggested by Junkins [7] and Junkins and Singla [8]

$$\nu(t, t_0) \equiv \sup_{i=1...n} \frac{\|\Phi_i(t, t_0) - \Phi(t, t_0)\|_F}{\|\bar{\Phi}(t, t_0)\|_F}$$
(10)

Here, $\Phi_i(t, t_0)$ is the state-transition matrix evaluated along the trajectory corresponding to the *i*th initial condition from a family of *n* neighboring initial conditions, and $|| ||_F$ indicates the Frobenius norm. It is noteworthy that the nonlinearity index is sensitive to the size of the neighborhood from which initial conditions are selected. The index provides the greatest insight when the size of the neighborhood is selected to match the worst-case variation or expected perturbation for a particular control or estimation problem [7, 8].

The selection of the neighboring initial conditions is clearly an important issue in computing the nonlinearity index for any system and nominal initial condition. In the following study, this selection was performed using a method suggested by Junkins and Singla of populating an *M*-dimensional sphere surrounding the nominal initial condition in state space [8]. The initial conditions are found by distributing points approximately uniformly on the *M*-dimensional sphere using an optimization process. This is done by considering the points as identical attracting points on the sphere and computing the configuration that minimizes the associated potential function. Initially the points are distributed randomly, and then iteratively the points are moved along the local gradient of the potential function.

Elastic Spherical Pendulum

In this section the nonlinearity index is computed for the elastic spherical pendulum shown in Fig. 1. The pendulum bob is considered as a particle with mass *m*.



FIG. 1. Elastic Spherical Pendulum.

The linear spring has a spring constant *k*. There is also a gravitational acceleration of $g\hat{e}_z$. Three representations of the physical system are considered: Cartesian coordinates, spherical coordinates, and Cartesian coordinates with the Cayley quasi velocities.

In Cartesian coordinates the position of the particle relative to the origin is given by $\mathbf{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$. The potential energy due to the elastic and gravitational potentials is given by $V = k(x^2 + y^2 + z^2)/2 - mgz$. From these the decoupled, linear equations of motion are found as

$$m\ddot{x} + kx = 0$$

$$m\ddot{y} + ky = 0$$

$$m\ddot{z} + kz - mg = 0$$
(11)

In spherical coordinates the position vector is given by $\mathbf{r} = r\hat{e}_r$, and the angular velocity of the body-fixed frame is given by $\omega_{B/I} = \dot{\phi}\hat{e}_z + \dot{\theta}\hat{e}_{\phi}$. The potential energy is given by $V = kr^2/2 - mgr \cos \theta$. Unlike the Cartesian coordinates, the equations of motion for the spherical coordinates are coupled and nonlinear and are given by

$$\ddot{r} = r\dot{\theta}^{2} + r\dot{\phi}^{2}\sin^{2}\theta - \frac{k}{m}r + g\cos\theta$$
$$\ddot{\theta} = \dot{\phi}^{2}\sin\theta\cos\theta - 2\frac{\dot{r}\dot{\theta}}{r} - \frac{g}{r}\sin\theta$$
$$\ddot{\phi} = -2\frac{\dot{r}\dot{\phi}}{r} - 2\dot{\phi}\dot{\theta}\frac{\cos\theta}{\sin\theta}$$
(12)

In deriving these equations, it is found that ϕ is a cyclic coordinate and the motion constant $h_z = r^2 \dot{\phi} \sin^2 \theta$ exists. This is the vertical component of the angular momentum about the origin.

The final representation of the elastic spherical pendulum that is considered is Cartesian coordinates in the Cayley form. This uses the Cartesian coordinates for generalized coordinates, $[\mathbf{q}] = [x \ y \ z]^T$, and the associated Cayley quasi velocities, $[\boldsymbol{\omega}] = [\boldsymbol{\omega}_1 \ \boldsymbol{\omega}_2 \ \boldsymbol{\omega}_3]^T$, for velocity-level coordinates. The Cayley quasi velocities for this three degree-of-freedom system are related to the generalized velocities as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}; \qquad \begin{bmatrix} A \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 + x^2 & xy - z & xz + y \\ yx + z & 1 + y^2 & yz - x \\ zx - y & zy + x & 1 + z^2 \end{bmatrix}$$
(13)

This relationship can be used to write the kinetic energy as a function of the generalized coordinates and quasi velocities as

$$T = \frac{1}{2} m \dot{\mathbf{q}}^{\mathrm{T}} \dot{\mathbf{q}} = \frac{1}{2} m \boldsymbol{\omega}^{\mathrm{T}} A^{\mathrm{T}} A \boldsymbol{\omega}$$
(14)

The dynamic equations are then developed by applying the equations of motion in equation (5) to obtain

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$$\dot{\omega}_{1} = -\left[\omega_{1}(\omega_{1}x + \omega_{2}y + \omega_{3}z) + \frac{2\frac{k}{m}x + 2gy}{1 + x^{2} + y^{2} + z^{2}}\right]$$
$$\dot{\omega}_{2} = -\left[\omega_{2}(\omega_{1}x + \omega_{2}y + \omega_{3}z) + \frac{2\frac{k}{m}y - 2gx}{1 + x^{2} + y^{2} + z^{2}}\right]$$
$$\dot{\omega}_{3} = -\left[\omega_{3}(\omega_{1}x + \omega_{2}y + \omega_{3}z) + \frac{2\frac{k}{m}z - 2g}{1 + x^{2} + y^{2} + z^{2}}\right]$$
(15)

The three representations are summarized in Table 1.

In order to integrate the state-transition matrix and compute the nonlinearity index, the Jacobian of each dynamical system was found. This was done by taking the partial derivatives of equation (11) and the associated kinematics (e.g., $\dot{x} = \dot{x}$), equation (12) and the associated kinematics, and equations (13) and (15) with respect to the corresponding state variables. Although these matrices are not shown here, it is important to note that the linear system corresponding to the Cartesian coordinate representation produces a constant Jacobian. For the other two representations the values of the Jacobian matrices vary with the state variables.

The nonlinearity index described above gives one measure for the nonlinearity of each of these representations. In computing this index, normalization is performed with respect to the nominal trajectory as represented by each set of coordinates. Therefore, the nonlinearity index represents a measurement of nonlinearity within the context of each individual coordinate system.

Another concept for measuring nonlinearity, however, is to check how well some property of interest related to the motion is captured by linear portions of the equations of motion. This is done by integrating the linearized departure motion from the nominal trajectory through

$$\dot{\mathbf{x}}_{dep} = F(\bar{\mathbf{x}})\mathbf{x}_{dep}; \quad \mathbf{x}_{dep}(t_0) = \mathbf{x}(t_0) - \bar{\mathbf{x}}(t_0)$$
(16)

The linear prediction of a neighboring trajectory is given by $\mathbf{x}_{dep}(t) + \bar{\mathbf{x}}(t)$ and can be analyzed to determine how well the linearized state equations capture the motion. For this example the total energy E = T + V and the vertical angular momentum h_z , both constants, are computed based on the linear approximation. Variations in these constants of the motion indicate error in the approximate solution.

Representation	Position-Level Coordinates	Velocity-Level Coordinates	Kinematics	Dynamics
Cartesian coordinates	<i>x</i> , <i>y</i> , <i>z</i>	<i>x</i> , <i>ý</i> , <i>ż</i>	linear	linear
Spherical coordinates	$r, heta, \phi$	$\dot{r}, \dot{ heta}, \dot{\phi}$	linear	nonlinear
Cayley form	<i>x</i> , <i>y</i> , <i>z</i>	$\omega_1, \omega_2, \omega_3$	nonlinear	nonlinear

Pendulum Numerical Results

The nonlinearity index of each representation was computed for the trajectory associated with the initial condition in Cartesian coordinates given as

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.1 \\ 1.0 \end{bmatrix}; \qquad \begin{bmatrix} \dot{x}_0 \\ \dot{y}_0 \\ \dot{z}_0 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.15 \\ 0.1 \end{bmatrix}$$
(17)

In order to investigate the behavior of each system in the neighborhood of this trajectory, a set of 500 initial conditions were selected on a six-dimensional sphere in the Cartesian coordinate state space with radius 0.01 surrounding the nominal initial point. The points were distributed approximately uniformly. In order to perform the computations for the spherical-coordinate and Cayley-form representations, these points were transformed to the corresponding variables using the appropriate nonlinear coordinate transformations. The parameter values m = k = g = 1 were used. All numerical integrations used fourth and fifth order Runge-Kutta integration with adaptive step size and 10^{-5} error tolerance. The nonlinearity index was then evaluated at time steps of 0.01.

As mentioned, the Jacobian matrix for the Cartesian coordinate representation is a constant. Therefore the state-transition matrix for these coordinates has the solution $\Phi(t, t_0) = \exp(F(t - t_0))$ and is independent of the initial condition. Therefore the nonlinearity index for this representation is identically zero, as expected for a linear system. Also, the linearized departure equations for the Cartesian coordinates are the true equations of motion, and therefore they exactly predict the correct energy and vertical angular momentum. The solution for the nominal trajectory in Cartesian coordinates over an interval of ten time units is shown in Fig. 2.

For the nonlinear systems associated with the spherical-coordinate and Cayleyform representations, the nonlinearity index was computed by integrating the trajectories and state-transition matrices over ten time units for each initial condition and then evaluating equation (10). The nominal trajectory in spherical-coordinate and Cayley-form representations is shown in Figs. 3 and 4. The nonlinearity indices



FIG. 2. Nominal Trajectory in Cartesian Coordinates.



FIG. 3. Nominal Trajectory in Spherical Coordinates.

found for each representation are shown in Figs. 5 and 6. The average value of the nonlinearity index over the time interval and the maximum value are shown for both spherical coordinates and the Cayley form in Table 2. These results show much lower nonlinearity indices for the Cayley form than the spherical coordinates.

In addition to the nonlinearity index, the errors in linear prediction of E and h_z were also computed. The linearized departure equations were integrated using the same nominal trajectory, set of initial conditions, and parameter values. The maximum error in E and h_z over the set of initial conditions was computed for each point in time. For the spherical coordinates the errors in these constants are shown in Fig. 7, and for the Cayley form the errors are shown in Fig. 8. The average values of the errors over the time interval and the maximum values are shown for both spherical coordinates and the Cayley form in Table 2. These results show that



FIG. 4. Nominal Trajectory for Cayley Quasi Velocities.



FIG. 5. Nonlinearity Index for Spherical Coordinates.

the linearized departure equations for the Cayley form perform much better than the spherical coordinates in predicting the correct values for the constants E and h_z .

Planar Orbital and Attitude Motion

A second example considered is the planar motion of a satellite about the Earth as shown in Fig. 9. This consists of two translational degrees of freedom representing the motion of the satellite in its orbital plane and one rotational degree of freedom representing rotations in that plane. The generalized coordinates are $[\mathbf{q}] = [\theta \ x \ y]^T$, where θ is measured in radians and x and y are measured in Earth radii (ER = 6378 km). The rates are computed using a time unit (TU) of 8000 sec, approximately equal to the orbital periods studied in the following section. For this example, the nonlinearity index is computed for the generalized-velocity and Cayley quasi-velocity representations.



FIG. 6. Nonlinearity Index for Cayley Form.

	Spherical Coordinates	Cayley Form	
v, average	0.4293	0.0085	
ν , maximum	11.0195	0.0135	
E, average error	$5.3768 imes 10^{-4}$	1.1347×10^{-5}	
E, maximum error	$8.1235 imes 10^{-3}$	1.6418×10^{-5}	
h_z , average error	$2.3004 imes 10^{-4}$	7.2015×10^{-6}	
h_z , maximum error	1.2375×10^{-3}	1.2244×10^{-5}	

TABLE 2. Summary of Pendulum Numerical Results



FIG. 7. Linearization Error in Motion Constants for Spherical Coordinates.



FIG. 8. Linearization Error in Motion Constants for Cayley Form.

The satellite will be modeled as a barbell configuration with point masses of mass *m* located at each end of a boom of total length 2*d*. In the Earth-centered, Earth-fixed frame the position of the satellite center of mass is given by $[\mathbf{r}] = [x \ y]^{\mathrm{T}}$. The positions of each mass relative to the center of mass are given by $[\rho_1] = d[\cos \theta \sin \theta]^{\mathrm{T}}$ and $[\rho_2] = -[\rho_1]$. The absolute positions are given by $\mathbf{r}_1 = \mathbf{r} + \rho_1$ and $\mathbf{r}_2 = \mathbf{r} + \rho_2$.

Each mass is subject to a gravitational force

$$\mathbf{f}_1 = -\frac{\mu m}{r_1^3} \mathbf{r}_1; \qquad \mathbf{f}_2 = -\frac{\mu m}{r_2^3} \mathbf{r}_2$$
 (18)

The equations of motion for the generalized velocity are therefore related to the total force and moment applied to the satellite through

$$\ddot{\theta} = \frac{\mu}{2d} \left(\frac{1}{r_2^3} - \frac{1}{r_1^3} \right) (y \cos \theta - x \sin \theta) \equiv \frac{M}{2md^2}$$
$$\ddot{x} = -\frac{\mu}{2r_1^3} (x + d \cos \theta) - \frac{\mu}{2r_2^3} (x - d \cos \theta) \equiv \frac{F_x}{2m}$$
$$\ddot{y} = -\frac{\mu}{2r_1^3} (y + d \sin \theta) - \frac{\mu}{2r_2^3} (y - d \sin \theta) \equiv \frac{F_x}{2m}$$
(19)

Note that these equations are nonlinear and fully coupled.

For the Cayley form, the equations of motion again come from the rotational kinematics and dynamics. The kinematic equations for this example are identical to equation (13) except the matrix A is defined in terms of (θ, x, y) instead of (x, y, z). The kinetic energy of the system in terms of the generalized velocities is described by



FIG. 9. Barbell Satellite in Earth Orbit.

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$$T_0(\dot{\mathbf{q}}) = md^2\dot{\theta}^2 + m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}\,\dot{\mathbf{q}}^{\mathrm{T}}J\dot{\mathbf{q}}$$
(20)

The matrix J is now the system mass matrix given by

$$[J] = 2 \begin{bmatrix} md^2 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$
(21)

Using the kinematic equations, the kinetic energy can be expressed as a function of the generalized coordinates and the Cayley quasi velocities

$$T_{1}(\mathbf{q},\boldsymbol{\omega}) = \frac{1}{2} \boldsymbol{\omega}^{\mathrm{T}}(A^{\mathrm{T}}JA)\boldsymbol{\omega}$$
(22)

The generalized forces in terms of the generalized velocities are equal to the moment and force components applied to the satellite: $[\mathbf{f}] = [M \quad F_x \quad F_y]^T$. The dynamic equations can now be developed using the rotational equations in equation (5) as

$$\dot{\omega}_{1} = -\omega_{1}(\omega_{1}\theta + \omega_{2}x + \omega_{3}y) + \frac{M + yd^{2}F_{x} - xd^{2}F_{y}}{md^{2}(1 + \theta^{2} + x^{2} + y^{2})}$$
$$\dot{\omega}_{2} = -\omega_{2}(\omega_{1}\theta + \omega_{2}x + \omega_{3}y) + \frac{-yM + d^{2}F_{x} + \theta d^{2}F_{y}}{md^{2}(1 + \theta^{2} + x^{2} + y^{2})}$$
$$\dot{\omega}_{3} = -\omega_{3}(\omega_{1}\theta + \omega_{2}x + \omega_{3}y) + \frac{xM - \theta d^{2}F_{x} + d^{2}F_{y}}{md^{2}(1 + \theta^{2} + x^{2} + y^{2})}$$
(23)

Again, in order to compute the nonlinearity index the Jacobians of both sets of equations of motion are necessary. Similar to the previous example, the generalized-velocity representation has linear kinematics, whereas the Cayley-form representation has nonlinear kinematics. In this example, however, both representations have nonlinear dynamics.

Again similar to the previous example, the nonlinearity is also measured by integrating the linearized departure motion from the nominal trajectory. For this system the total energy E and the angular momentum H about the origin, are the constants

$$\frac{E}{m} = d^2 \dot{\theta}^2 + \dot{x}^2 + \dot{y}^2 - \mu \left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$
(24)

$$\frac{H}{2m} = x\dot{y} - \dot{x}y + d^2\dot{\theta}$$
(25)

The performance of the linearized equations in capturing these constants, as well as the nonlinearity index for both representations, are described in the following section.

Satellite Numerical Results

The nonlinearity index was computed for this satellite system using three different nominal initial conditions. These corresponded to orbital eccentricities of e = 0.0, 0.1, and 0.2. The perigee radius, r_p , for all three orbits was set to 8000 km

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or approximately 1.254 ER. The initial orientation of the satellite was set to 10° or approximately 0.1745 rad. The initial rotational rate was set to the orbital rotation rate of the circular orbit corresponding to $r = r_p$.

$$\begin{bmatrix} \theta_0 \\ x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 10^\circ \\ r_p \\ 0 \end{bmatrix}; \qquad \begin{bmatrix} \dot{\theta}_0 \\ \dot{x}_0 \\ \dot{y}_0 \end{bmatrix} = \begin{bmatrix} \sqrt{\theta/r_p^3} \\ 0 \\ \sqrt{\mu(1+e)/r_p} \end{bmatrix}$$
(26)

Again, a set of 500 initial conditions were chosen by selecting perturbations from these nominal initial conditions on a six-dimensional unit sphere. The perturbations in the generalized coordinates and velocities were then scaled by 10^{-3} . These perturbations were chosen in consideration of the relative magnitudes of the state variables, which were roughly equal in the selected units. The gravitational parameter for the Earth $\mu = 98.319 \text{ ER}^3/\text{TU}^2$ was used, and the satellite model used the somewhat unrealistic value of d = 1 km.

The nonlinearity index was computed for both the generalized-velocity and Cayleyform representations. The trajectories and state-transition matrices for both were integrated over two orbital periods of the nominal initial condition. The resulting nonlinearity indices are shown in Figs. 10 to 12. Over the time intervals these values are generally increasing, as expected. Both representations also demonstrate greater nonlinearity with higher eccentricity. The nonlinearity indices for the Cayley form, however, tend to increase at a somewhat lower rate. This is more evident as the simulation time progresses.

The predictions for the motion constants E/m and H/2m produced by the linearized departure equations reflect the results with the nonlinearity index. The departure equations were integrated using the same family of initial perturbations as used in the nonlinearity index computations. Similar to the nonlinearity index computations, for each time step the worst-case error was selected from the set of neighboring trajectories. These worst-case errors were then averaged over the entire time interval. The resulting average errors are shown in Table 3. The errors increase with higher eccentricity, as expected, and demonstrate a slight improvement in accuracy associated with the Cayley form. The first grouping of E/m and H/2m pertains to



FIG. 10. Nonlinearity Index for e = 0.



FIG. 11. Nonlinearity Index for e = 0.1.



FIG. 12. Nonlinearity Index for e = 0.2.

TABLE 3. Errors in Linear Predictions for E/m and H/2m

	E/m	H/2m	E/m	H/2m	E/m	H/2m
Gen. Vel.	0.061%	0.031%	0.079%	0.038%	0.110%	0.049%
Cayley Form	0.060%	0.030%	0.077%	0.037%	0.107%	0.048%

eccentricity equal to zero; the second grouping pertains to eccentricity equal to 0.1; and the final grouping pertains to eccentricity equal to 0.2.

Discussion

The results for the spherical pendulum show good agreement between the two types of nonlinearity measurement. Both measurements show lower nonlinearity for the Cayley form than the spherical coordinates and also agree in several features in the time history. For the spherical-coordinate representation the nonlinearity index and motion-constant errors experience sharp peaks at the points along the trajectory where the coordinate ϕ goes through large changes in value. At these points the trajectory approaches the singularity in the spherical coordinates. In particular, the coordinate ϕ is undefined when $\theta = 0$. The constant of motion described above shows that $\dot{\phi}$ can diverge as θ approaches zero. (Of course, another singularity exists such that both θ and ϕ are undefined for r = 0.) The nonlinearity index indicates that the spherical-coordinate representation is, in general, moderately nonlinear and highly nonlinear in the neighborhood of the singular configuration.

Alternatively, the Cayley-form representation is singularity free; neither the Cartesian position-level coordinates nor the Cayley kinematics suffer from singularities. Compared with the spherical-coordinate representation, the nonlinearity index for the Cayley form shows only mild nonlinearity. Related to the issue of singularities is the fact that the Cayley form introduces polynomial nonlinearities into the kinematics and dynamics, whereas the spherical coordinates produce trigonometric nonlinearities in the dynamic equations.

The results for the satellite motion, however, compare two representations that are both singularity free. The Cayley form again exhibited lower values of nonlinearity index. Here, the lower nonlinearity of the Cayley form resulted from properties of the Cayley kinematics, i.e., reduced sensitivity to motion with large generalized-coordinate values. Whereas this paper presented results only for particular examples, these results illustrate several advantages of the Cayley representation of dynamic systems. Additionally, these examples illustrate the use of the nonlinearity index to address the issue of "how nonlinear is it?" for a wide class of nonlinear dynamic systems.

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