This note illustrates how one can infer the complete from of a distribution from its dependence on the random variable and how this can be helpful in other calculations.

- A Beta distribution, $\text{Beta}(p|a,b)$, over $p \in [0,1]$ has the following form and properties:

$$\begin{aligned}
Pr(p) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\, p^{a-1}(1-p)^{b-1} \\
\Gamma(x) &= \int_0^\infty \mu^x e^{-\mu} d\mu \\
\Gamma(1) &= 1 \\
\Gamma(x+1) &= x\Gamma(x) \quad \text{ and for integers } \Gamma(x+1) = x! \\
E[p] &= \frac{a}{a+b} \\
Mode[p] &= \frac{a-1}{a+b-2}
\end{aligned}$$

- Suppose that we know that $Pr(p) \propto p^2(1-p)^2$; can we find the precise form of the distribution?

  Yes, the missing constant is $\frac{\Gamma(3+3)}{\Gamma(3)\Gamma(3)} = \frac{5!}{2!\ 2!} = 30$ and the distribution is $Pr(p) = 30p^2(1-p)^2$.

- What is the value of $\int_0^1 p^3(1-p)^5 dp$?

  We can calculate the integral directly but instead we can infer the value through the constant factor of the Beta distribution.
  $\int_0^1 p^3(1-p)^5 dp = \frac{\Gamma(4)\Gamma(6)}{\Gamma(4+6)} \int_0^1 \frac{\Gamma(4+6)}{\Gamma(4)\Gamma(6)} p^3(1-p)^5 dp = \frac{\Gamma(4)\Gamma(6)}{\Gamma(4+6)} \cdot 1 = \frac{3!\ 5!}{9!} = \frac{1}{504}$

- A univariate Normal distribution, $\mathcal{N}(x|\mu,\sigma^2)$, over $x \in R$ has the following form and properties:

$$\begin{aligned}
Pr(x) &= \frac{1}{\sqrt{2\pi}\ \sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\
E[x] &= Mode[x] = \mu \\
Var[x] &= \sigma^2
\end{aligned}$$

- Suppose that we know that $Pr(x) \propto e^{5x-8x^2}$; can we find the precise form of the distribution?

  Yes, here too we can calculate the constant term. The distribution depends on x via $e^{-\frac{(x^2-2\mu x)}{2\sigma^2}} = e^{-\frac{x^2}{2\sigma^2}}\ e^{\frac{\mu x}{\sigma^2}}$ and the remaining terms $\frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{\mu^2}{2\sigma^2}}$ do not depend on x.

  From this we conclude that $8 = \frac{1}{2\sigma^2}$ or $\sigma^2 = \frac{1}{16}$ and that $5 = \frac{\mu}{\sigma^2} = 16\mu$ and $\mu = \frac{5}{16}$. Therefore the distribution is $\mathcal{N}(x|\frac{5}{16}, \frac{1}{16})$. This trick is known as "completing the square".