Outline:

- review of HW1.
- when does a dynamical system have a unique solution?
- state-space and transfer-function representations of LTI systems.
- state-space canonical forms.
- matrix computation: determinants.

1 Fundamental Theorem of Differential Equations

Knowing the existance of a solution is the first step towards getting the answer. For ME 232, the following theorem addresses the question of whether a dynamical system has a *unique* solution or not.

Theorem 1. *Consider* $\dot{x} = f(x,t)$, $x(t_0) = x_0$, *with:*

- $f(x,t)$ piecewise continuous in t
- $f(x,t)$ Lipschitz continuous in x

then there exists a *unique* function of time $\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ which is continuous almost everywhere and satisfies

- $\phi(t_0) = x_0$
- $\dot{\phi}(t) = f(\phi(t), t)$, $\forall t \in \mathbb{R}_+ \backslash D$, where D is the set of discontinuity points for f as a function of t.

Note:

- piecewise continuous: continuous except at finite points of discontinuity.
  - exercise: are these functions piecewise continuous?–$f(t) = |t|$ and

$$f(x,t) = \begin{cases} A_1 x, & t \le t_1 \\ A_2 x, & t > t_1 \end{cases}$$

- Lipschitz continuous: if $f(x,t)$ satisfies the following cone-shape constraint:

$$||f(x,t) - f(y,t)|| \le k(t) ||x - y||$$

  where $k(t)$ is piecewise continuous.
  - exercise: is $f(x) = Ax + B$ Lipschitz continuous?

2 State-space and transfer-function descriptions of LTI systems

- why are we learning them? [history of control engineering and connections with future courses]
- relations between the two:

Table 1: Relations between state-space (ss) and transfer-function (tf) system representations

ss	tf
$\dot{x}(t) = Ax(t) + Bu(t)\,,\ y(t) = Cx(t) + Du(t)$	$G(s) = C(sI - A)^{-1}B + D$
$x(k+1) = Ax(k) + Bu(k)\,,\ y(k) = Cx(k) + Du(k)$	$G(z) = C(zI - A)^{-1}B + D$

3 Canonical forms

$$G=\frac{b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}$$

- Controllable canonical form

$$A=\begin{bmatrix} & 1 & \\ & & 1 \\ -a_0 & -a_1 & -a_2\end{bmatrix},\ B=\begin{bmatrix}0\\0\\1\end{bmatrix},\ C=\begin{bmatrix} b_0 & b_1 & b_2\end{bmatrix},\ D=0$$

  - understanding the formula: the transfer function is given by $G(s)=C(sI-A)^{-1}B+D$, where the poles of the system come from $\det(sI-A)=0$. Suppose we don't know the order of a_0, a_1 and a_2 in the last row of A, and use $\star$ as a temporary representation in A. Looking at

$$\det(sI-A)=\det\begin{bmatrix} s & -1 & \\ & s & -1 \\ \star & \star & s+\star\end{bmatrix}$$

    we see that the only way for s^2 to appear is from the term $s^2(s+\star)$ in the determinant computation. Hence the location of $-a_2$ has to be at the bottom right corner of A.
  - exercise: write down the controllable canonical form for the following systems
    * $G(s)=\frac{s^2+1}{s^3+2s+10}$
    * $G(s)=\frac{b_0s^2+b_1s+b_2}{s^3+a_0s^2+a_1s+a_2}$

- Observable canonical form

$$A=\begin{bmatrix} -a_2 & 1 & \\ -a_1 & & 1 \\ -a_0 & & \end{bmatrix},\ B=\begin{bmatrix}b_2\\b_1\\b_0\end{bmatrix},\ C=\begin{bmatrix} 1 & 0 & 0\end{bmatrix},\ D=0$$

- Diagonal form: for systems described by

$$G(s)=\frac{k_1}{s-p_1}+\frac{k_2}{s-p_2}+\frac{k_3}{s-p_3}$$

  one state-space form is

$$A=\begin{bmatrix} p_1 & & \\ & p_2 & \\ & & p_3\end{bmatrix},\ B=\begin{bmatrix}1\\1\\1\end{bmatrix},\ C=\begin{bmatrix} k_1 & k_2 & k_3\end{bmatrix},\ D=0$$

- Jordan form: if

$$G(s)=\frac{k_1}{s-p_1}+\frac{k_2}{(s-p_m)^2}+\frac{k_3}{s-p_m}$$

  then one state-space form of the system is

$$A=\begin{bmatrix} p_1 & & \\ & p_m & 1 \\ & & p_m\end{bmatrix},\ B=\begin{bmatrix}1\\0\\1\end{bmatrix},\ C=\begin{bmatrix} k_1 & k_2 & k_3\end{bmatrix},\ D=0$$

  - understanding the formula: why $B=[1,0,1]^T$? What if

$$G(s)=\frac{k_1}{(s-p_1)^2}+\frac{k_1}{s-p_1}+\frac{k_2}{(s-p_m)^2}+\frac{k_3}{s-p_m}$$

- Modified Jordan form: this is for systems with complex poles:

$$G(s)=\frac{k_1}{s-p_1}+\frac{\alpha s+\beta}{(s-\sigma)^2+\omega^2}$$

  we have

$$A=\begin{bmatrix} p_1 & & \\ & \sigma & \omega \\ & -\omega & \sigma\end{bmatrix},\ B=\begin{bmatrix}1\\0\\1\end{bmatrix},\ C=\begin{bmatrix} k_1 & k_2 & k_3\end{bmatrix},\ D=0$$

4 Review and preparation for the next few lectures

4.1 Computing determinants

- 2×2 matrices:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- 3×3 matrices:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & k \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & k \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$
$$= aek + bfg + cdh - gec - bdk - ahf$$