Loop Quantization
versus
Fock Quantization
of p-Form Electromagnetism
on Static Spacetimes

by

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Definition 1 (stationary and static spacetimes). *A Lorentzian manifold without timelike loops (also called a* spacetime*) is* stationary *if, and only if, it admits a one-parameter group of isometries with smooth, timelike orbits. A stationary spacetime is* static *if, in addition, it is foliated by a family of spacelike hypersurfaces everywhere orthogonal to the orbits of the isometries.*

Definition 2 (globally hyperbolic spacetime). *A piecewise-smooth curve in a spacetime M is* causal *if its tangent vector is everywhere timelike. A set is* achronal *if there are no causal curves between any two of its points. The* domain of dependence *of a set consists of all points $p \in M$ such that every inextensible causal curve through p intersects the set. A* Cauchy surface *in a spacetime M is a closed achronal set whose domain of dependence is all of M. A spacetime is* globally hyperbolic *if, and only if, it admits a Cauchy surface.*

Definition 3 (linear phase space). *A* linear phase space *is a reflexive real topological vector space* $\mathbf{P}$ *whose dual* $\mathbf{P}^*$ *is a* symplectic vector space. *That is,* $\mathbf{P}^*$ *is a topological vector space equipped with a* symplectic structure*: a continuous, skew-symmetric bilinear form* ω *which is* weakly nondegenerate *in the sense that the duality map* $*\colon \mathbf{P}^* \to \mathbf{P}$ *given by*

$$\omega(f, g) = f(g^*) \qquad \text{for all} \quad f, g \in \mathbf{P}^*$$

is injective.

Definition 4 (automorphism of a linear phase space). *An automorphism of the linear phase space* $\mathbf{P}$ *is a continuous invertible linear map* $T\colon \mathbf{P} \to \mathbf{P}$ *whose dual map* $T^*\colon \mathbf{P}^* \to \mathbf{P}^*$ *preserves the symplectic structure on* $\mathbf{P}^*$.

Definition 5 (Heisenberg system). *A* Heisenberg system *on a symplectic vector space* $(\mathbf{P}^*, \omega)$ *is a real-linear map* $\Phi\colon f \mapsto \Phi(f)$ *from* $\mathbf{P}^*$ *to the self-adjoint operators on some complex Hilbert space* $\mathbf{K}$*, satisfying the* Heisenberg commutation relations

$$[\Phi(f), \Phi(g)] = i\omega(f,g)\mathbf{1}_{\mathbf{K}}. \qquad \textit{for all} \quad f, g \in \mathbf{P}^*$$

as an operator equation holding on the common domain of $\Phi(f)\Phi(g)$ *and* $\Phi(f)\Phi(g)$*, which is assumed to be dense. The operator* $\Phi(f)$ *is called the* Heisenberg operator *associated to* $f \in \mathbf{P}^*$*.*

Definition 6 (Weyl algebra). *The* Weyl algebra *on a symplectic vector space space* $(\mathbf{P}^*, \omega)$*, is the complex* $*$*-algebra* $\mathcal{W}(\mathbf{P}^*, \omega)$ *generated by the set* $\mathcal{W}(\mathbf{P}^*) = \{\mathcal{W}(f)\}_{f \in \mathbf{P}^*}$*, of* Weyl operators*, modulo the* unitarity relations

$$\mathcal{W}(f)^* = \mathcal{W}(-f) \qquad \textit{for all} \quad f \in \mathbf{P}^*$$

and the Weyl relations

$$\mathcal{W}(f)\mathcal{W}(g) = e^{\omega(f,g)/2i}\mathcal{W}(f+g) \qquad \textit{for all} \quad f, g \in \mathbf{P}^*.$$

Definition 7 (Weyl system). *A* Weyl system *on the symplectic vector space* $(\mathbf{P}^*, \omega)$ *is a continuous mapping* $W\colon \mathbf{P}^* \to U(\mathbf{K})$*, where* $U(\mathbf{K})$ *is the group of unitary operators on the complex Hilbert space* $\mathbf{K}$ *with the strong operator topology, and* W *satisfies the Weyl relations*

$$W(f)W(g) = e^{\omega(f,g)/2i}W(f+g) \qquad \textit{for all} \quad f, g \in \mathbf{P}^*.$$

Definition 8 (general boson field). *If* $(\mathbf{P}^*, \omega)$ *is a symplectic vector space, the* general boson field *over it is the pair* $(\mathcal{W}, \gamma)$ *where* $\mathcal{W}\colon f \mapsto \mathcal{W}(f)$ *is the map from* $\mathbf{P}^*$ *to* $\mathcal{W}(\mathbf{P}^*, \omega)$*, and* γ *is the representation of automorphisms of* $\mathbf{P}$ *by* $*$*-automorphisms of* $\mathcal{W}(\mathbf{P}^*, \omega)$ *mentioned in Lemma 19.*

Definition 9 (GNS state). *A* state *on a* $*$*-algebra* A *is a linear functional*

$$\langle\ \rangle\colon A \to \mathbb{C}$$

which is nonnegative

$$\langle a^* a\rangle \geq 0 \qquad \textit{for all} \quad a \in A,$$

and normalized

$$\langle 1\rangle = 1.$$

Definition 10 (characteristic functional). *If* $\langle\ \rangle$ *is a state on the Weyl algebra* $\mathcal{W}(\mathbf{P}^*, \omega)$*, its* characteristic functional $\mu\colon \mathbf{P}^* \to \mathbb{C}$ *is given by*

$$\mu(f) \colon= \langle \mathcal{W}(f)\rangle \qquad \textit{for all} \quad f \in \mathbf{P}^*. \tag{0.1}$$

We say the state $\langle\ \rangle$ *is* regular *if, for every* $f \in \mathbf{P}^*$*, the function*

$$t \mapsto \mu(tf) \qquad (t \in \mathbb{R})$$

is twice differentiable at $t = 0$.

Definition 11 (relative coherent states). *Given a regular state* $\langle\ \rangle$ *on* $\mathcal{W}(\mathbf{P}, \omega)$*, the element* $x \in \mathbf{P}$ *such that*

$$i\partial_f \mu(0) = f(x) \qquad \textit{for all} \quad f \in \mathbf{P}^*$$

is called the background *for* $\langle\ \rangle$*. The image of* $\mathcal{W}(f)$ *inside* $\mathbf{K}$ *by the GNS construction, denoted by* $|x + f^*\rangle$*, is called a* coherent state relative *to the state* $\langle\ \rangle$*. We denote the set of relative coherent states by* $\Psi = \{|x + f^*\rangle : f \in \mathbf{P}^*\}$.

Definition 12 (free boson field). *The* free boson field *over a complex Hilbert space* $\mathbf{H}$ *consists of*

1. *a complex Hilbert space* $\mathbf{K}$
2. *a Weyl system* $W\colon \mathbf{H} \to U(\mathbf{K})$
3. *a continuous representation* $\Gamma\colon U(\mathbf{H}^\dagger) \to U(\mathbf{K})$ *satisfying*

$$\Gamma(U) W(z) \Gamma(U)^{-1} = W(Uz) \qquad \textit{for all} \quad z \in \mathbf{H}$$

4. *a unit vector* $\nu \in \mathbf{K}$ *which is invariant under* $\Gamma(U)$ *for all* $U \in U(\mathbf{H}^\dagger)$ *and a cyclic vector of* $W(\mathbf{H})$

such that Γ *is positive in the sense that, if the one-parameter group* $U(t) \subset U(\mathbf{H}^\dagger)$ *has a nonnegative self-adjoint generator* A*, then* $\partial\Gamma(A)$*, which denotes the self-adjoint generator of the group* $\Gamma(U(t))\colon \mathbf{K} \to \mathbf{K}$*, is a nonnegative self-adjoint operator on* $\mathbf{K}$.

Definition 13 (Wick power). *If* $f \in \mathbf{H}$*, the* n*th* Wick power *or* normal-ordered power *of the Heisenberg operator* $\Phi(f)$ *is the operator on* $\mathbf{K}$ *given by*

$$:\!\Phi(f)^n\!: = \frac{1}{2^{n/2}} \sum_{m=0}^{n} \binom{n}{m} a^\dagger(f)^m a(f)^{n-m}.$$

Definition 14 (quasioperator). *Let* $\mathbf{K}_0$ *be a topological vector space with a dense continuous inclusion into the Hilbert space* $\mathbf{K}$*. A* quasioperator *on* $\mathbf{K}$ *with domain* $\mathbf{K}_0$ *is a continuous sesquilinear form* $Q\colon \mathbf{K}_0 \times \mathbf{K}_0 \to \mathbb{C}$*, antilinear in the first argument and linear in the second.*

Definition 15 (smooth coherent states). *Let* $\mathbf{P}$ *be the oscillating phase space of p-form electromagnetism, and let* $\mathbf{K}$ *be the associated Fock space. We say that* $X = [A] \oplus E \in \mathbf{P}$ *is a* smooth field configuration*, and write* $X \in \mathbf{P}_0$*, if* $[A]$ *and* E *are infinitely-differentiable. A coherent state* $|X\rangle$ *with* $X \in \mathbf{P}_0$ *is called a* smooth coherent state*. We denote by* $\mathbf{K}_0$ *the span of the smooth coherent states.*

Lemma 1. *The completion of* $C_0^\infty \Omega_S^k$ *with respect to the inner product* $(\ ,\ )$ *is* $L^2\Omega_S^k$.

Lemma 2. *A densely defined operator* T *is closable if, and only if,* T^* *is densely defined. In that case,* $\overline{T} = T^{**}$.

Proposition 3 (Gaffney). *If* S *is a complete oriented Riemannian manifold, then*

$$(\delta^*\alpha, \beta) = (\alpha, \mathrm{d}^*\beta)$$

whenever $\alpha \in \operatorname{dom} \delta^*$ *and* $\beta \in \operatorname{dom} \mathrm{d}^*$.

Corollary 4. *If* S *is a complete oriented Riemannian manifold, then*

$$\overline{\mathrm{d}} = \delta^* \qquad \text{and} \quad \overline{\delta} = \mathrm{d}^*.$$

Theorem 5. *Let S be a smooth manifold equipped with a complete Riemannian metric g. Then the formally adjoint operators*

$$C_0^\infty \Omega_S^k \underset{\mathrm{d}_k^*}{\overset{\mathrm{d}_k}{\rightleftarrows}} C_0^\infty \Omega_S^{k+1}$$

have mutually adjoint closures

$$L^2 \Omega_S^k \underset{\mathrm{d}_k^*}{\overset{\mathrm{d}_k}{\rightleftarrows}} L^2 \Omega_S^{k+1} \ .$$

These closed operators satisfy

$$\operatorname{ran} \mathrm{d}_{k-1} \subseteq \ker \mathrm{d}_k, \qquad \operatorname{ran} \mathrm{d}_k^* \subseteq \ker \mathrm{d}_{k-1}^*$$

and there is a Hilbert-space direct-sum decomposition

$$L^2 \Omega^k = \overline{\operatorname{ran} d_{k-1}} \oplus \ker \Delta_k \oplus \overline{\operatorname{ran} \delta_k}.$$

where the Laplacian on k-forms,

$$\Delta_k = \delta_k d_k + d_{k-1} \delta_{k-1},$$

is a nonnegative densely defined self-adjoint operator on $L^2 \Omega^k$.

Proposition 6 (Kodaira decomposition). *If*

$$H \xrightarrow{S} H' \xrightarrow{T} H''$$

are densely defined closed operators and $\operatorname{ran} S \subseteq \ker T$*, then*

$$H' = \overline{\operatorname{ran} T^*} \oplus \ker(T^*T + SS^*) \oplus \overline{\operatorname{ran} S}.$$

Lemma 7. *If*

$$H \xrightarrow{T} H'$$

is a densely defined operator, then

$$\ker T^* = (\operatorname{ran} T)^\perp \qquad \textit{and} \quad \ker T = (\operatorname{ran} T^*)^\perp \cap \operatorname{dom} T.$$

Lemma 8. *If*

$$H \xrightarrow{S} H' \xrightarrow{T} H''$$

are densely defined operators and $\operatorname{ran} S \subseteq \ker T$*, then*

$$\operatorname{ran} T^* \subseteq \ker S^*.$$

Corollary 9. *If*

$$H \xrightarrow{S} H' \xrightarrow{T} H''$$

are densely defined closable operators and $\operatorname{ran} S \subseteq \ker T$*, then*

$$\operatorname{ran} \overline{S} \subseteq \ker \overline{T}.$$

Result 10. *Let M be a $(3+1)$-dimensional static, globally hyperbolic spacetime, with metric*

$$g_M = e^{2\Phi}(-\mathrm{d}t^2 + g).$$

Then, electromagnetism on M with gauge group $\mathbb{R}$ has as its phase space the real Hilbert space

$$\mathbf{P} = \frac{\mathrm{dom}\{\mathrm{d}\colon L^2\Omega^1_S \to L^2\Omega^2_S\}}{\overline{\mathrm{ran}}\{\mathrm{d}\colon L^2\Omega^0_S \to L^2\Omega^1_S\}} \oplus \ker\{\mathrm{d}^*\colon L^2\Omega^1_S \to L^2\Omega^0_S\},$$

with continuous symplectic structure

$$\omega(X, X') = (E, A') - (E', A)$$

where $X = [A] \oplus E$ and $X' = [A'] \oplus E'$ lie in $\mathbf{P}$, and

$$(\alpha, \beta) = \int_S g(\alpha, \beta)\mathrm{vol}$$

is the canonical inner product induced on Ω^k_S by the optical metric g on S. The Hamiltonian is the continuous quadratic form

$$H[X] = \frac{1}{2}[(E, E) + (\mathrm{d}A, \mathrm{d}A)].$$

There phase space splits naturally into two sectors,

$$\mathbf{P} = \mathbf{P}_o \oplus \mathbf{P}_f,$$

and the direct summands

$$\mathbf{P}_f = \mathbf{P} \cap \ker \Delta \qquad \textit{and} \quad \mathbf{P}_o = \mathbf{P} \cap \mathrm{ran}\, \mathrm{d}^*_1$$

are preserved by time evolution. On $\mathbf{P}_o$, time evolution takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_o(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{\Delta}) & \sin(t\sqrt{\Delta}) \,/\, \sqrt{\Delta} \\ -\sqrt{\Delta}\, \sin(t\sqrt{\Delta}) & \cos(t\sqrt{\Delta}) \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}$$

while on $\mathbf{P}_f$ it takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_f(t) \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A \\ E \end{pmatrix}.$$

Theorem 11. *Let S be a smooth n-dimensional manifold equipped with a complete Riemannian metric g, and let Φ be a smooth real-valued function on S. Fix an integer $0 \leq p \leq n$. Then for any integer k, the operators*

$$C_0^\infty \Omega_S^k \underset{D_k^\dagger}{\overset{D_k}{\rightleftarrows}} C_0^\infty \Omega_S^{k+1}$$

*defined in equations (**??**) and (**??**) have mutually adjoint closures, which we write as*

$$L^2 \Omega_S^k \underset{D_k^*}{\overset{D_k}{\rightleftarrows}} L^2 \Omega_S^{k+1}$$

These closures satisfy

$$\operatorname{ran} D_{k-1} \subseteq \ker D_k, \qquad \operatorname{ran} D_k^* \subseteq \ker D_{k-1}^*,$$

and we obtain a direct sum decomposition

$$L^2 \Omega^k = \overline{\operatorname{ran} D_{k-1}} \oplus \ker L_k \oplus \overline{\operatorname{ran} D_k^*}.$$

where the twisted Laplacian on k-forms,

$$L_k = D_k^* D_k + D_{k-1} D_{k-1}^*,$$

is a nonnegative densely defined self-adjoint operator on $L^2 \Omega^k$.

Lemma 12 (Chernoff). *If the metric $c^{-2}g$ makes S into a complete Riemannian manifold, the symmetric hyperbolic system $\partial_t\alpha = T\alpha$ with initial data in $C_0^\infty E$ has a unique solution on $\mathbb{R}\times S$ which is in $C_0^\infty E$ for all $t\in\mathbb{R}$. Moreover, if T is formally skew-adjoint ($T+T^\dagger=0$), then $-iT$ and all its powers are essentially self-adjoint on $C_0^\infty E$.*

Lemma 13. *Let H_1 and H_2 be Hilbert spaces and let*

$$H_1 \underset{B}{\overset{A}{\rightleftarrows}} H_2$$

be densely defined operators that are formal adjoints of one another:

$$\langle A\phi,\psi\rangle_1 = \langle\phi, B\psi\rangle_2 \qquad \textit{for all} \quad \phi\in\operatorname{dom}A, \psi\in\operatorname{dom}B.$$

Let $H=H_1\oplus H_2$ and let S be the densely defined operator

$$\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix}$$

on H. If S is essentially self-adjoint, then A and B have mutually adjoint closures.

Lemma 14. *Suppose S is a complete Riemannian manifold and Φ a smooth real-valued function on S. Let*

$$T\colon L^2\Omega_S^k\oplus L^2\Omega_S^{k+1}\to L^2\Omega_S^k\oplus L^2\Omega_S^{k+1}$$

be the densely defined operator

$$\begin{pmatrix} 0 & iD_k^\dagger \\ iD_k & 0 \end{pmatrix}.$$

Then $-iT$ and all its powers are essentially self-adjoint on $C_0^\infty\Omega^k\oplus C_0^\infty\Omega^{k+1}$.

Corollary 15. *Under the same hypothesis as Lemma 14, the operators*

$$C_0^\infty\Omega_S^k \underset{D_k^\dagger}{\overset{D_k}{\rightleftarrows}} C_0^\infty\Omega_S^{k+1}$$

have mutually adjoint closures, and the operators $D_k^\dagger D_k$ and $D_k D_{k-1}^\dagger$ are essentially self-adjoint on $C_0^\infty\Omega^k$.

Result 16. *Let M be a $(n+1)$-dimensional static globally hyperbolic spacetime, with metric*

$$g_M = e^{2\Phi}(-\mathrm{d}t^2 + g).$$

Then, p-form electromagnetism on M with gauge group $\mathbb{R}$ has as its phase space the real Hilbert space

$$\mathbf{P} = \frac{\mathrm{dom}\{D_p\colon L^2\Omega_S^p \to L^2\Omega_S^{p+1}\}}{\overline{\mathrm{ran}}\{D_{p-1}\colon L^2\Omega_S^{p-1} \to L^2\Omega_S^p\}} \oplus \ker\{D_{p-1}^*\colon L^2\Omega_S^p \to L^2\Omega_S^{p-1}\},$$

where

$$D_p = e^{\frac{1}{2}(n-2p-1)\Phi}\mathrm{d}_p e^{-\frac{1}{2}(n-2p-1)\Phi}$$

is the twisted exterior derivative*. The phase space admits a continuous symplectic structure*

$$\omega(X, X') = (E, A') - (E', A)$$

where $X = [A] \oplus E$ and $X' = [A'] \oplus E'$ lie in $\mathbf{P}$ and

$$(\alpha, \beta) = \int_S g(\alpha, \beta)\mathrm{vol}$$

is the canonical inner product induced on Ω_S^k by the optical metric g on S. The Hamiltonian is the continuous quadratic form

$$H[X] = \frac{1}{2}[(E, E) + (D_p A, D_p A)].$$

The phase space splits naturally into two sectors,

$$\mathbf{P} = \mathbf{P}_o \oplus \mathbf{P}_f,$$

and the direct summands

$$\mathbf{P}_f = \mathbf{P} \cap \ker L \qquad \textit{and} \quad \mathbf{P}_o = \mathbf{P} \cap \mathrm{ran}\, D_p^*$$

are preserved by time evolution. On $\mathbf{P}_o$, time evolution takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_o(t)\begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} \cos(t\sqrt{L_p}) & \sin(t\sqrt{L_p}) \,/\, \sqrt{L_p} \\ -\sqrt{L_p}\,\sin(t\sqrt{L_p}) & \cos(t\sqrt{L_p}) \end{pmatrix}\begin{pmatrix} A \\ E \end{pmatrix}$$

while on $\mathbf{P}_f$ it takes the form

$$\begin{pmatrix} A \\ E \end{pmatrix} \mapsto T_f(t)\begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A \\ E \end{pmatrix}.$$

Lemma 17. *If* $W\colon \mathbf{P}^* \to U(\mathbf{K})$ *is a Weyl system on the symplectic vector space* $(\mathbf{P}^*, \omega)$ *then* $\Phi\colon \mathbf{P}^* \to L(\mathbf{K})$ *is a Heisenberg system on* (Φ^*, ω)*. In addition, for all* $x, y \in \mathbf{P}$*, the operator* $\Phi(f) + i\Phi(g)$ *is closed and* $\Phi(f+g)$ *is the closure of* $\Phi(f) + \Phi(g)$*.*

Lemma 18. *Suppose that* $\gamma\colon \mathcal{W}(\mathbf{P}^*, \omega) \to \mathcal{W}(\mathbf{P}^*, \omega)$ *is a* $*$*-algebra endomorphism such that*

$$\text{for every} \quad f \in \mathbf{P}^*, \qquad \gamma(\mathcal{W}(f)) = \mathcal{W}(g) \qquad \text{for some} \quad g \in \mathbf{P}^*,$$

and suppose furthermore that the map $T^*\colon (\mathbf{P}^*, \omega) \to (\mathbf{P}^*, \omega)$ *given by* $T^* f = g$ *is continuous. Then,* T^* *is in fact linear and preserves the symplectic structure* ω*. If, in addition,* γ *is an automorphism, then* T^* *is invertible, that is,* T *is an automorphism of the linear phase space* $\mathbf{P}$*.*

Lemma 19. *If* $T\colon \mathbf{P} \to \mathbf{P}$ *is an automorphism of the linear phase space* $\mathbf{P}$*, then there exists a unique* $*$*-algebra automorphism* $\gamma(T)\colon \mathcal{W}(\mathbf{P}^*, \omega) \to \mathcal{W}(\mathbf{P}^*, \omega)$ *determined by*

$$\gamma(T)\colon \mathcal{W}(T^* f) \mapsto \mathcal{W}(f) \qquad \text{for all} \quad f \in \mathbf{P}^*$$

and such that $\gamma(ST) = \gamma(S)\gamma(T)$*.*

Theorem 20. *Let $(\mathbf{P}^*, \omega)$ be a symplectic vector space. Then, given a regular state $\langle\ \rangle$ on $\mathcal{W}(\mathbf{P}^*, \omega)$ with characteristic function μ, there is an $x \in \mathbf{P}$ such that*

$$i\partial_f \mu(0) = f(x) \qquad \text{for all} \quad f \in \mathbf{P}^*.$$

Then, the collection of formal symbols $\Psi = \{|x + f^*\rangle : f \in \mathbf{P}^*\}$ generates a complex vector space with the following properties:

1. *the sesquilinear form*

$$\langle x + f^* \mid x + g^* \rangle = e^{\omega(g,f)/2i} \mu(g - f) \tag{0.2}$$

   makes the span of Ψ into a complex pre-Hilbert space whose Hilbert space completion is denoted $\mathbf{K}$

2. *there is a Weyl system $W : \mathbf{P}^* \to U(\mathbf{K})$ on $(\mathbf{P}^*, \omega)$, given by*

$$W(f)\,|x + g^*\rangle = e^{\omega(f,g)/2i}\,|x + f^* + g^*\rangle \qquad \text{for all} \quad f, g \in \mathbf{P}^* \tag{0.3}$$

3. *the unit vector $|x\rangle \in \mathbf{K}$ is a cyclic vector of the Weyl system $W(\mathbf{P}^*, \omega)$*

4. *the associated Heisenberg system $\Phi : \mathbf{P}^* \to L(\mathbf{K})$ satisfies*

$$\langle x + g^* |\, \Phi(f)\, | x + g^* \rangle = f(x + g^*) \qquad \text{for all} \quad f, g \in \mathbf{P}^*.$$

Lemma 21. *Suppose that a regular state* $\langle\ \rangle$ *is given on the Weyl algebra* $\mathcal{W}(\mathbf{P},\omega)$ *and the GNS construction is performed resulting in the Hilbert space* $\mathbf{K}$*, as just described. Then, Equation (**??**) defines a map* $W\colon \mathbf{P}^* \to U(\mathbf{K})$ *which is a Weyl system on* $(\mathbf{P}^*,\omega)$*. In addition, the unit vector* $\psi_0 \in \mathbf{K}$ *is a cyclic vector of the Weyl system* $W\colon \mathbf{P}^* \to U(\mathbf{K})$*.*

Lemma 22. *In the hypotheses of Lemma 21,* $\langle \psi_g \mid \Phi(f)\psi_g \rangle$ *and* $\|\Phi(f)\psi_g\|$ *are both finite for all* $f, g \in \mathbf{P}^*$*. Moreover,*

$$\langle \psi_g \mid \Phi(f)\psi_g \rangle = \omega(f,g) + \langle \psi_0 \mid \Phi(f)\psi_0 \rangle$$

and

$$\|\Phi(f)\psi_g\|^2 - \|\Phi(f)\psi_0\|^2 = \langle \psi_g \mid \Phi(f)\psi_g \rangle^2 - \langle \psi_0 \mid \Phi(f)\psi_0 \rangle^2.$$

Lemma 23. *Assume that* $\langle\ \rangle$ *is a regular state on* $\mathcal{W}(\mathbf{P}^*,\omega)$*, with background* $x \in \mathbf{P}$*. Given any automorphism* $T\colon \mathbf{P} \to \mathbf{P}$ *of the linear phase space* $\mathbf{P}$*, there is a densely defined linear map* $\Gamma(T)\colon \mathbf{K} \to \mathbf{K}$ *such that*

$$\Gamma(T)\,|x + Tf^*\rangle = |x + f^*\rangle\,. \tag{0.4}$$

This map intertwines the unitary operators $W(f)$*, that is,*

$$\Gamma(T)W(T^*f) = W(f)\Gamma(T) \qquad \text{for all} \quad f \in \mathbf{P}^*, \tag{0.5}$$

and satisfies $\Gamma(ST) = \Gamma(S)\Gamma(T)$*.*

Lemma 24. *In the hypotheses of Lemma 23, the operator* $\Gamma(T)$ *extends uniquely to a unitary operator on* $\mathbf{K}$ *if, and only if,* T *preserves* $\langle\ \rangle$ *in the sense that*

$$\mu(T^*h) = \mu(h) \qquad \text{for all} \quad h \in \mathbf{P}^*.$$

Theorem 25. *Let $(\mathbf{P}, \omega)$ be a linear phase space, let $\langle \ \rangle$ be a regular GNS state on the Weyl algebra $\mathcal{W}(\mathbf{P}, \omega)$ with characteristic function μ. Let the background $x \in \mathbf{P}$ associated to μ be defined by*

$$i\partial_f \mu(0) = f(x) \qquad \text{for all} \quad f \in \mathbf{P}^*,$$

and let $\Psi = \{|x + f^*\rangle \mid f \in \mathbf{P}^*\}$. Then,

1. *the sesquilinear form*

$$\langle x + f^* \mid x + g^* \rangle = e^{i\omega(f,g)/2} \mu(g - f)$$

   makes the span of Ψ into a complex pre-Hilbert space whose Hilbert-space completion is denoted $\mathbf{K}$

2. *there is a Weyl system $W \colon \mathbf{P}^* \to U(\mathbf{K})$ on $(\mathbf{P}^*, \omega)$, given by*

$$W(f)\, |x + g^*\rangle = e^{\omega(f,g)/2i}\, |x + g^* + f^*\rangle \qquad \text{for all} \quad f \in \mathbf{P}, g \in \mathbf{P}^*$$

3. *the associated Heisenberg system $\Phi \colon \mathbf{P}^* \to L(\mathbf{K})$ satisfies*

$$\langle x + g^* |\, \Phi(f)\, | x + g^* \rangle = f(x + g^*) \qquad \text{for all} \quad f, g \in \mathbf{P}^*$$

4. *there is a group homomorphism Γ mapping automorphisms $T \colon \mathbf{P} \to \mathbf{P}$ to invertible linear operators on $\mathbf{K}$, given by*

$$\Gamma(T)\, |x + Tf^*\rangle = |x + f^*\rangle \qquad \text{for all} \quad f \in \mathbf{P}^*$$

   and satisfying

$$\Gamma(T) W(T^* f) = W(f) \Gamma(T) \qquad \text{for all} \quad f \in \mathbf{P}^*$$

5. *the unit vector $|x\rangle \in \mathbf{K}$ is a cyclic vector of the Weyl system $W(\mathbf{P}^*, \tilde{\omega})$*

6. *$\Gamma(T)$ is unitary if, and only if, μ is constant on orbits of T.*

Theorem 26. *Let* $\mathbf{H}$ *be a complex Hilbert space with inner product* $\langle\ ,\ \rangle$ *and norm* $\|\ \|$. *Define* h, *and* ω *on* $\mathbf{H} \cong \mathbf{P}^*$ *and* $*\colon \mathbf{H} \to \mathbf{P}$ *as above. Then, the representation of the general boson field on* $\mathcal{W}(\mathbf{P}^*, \omega)$ *given by the regular state with characteristic functional*

$$\mu(f) = e^{-\|f\|^2/4} \qquad \text{for all} \quad f \in \mathbf{H}$$

is the free boson field on $\mathbf{H}$, *with*

1. $\mathbf{K}$ *being the completion of the span of* $\Psi = \{|f^*\rangle : f \in \mathbf{H}\}$ *with respect to the complex inner product*

$$\langle f^* \mid g^* \rangle = e^{\omega(g,f)/2i} e^{-\|g-f\|^2/4}$$

2. W *being the Weyl system on* $\mathcal{W}(\mathbf{H}, \omega)$ *given by*

$$W(f)\,|g^*\rangle = e^{ig^*(f)/2}\,|g^* + f^*\rangle \qquad \text{for all} \quad f, g \in \mathbf{H}$$

3. Γ *being defined by*

$$\Gamma(U)\,|f^*\rangle = |(Uf)^*\rangle \qquad \text{for all} \quad f \in \mathbf{H}$$

4. $\nu = |0\rangle$

In addition, the mean and variance of $\Phi(g)$ *in the state* $|x\rangle$ *are*

$$\langle f^* |\, \Phi(g)\, | f^* \rangle = \omega(g, f) \qquad \text{and} \quad \mathrm{Var}_{f^*}(g) = \frac{1}{2}\|g\|^2 \qquad \text{for all} \quad x, f \in \mathbf{H}.$$

Lemma 27. *If* $f, g \in \mathbf{H}$ *then*

$$a(g)\,|h^*\rangle = \frac{\langle g, h\rangle}{i\sqrt{2}}\,|h^*\rangle\,.$$

Lemma 28. *For all* $n \in \mathbb{N}$ *and all* $f \in \mathbf{H}$*, the Wick power* $:\!\Phi(f)^n\!:$ *is densely defined on* $\mathbf{K}$*.*

Lemma 29. *The matrix elements of Wick powers on coherent states satisfy*

$$\frac{\langle f^*|\,:\!\Phi(g)^n\!:\,|h^*\rangle}{\langle f^* \mid h^*\rangle} = \Big(\frac{\langle f^*|\,\Phi(g)\,|h^*\rangle}{\langle f^* \mid h^*\rangle}\Big)^n$$

whenever $f, g, h \in \mathbf{H}$*.*

Lemma 30. *Let* $\mathbf{H}_0 \subseteq \mathbf{H}$ *be a topological vector space with a dense continuous inclusion into* $\mathbf{H}$*. Then, if* $f_n \in \mathbf{H}_0$ *for all* n *and* $\lim_{n\to\infty} f_n = f$ *in the topology of* $\mathbf{H}$*, then*

$$\lim_{n\to\infty} |f_n^*\rangle = |f^*\rangle$$

in the topology of $\mathbf{K}$*.*

Lemma 31. *For every* $g \in \mathbf{H}_0^\dagger$ *there is a unique quasioperator* $\Phi(g)$ *on* $\mathbf{K}$ *with domain* $\mathbf{K}_0$ *such that*

$$\frac{\langle f^*|\,\Phi(g)\,|h^*\rangle}{\langle f^* \mid h^*\rangle} = \frac{i}{2}[\langle f, g\rangle - \langle g, h\rangle] \qquad \textit{for all} \quad f, h \in \mathbf{H}_0.$$

Lemma 32. *For every* $g \in \mathbf{H}_0^\dagger$ *there is a unique quasioperator* $:\!\Phi(g)^n\!:$ *on* $\mathbf{K}$ *with domain* $\mathbf{K}_0$ *such that*

$$\frac{\langle f^*|\,:\!\Phi(g)^n\!:\,|h^*\rangle}{\langle f^* \mid h^*\rangle} = \Big(\frac{\langle f^*|\,\Phi(g)\,|h^*\rangle}{\langle f^* \mid h^*\rangle}\Big)^n \qquad \textit{for all} \quad f, h \in \mathbf{H}_0.$$

Corollary 33. *Let* $F\colon \mathbb{C}^n \to \mathbb{C}$ *be an entire function. Then, for all* $g \in \mathbf{H}_0^\dagger$*, there is a unique quasioperator* $:\!F(\Phi(g))\!:$ *on* $\mathbf{K}$ *with domain* $\mathbf{K}_0$ *satisfying*

$$\frac{\langle f^*|\,:\!F(\Phi(g))\!:\,|h^*\rangle}{\langle f^* \mid h^*\rangle} = F\Big(\frac{\langle f^*|\,\Phi(g)\,|h^*\rangle}{\langle f^* \mid h^*\rangle}\Big) \qquad \textit{for all} \quad f, h \in \mathbf{H}_0.$$

Lemma 34.

$$:\!W(g)\!: = \frac{W(g)}{\langle 0|\,W(g)\,|0\rangle} \qquad \textit{for all} \quad g \in \mathbf{H}_0^\dagger$$

as an equation between quasioperators on $\mathbf{K}$ *with domain* $\mathbf{K}_0$*.*

Theorem 35. *Let $\mathbf{E}_o$ be a real Hilbert space with inner product $(\ \mid\ )$, let L be a nonnegative self-adjoint operator on $\mathbf{E}_o$ with vanishing kernel, and consider the real Hilbert space*

$$\mathbf{A}_o := \{A \in \mathbf{E}_o : \|A\|^2 + \|L^{1/2}A\|^2 < \infty\}.$$

Define time evolution on $\mathbf{P}_o = \mathbf{A}_o \oplus \mathbf{E}_o$ by

$$\partial_t(A \oplus E) = E \oplus -LA,$$

which preserves the canonical symplectic structure on $\mathbf{A}_o \oplus \mathbf{E}_o$, namely

$$\omega(A \oplus E, A' \oplus E') = (A \mid E') - (A' \mid E).$$

Then, there is a densely-defined complex structure $J: \mathbf{Y} \to \mathbf{Y}$ given by $J = -L^{-1/2}K$, or

$$J(A \oplus E) := -L^{-1/2}E \oplus L^{1/2}A,$$

commuting with K and whose domain

$$\mathbf{Y} := \{A \oplus E \in \mathbf{P}_o : \|A\|^2 + \|L^{1/2}A\|^2 + \|E\|^2 + \|L^{-1/2}E\|^2 < \infty\}$$

is dense in $\mathbf{P}_o$, preserved by time evolution and satisfying

$$\|Jx\|_{\mathbf{Y}} = \|x\|_{\mathbf{Y}} \qquad \text{and} \quad \omega(Jx, Jy) = \omega(x, y) \qquad \text{for all} \quad x, y \in \mathbf{Y}.$$

Finally, the completion of $\mathbf{Y}$ with respect to the norm

$$\|x\|^2_{\mathbf{H}} := \omega(x, Jy)$$

is a complex Hilbert space $\mathbf{H}$ with inner product

$$\langle x, y \rangle := \omega(x, Jy) + i\omega(x, y)$$

Time evolution defined on $\mathbf{Y}$ then extends to a strongly-continuous one-parameter group of unitary operators on $\mathbf{H}$, with nonnegative, self-adjoint generator $H = L^{1/2}$.

Theorem 36. *Let $T_o(t)$ be a one-parameter group of symplectic transformations on the linear symplectic space $(\mathbf{P}, \omega)$. Then there is at most one complex structure J on $\mathbf{P}$ which is invariant, positive, symplectic and such that the self-adjoint generator H of $T_o(t)$ in the completion of $\mathbf{P}$ as a complex Hilbert space, $\mathbf{H}$, is nonnegative and with vanishing kernel.*

Theorem 37. *Let* $|X(t)\rangle = \Gamma(U(t))\,|X\rangle$ *for all* $X \in \mathbf{P}$. *Then,*

$$\begin{aligned}
\frac{\partial}{\partial t}\langle X'(t)|\,\hat{A}\,|X(t)\rangle &= \langle X'(t)|\,\hat{E}\,|X(t)\rangle \\
\frac{\partial}{\partial t}\langle X'(t)|\,\hat{E}(x)\,|X(t)\rangle &= -\langle X'(t)|\,L_p\hat{A}\,|X(t)\rangle
\end{aligned}$$

Corollary 38.

$$\frac{\partial}{\partial t}\frac{\langle X'|\, e^{i\oint_\gamma \hat{A}}\, |X\rangle}{\langle X' \mid X\rangle} = \frac{i\,\langle X'|\oint_\gamma \hat{E}\,|X\rangle}{\langle X' \mid X\rangle}\exp\frac{i\,\langle X'|\oint_\gamma \hat{A}\,|X\rangle}{\langle X' \mid X\rangle}.$$