Packing Density as a Function on the Voronoi Graph of Perfect Forms

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Abstract

We observe that for small n *the ball packing density has a* unique *local maximum on the reduced Voronoi graph, whose vertices are the arithmetic equivalence classes of perfect quadratic forms and whose edges are the walls between the corresponding perfect domains.*

1 Introduction

Consider $\mathbb{R}^n$ $(n \geq 0)$ as a topological group with respect to vector addition. A *lattice* is a discrete subgroup of $\mathbb{R}^n$; if $o \in \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}^n$ is a lattice, then the set $\{o + v \mid o \in \mathbb{R}^n,\ v \in \Lambda\}$ is called a *point lattice.* Let $\mathcal{Q}$ be a positive definite quadratic form on $\mathbb{R}^n$. Denote by $m(\mathcal{Q})$ the minimum of $\mathcal{Q}$ on $\Lambda \setminus 0$. A vector v of Λ is called minimal for $\mathcal{Q}$ if $\mathcal{Q}[v] = m(\mathcal{Q})$. Consider a linear inhomogeneous system the system of equations

$$X[v] = m(\mathcal{Q}),$$

where v runs over all minimal vectors of Λ and the coefficients of an unknown quadratic form X are considered as variables. The form $\mathcal{Q}$ on Λ is called *perfect* if every non-zero solution of the above system is of the form $X = c\mathcal{Q}$, where $c \in \mathbb{R}_{>0}$.

We will refer to a lattice of rank (dimension) n as an n-lattice. Same convention applies to polytopes, subspaces, etc. With any n-lattice Λ in $\mathbb{R}^n$ we can consider a set of equal n-balls centered at the elements of Λ. Suppose the interiors of the balls do not overlap. Such an arrangement of balls is called a lattice ball packing (and, in the case of a concrete lattice *Lambda*, the *Lambda*-packing). The density of the packing is given by

$$\frac{\operatorname{Vol} B}{\operatorname{Vol} \Pi},$$

where B stands for a ball in the Λ-packing and Π for a fundamental parallelepiped of the lattice. A problem, studied by many, is to find a lattice that gives the highest possible density among all lattices of its dimension. Obviously, it is enough to restrict our attention to packings of balls that are tight, i.e. no increase of the ball size is possible without violating the condition that the interiors of the balls do not overlap. Geometrically, this means each ball touches at least two other balls. Again, it is obvious that we can further restrict to packings where each ball touches at least $2n$ balls. Furthermore, we only need to examine those Λ's where any small perturbation of the angles and lengths does not increase the density. Such lattices are called *extreme.*

An important observation that goes back to Gauss (for $n = 2$) and Minkowski (for general dimension) is that isometry classes of lattices in $\mathbb{R}^n$ are in 1-to-1 correspondence with arithmetic equivalence classes of positive definite quadratic forms ($\mathcal{Q} \cong \mathcal{Q}'$ iff there is $\tau \in GL_n(\mathbb{Z})$ such that

$\mathcal{Q}'[\tau x] = \mathcal{Q}[x]$). Namely, given a basis of an n-lattice, the corresponding quadratic form is given by $x \mapsto x^t B^t B x$, where x is a column vector and B is the matrix of the basis. Conversely, any symmetric positive definite $n \times n$ matrix admits a factorization $M^t M$, where $\operatorname{rank} M = n$. Then the columns of M can be interpreted as basis vectors for a lattice.Thus, the packing problem can be rephrased in terms of quadratic forms. When dealing with quadratic forms we fix the lattice Λ as $\mathbb{Z}^n$ and change the coefficients of the (variable) form. Quadratic forms can be considered as vectors in $\mathbb{R}^N$, where $N = \frac{n(n+1)}{2}$. This interpretation endows the set of all quadratic forms in n-variables with a dot product. If A and B are the matrices of quadratic forms, then the dot product $A \cdot B$ is defined as $\mathsf{Tr}(AB)$. Hence, for a fixed n we can talk about locally best packing forms as opposed to globally best packing forms.

Korkin & Zolotareff (1873) proved that a form, which is a local maximum of the packing density, must be perfect. Using this characterization they were able to find all the best packing lattices (forms) for $n \leq 8$. They also constructed a few infinite series (in dimension) of perfect forms. In particular, they described all root lattices. It is not that they were interested in reflections, it just happens that all root lattices, except for $\mathbb{Z}^n$ are perfect. Not all perfect forms are extreme (locally maximal). In order for a perfect form to be extreme it must be eutactic, which is a classical theorem of Voronoi (there is another criterion due to Korkin and Zolotareff, see Ryshkov and Baranovskii (1979) or Martinet (2003)). A form $\mathcal{Q}$ is called eutactic if its dual can be written as a strictly positive linear combination of rank one forms built from the minimal vectors of $\mathcal{Q}$ (i.e. forms of type $(v \cdot x)^2$, where v is a minimal vector of $\mathcal{Q}$ and x is a variable vector).

For each n starting from 4 there is more than one extreme form. This makes it difficult to rigorously solve the problem of the best lattice packing; even if we suspect that certain lattice is absolutely the best, we have to prove that other local maxima do not beat our candidate. From a formal point of view we are solving a *non-convex* optimization problem on the cone of positive quadratic forms. Since each form has infinitely many equivalent ones, it is enough to find all perfect forms up to arithmetic equivalence. The number of arithmetic equivalence classes of perfect forms is finite in each dimension (a theorem of Voronoi, 1908). Voronoi gave an algorithm that finds all perfect forms in a given dimension. Voronoi thought of each perfect form as of a cone whose extreme rays are rank one forms corresponding to the minimal vectors of the form. Two cones are called adjacent if they share an $(N-1)$-face (*facet*). In the literature on perfect forms these facets are usually called *walls*. The graph whose vertices are perfect forms in n variables and whose edges are common walls of these cones is called the *Voronoi graph* for dimension n. The *reduced Voronoi graph* has one vertex for each arithmetic equivalence class and one edge for each arithmetic equivalence class of a pair of perfect cones with a common facet. It may happen that on each side of a wall we find arithmetically equivalent forms. That is why for $n \geq 2$ the reduced Voronoi graph has loops.

2 Packing density as a function on Voronoi's graph

Let (G, V) be a graph. We say that a vertex $v \in V$ is a local maximum of a function $f : V \to \mathbb{R}$ if for any w adjacent to v we have $f(w) \leq f(v)$. We observed that for small values of n ($n \leq 8$) the density function has a unique local maximum on the reduced Voronoi graph. In other words, the only local maximum of the packing density is the global maximum. In dimension 9 the situation seems to persist (as of this time perfect forms have been classified through the dimension 8).

Conjecture 1 *The packing density has a unique local maximum on the reduced Voronoi graph.*

In dimensions where this conjecture holds the best packing lattice can be found by a greedy algorithm that stops at the first encountered local maximum. This means once local optimality of a vertex v is established, one does not need to examine the neighborhoods of vertices adjacent to v. Hypothetically, this means that in certain dimensions we might be able to find the densest packing lattice without complete enumeration of perfect forms. Of course, for some vertices of the reduced Voronoi graph finding all the adjacent vertices can be very difficult, as demonstrated by the case of E_8 (Dutour, Schuermann, Valentin, 2007).

2.1 $n \leq 3$

In each of these dimensions the perfect form is unique and is equivalent to A_n.

2.2 $n \leq 6$

In our pictures we use two notations, Coxeter's (1951) notation and Conway-Sloane's (1988) notation. Coxeter's notation covers only root lattices and their centerings. Conway-Sloane's notation is the simplest one; in each dimension n perfect lattices are named P_n^1 through P_n^{33} in the order of decreasing density, e.g. for $n = 6, 7, 8$ P_n^1 stands for E_6, E_7, E_8 respectively. *We will omit the dimensional subindex when the dimension is clear from the context.* There are two other systems of notation, the original one by Korkin and Zolotareff (1873) and the one by Voronoi (1908), as extended by Barnes (1957). Korkin-Zolotareff notation is not very convenient. Voronoi-Barnes notation is quite systematic (see Barnes (1952) and Anzin (2003)).

In all of the following pictures there is an arrow from a vertex X to a vertex Y if and only if the density of X is strictly less than the density of Y (except for Figures 2 and 3 where the arrow from P^{28} to P^{27} corresponds to an equality.) It is known from Voronoi (1908) that in all dimensions A_n is adjacent only to D_n.

2.3 $n = 7$

In dimension 7 all forms, except for $A_7 = P^{33}$, are connected to $E_7 = P^1$. It is worth to mention that $P^2 = E_7^*$ and $P^4 = D_7$. For $n = 7$ we do not show loops and multiple edges. See Figures 2-4 for some other visualization of this graph.

2.4 $n = 8$

The situation in dimension 8 almost repeats the one in dimension 7. Namely, all forms except for A_8 and the form number 8190 (this is not the form number in Conway-Sloane's notation) on the list of 8-dimensional perfect forms compiled by Dutour, Schuermann, and Vallentin are connected to E_8. A simple check shows that the latter form is connected to forms of higher density.

2.5 $n = 9$

One change that is worth mentioning is that in dimension 9 the densest known form is not connected to D_9, unlike the cases of $n \leq 8$ (Anzin, 2003).

3 Discussion

We suspect that there is hidden convexity structure on perfect (or just extreme ?) forms, at least in low dimensions. Perhaps the best way to look for this structure is by recasting the description of

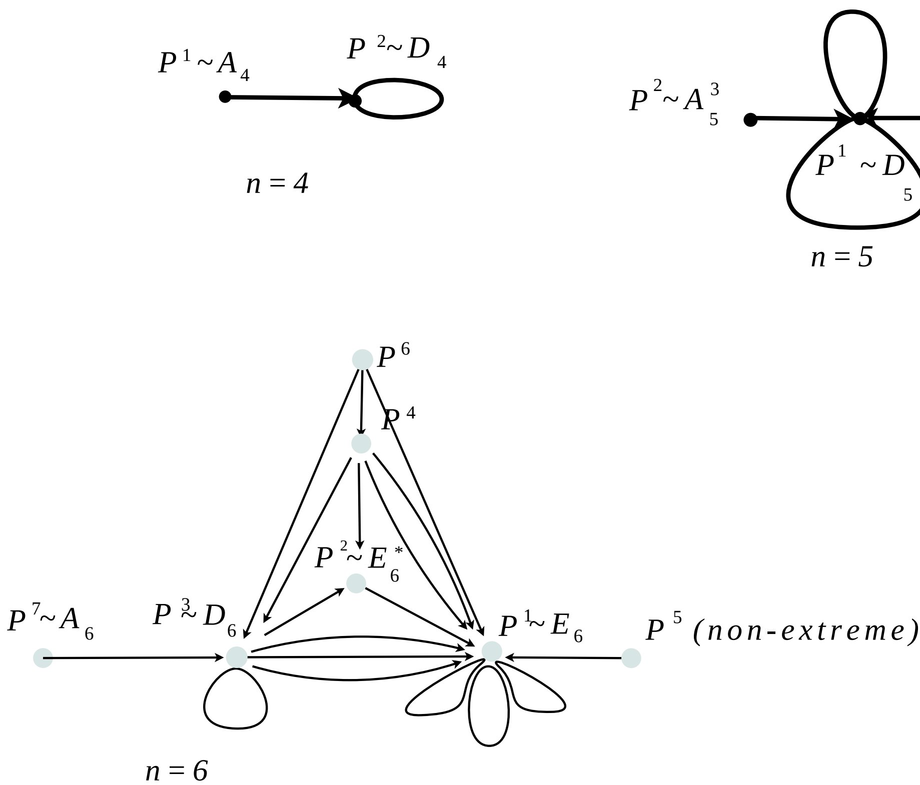

Figure 1: Reduced Voronoi graph for $n = 4, 5, 6$. Arrows point in the direction of increasing packing density.

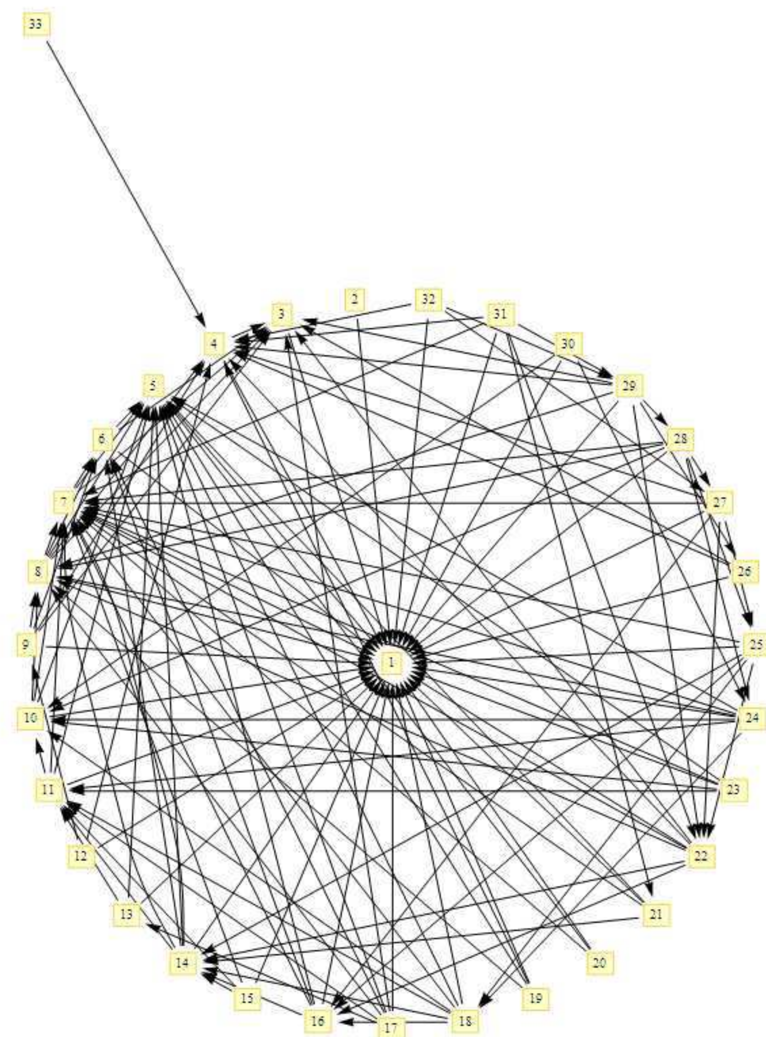

Figure 2: Reduced Voronoi graph for $n = 7$ (loops are not shown). Arrows point in the directions of increasing packing density. Although there an arrow from P^{28} to P^{27} is shown, they have the same density.

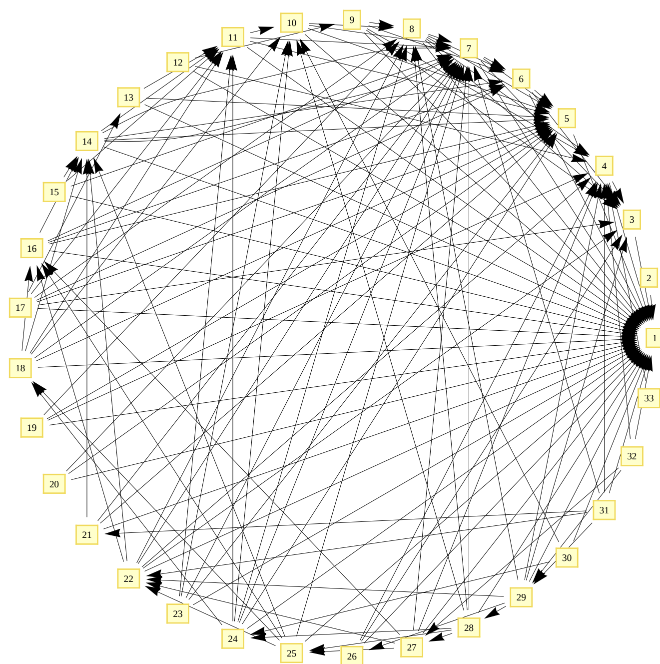

Figure 3: Circular visualization of reduced Voronoi graph for $n = 7$ (loops are not shown). Arrows point in the directions of increasing packing density. Although there an arrow from P^{28} to P^{27} is shown, they have the same density.

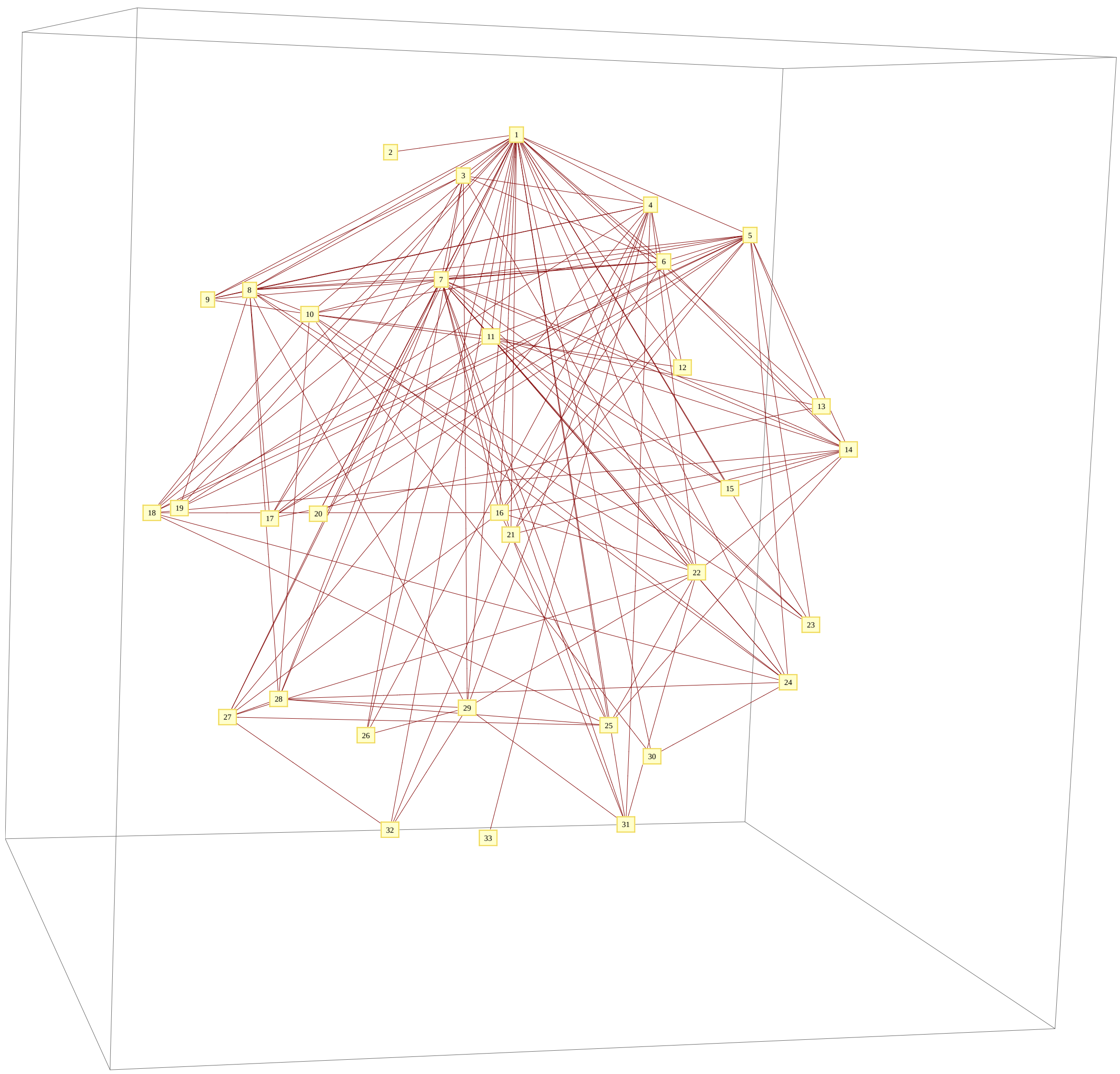

Figure 4: 3D visualization of reduced Voronoi graph for $n = 7$ (loops are not shown). Vertices with with higher altitude have higher density.

perfect domains in terms of Ryshkov's polyhedron $\mu(n)$ (Ryshkov, 1970). The facets of polyhedron μ are defined by inequalities $X[v] \geq 1$, where v runs over all primitive integer vectors and X is a variable positive quadratic form. Polyhedron μ is known to be combinatorially dual to Voronoi's polyhedron Π. Indeed, if $\mathbb{R}^N$ is regarded as a *vector* space, the vertices of μ are perpendicular to the facets (not the walls!) of Π. However, polyhedron μ seems to provide more geometric insight than perfect cones. The relationship between μ and Π as *affine* objects calls for further investigation.

References

[Az] M.M. Anzin, Investigation of the third perfect form of G. F. Voronoĭ, $\varphi_2^{(n)} = \varphi_1^{(n)} - \rho_n x_1 x_3$. (Russian) Dedicated to the 75th birthday of Aleksandr Vasil'evich Malyshev. *ChebyshevskiĭSb.* 4 (2003), no. 3(7), 24–58.

[Bar] E. S. Barnes, The complete enumeration of extreme senary forms. *Philos. Trans. Roy. Soc. London.* Ser. A. **249** (1957), 461–506.

[CS] J. H. Conway, N. J. A. Sloane, Low-dimensional lattices. III. Perfect forms. *Proc. Roy. Soc. London Ser. A* **418** (1988), no. 1854, 43–80.

[1] Coxeter, H. S. M. Extreme forms. *Canadian J. Math.* **3**, (1951). 391–441.

[DSV] M. Dutour, A. Schuermann, F. Vallentin, (2006) Classification of eight dimensional perfect forms, *Electron. Res. Announc. Amer. Math. Soc.* **13** (2007), 21-32. Preprint arXiv:math/0609388v2[math.NT].

[R70] S. S. Ryshkov, The polyhedron (m) and certain extremal problems of the geometry of numbers. (Russian) *Dokl. Akad. Nauk SSSR,* **194**, (1970), 514–517.

[RB79] S. S. Ryshkov, E. P. Baranovskii, Classical methods in the theory of lattice packings, *Russian Mathematical Surveys* **34**, (1979).

[GL] Gruber, P. M.; Lekkerkerker, C. G. *Geometry of numbers.* Second edition. North-Holland Mathematical Library, 37. North-Holland Publishing Co., Amsterdam-New York, (1987)

[KZ] A. Korkine, G. Zolotareff, Sur les formes quadratiques, *Math. Ann.* 6 (1873), 366-389.

[M] J. Martinet, *Perfect lattices in Euclidean spaces.* Fundamental Principles of Mathematical Sciences, **327**, Springer-Verlag, Berlin, (2003).

[VorI08] G. F. Voronoi, Nouvelles applications des paramèters continus à la théorie des formes quadratiques, Prèmier memoire, *J. Reine Angew. Math.*, **133** (1908), 97–178.