

ON QUADRATIC SCHRÖDINGER EQUATIONS IN \mathbf{R}^{1+1} : A NORMAL FORM APPROACH

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ABSTRACT. For the Schrödinger equation $u_t + iu_{xx} = \langle \nabla \rangle^\beta [u^2]$, $\beta \in (0, 1/2)$, we establish local well-posedness in $H^{\beta-1+}$ (note that if $\beta = 0$, this matches, up to an endpoint, the sharp result of Bejenaru-Tao, [4]). Our approach differs significantly from the previous one - we use normal form transformation to analyze the worst interacting terms in the non-linearity and then show that the remaining terms are (much) smoother. In particular, this allows us to conclude that $u - e^{-it\partial_x^2} u(0) \in H^{-\frac{1}{2}}(\mathbf{R}^1)$, even though $u(0) \in H^{\beta-1+}$.

In addition, as a byproduct of our normal form analysis, we obtain a Lipschitz continuity property in $H^{-\frac{1}{2}}$ of the solution operator (which originally acts on $H^{\beta-1+}$), which is new even in the case $\beta = 0$. As an easy corollary, we obtain local well-posedness results for $u_t + iu_{xx} = \langle \nabla \rangle^\beta z \langle \nabla \rangle^\beta \bar{z}$.

Finally, we sketch an approach to obtain similar results for the equations $u_t + iu_{xx} = \langle \nabla \rangle^\beta [u\bar{u}]$ and $u_t + iu_{xx} = \langle \nabla \rangle^\beta [\bar{u}^2]$.

1. INTRODUCTION

In this paper, we will be concerned with local solutions of the quadratic Schrödinger equations

$$(1) \quad \begin{cases} u_t + iu_{xx} = Q(u, u) : & (t, x) \in \mathbf{R}_+^1 \times \mathbf{R}^1 \\ u(0, x) = u_0 \end{cases}$$

The problem has received a lot of attention in the last twenty years and a full account of the appropriate results and open questions is beyond of the scope of the current project. We will however outline a selected list of recent works, which has some bearing on the problem that we are studying.

The classical results of the subject go back to Tsutsumi, [22], which establishes local well-posedness for data in H^s , $s \geq 0$ for all quadratic nonlinearities (i.e. $|Q(u, u)| \leq C|u|^2$). This is in a way optimal, since for Hamiltonian models (i.e. with $Q(e^{i\theta}u, e^{i\theta}u) = e^{i\theta}Q(u, u)$), it is well-known that there is ill-posedness in H^s , $s < 0$ - this is in the work of Kenig-Ponce-Vega, [13], see also Christ, [6] and Christ-Colliander-Tao, [7] for further results in this direction.

For the non-Hamiltonian model, several different models have been considered in the literature, the most popular being $Q(u, u) = u^2, u\bar{u}, \bar{u}^2$. Each of these comes with its own specifics and the corresponding local well-posedness results reflect that. Regarding the cases u^2, \bar{u}^2 , it has been show by Kenig-Ponce-Vega [13], that these are well posed in

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$H^{-\frac{3}{4}+}$ by means of bilinear estimates in $X^{s,b}$ spaces. Moreover, such bilinear estimates necessarily fail at the critical index $-3/4$, [13, 15]. Regarding the nonlinearity $u\bar{u}$, it has been shown, that the problem is well-posed in $H^{-\frac{1}{4}+}$, [13] and this turns out to be sharp¹, [14]. On the other hand, the results for $u\bar{u}$ may be pushed down to the really sharp index $s = -1/2+$, if one is willing to put some homogeneous Sobolev space requirements on the low-frequency portion of the data, [14]

Regarding the nonlinearity u^2 , the results of [13] were extended to the sharp index $s \geq -1$, by Bejenaru-Tao [4], see also the work Bejenaru-Da Silva, [3] for the same result in two spatial dimensions. As we have mentioned, the spaces $X^{s,b}$ by themselves, could not accommodate such low regularity of solutions, so the authors had to come up with further refinements of these spaces, in which they were able to perform their fixed point arguments, see also Nishimoto, [17] for interesting commentary on these developments. For the nonlinearity \bar{u}^2 , it has been shown that similar techniques may be used to obtain l.w.p. in H^s , $s \geq -1$, Nishimoto, [16]. It also should be noted that in all of these papers (with the exception of [17]), it is hard to show optimal l.w.p. for Schrödinger equations with nonlinearity of the form $Q(u, u) = c_1 u^2 + c_2 \bar{u}^2$, due to the specifics of the approaches. The result in [17] achieves this goal, at the expense of further layer of complexity, involved in the definition of the spaces and the corresponding bilinear estimates that need to be shown.

Our main result concerns the following specific generalization of the quadratic Schrödinger equation (1)

$$(2) \quad \begin{cases} u_t + iu_{xx} = \langle \nabla \rangle^\beta [u^2] : & (t, x) \in \mathbf{R}_+^1 \times \mathbf{R}^1 \\ u(0, x) = u_0 \in H^{-\alpha}, \end{cases}$$

where $\beta \geq 0$. This model has been well-studied over the years, mainly the case $\beta = 1$. We should first mention, that the corresponding equation is ill-posed, in the sense that the flow map $u_0 \rightarrow u$ experiences norm inflation, Christ [6], see also [7] for related results. In the work of Stefanov, [20] existence of weak solutions in H^1 was shown, under the additional smallness requirement $\sup_x |\int_{-\infty}^x u_0(y) dy| \ll 1$. Similar results² (in \mathbf{R}^n , $n \geq 1$), were obtained for the more general Ginzburg-Landau equation in the work of Han-Wang-Guo, [11]. Finally, we mention some recent local well-posedness results, which were obtained for (not necessarily small) data in weighted Sobolev spaces by Bejenaru, [1, 2] and Bejenaru-Tataru, [5].

While some of the positive results mentioned above surely will extend to the case $\beta \in (0, 1)$, it seems that this model has not been considered in the literature. One of the purposes of the current paper is to address the question for local well-posedness of this problem. An alternative goal is to develop an alternative proof of the known results in the case $\beta = 0$, which is within the framework of the standard $X^{s,b}$ spaces. We achieve that by the technique of normal forms. This method has been used extensively in the last twenty years, to treat global small solutions of models with low-power nonlinearities - see the pioneering work of Shatah, [18] and more recent developments in Germain-Masmoudi-Shatah [9], [10] and Shatah, [19].

¹At least as far as the uniform continuity of the solution map goes

²again for data small in L^1 sense and so that it belongs to some smooth modulation spaces

Informally, the method starts with a preconditioning of the equation first (via a change of variables - normal form). The particular type of the normal form is a bilinear pseudo-differential operator, which solves explicitly the Schrödinger equation with right-hand side, which consists of the most troublesome terms in the nonlinearity, see (34) below. Then, one needs to argue that the remaining terms of the solution (which are obtained through a fixed point argument, involving themselves, the normal form and various interactive terms) are better behaved - in our case, they are (much) smoother than the free solution. As a byproduct of this approach, we obtain more precise information on these correction terms (see (3)). *To the best of our knowledge, this paper is the first one that treats low regularity solutions, with the method of normal forms.*

Our main results recover (up to an endpoint) the sharp results of [4], [17] in the case $\beta = 0$, but of course covers also the new cases $\beta \in (0, 1/2)$, where the results are also arguably sharp. We also obtain the Lipschitz property (4) of the solution map, which is a new feature, even in the case $\beta = 0$.

More specifically,

Theorem 1. *Let $\beta \in [0, \frac{1}{2})$ and $\alpha : \frac{1}{2} < \alpha < 1 - \beta$. Then the equation (2) is locally well-posed in $H^{-\alpha}(\mathbf{R}^1)$. More specifically, for every $u_0 \in H^{-\alpha}(\mathbf{R}^1)$, there exists a non-trivial $T > 0$, so that the equation (2) has a unique solution $u \in C([0, T], H^{-\alpha})$.*

Moreover, for fixed $\delta : 0 < \delta < \frac{1-\alpha-\beta}{10}$, there is the following decomposition

$$(3) \quad u = e^{-it\partial_x^2} u_0 + h + w,$$

where $h \in L_t^\infty H_x^{\frac{1}{2}-\alpha} \cap X^{1-\alpha-\delta, \delta}$, $w \in X^{-\frac{1}{2}, \frac{1}{2}+\delta}$. In particular $u - e^{-it\partial_x^2} u_0 \in L_t^\infty H_x^{-\frac{1}{2}}$.

We also have the following Lipschitz property of the solution map of (2). Let $N > 0$ and $u_0, v_0 \in H^{-\alpha}(\mathbf{R}^1) : \|u_0\|_{H^{-\alpha}} < N, \|v_0\|_{H^{-\alpha}} < N$, so that $u_0 - v_0 \in H^{-\frac{1}{2}}$. Then the corresponding solutions (defined on a common non-trivial time interval $(0, T_N)$) satisfy

$$(4) \quad \|u - v\|_{L_T^\infty H_x^{-\frac{1}{2}}} \leq C_N \|u_0 - v_0\|_{H_x^{-\frac{1}{2}}}.$$

where C_N depends only on N .

Remarks:

- There are of course well-posedness results in the cases $\alpha \in (0, 1/2)$ and they are easier to obtain. We chose to include only those with $\alpha > 1/2$ in order to simplify our exposition.
- The Lipschitz property (4) is new even in the case $\beta = 0$.
- We do not obtain l.w.p. for the case $\alpha = 1 - \beta$, which in the case $\beta = 0$, will correspond to the endpoint case of $s = 1$, considered in [4]. Our arguments would imply such a statement, at least in the case of a Besov-1 space version of the main result.
- While our arguments fail at $\beta \geq 1/2$, we cannot claim sharpness in this regard. However, we very strongly suspect that this is the case. That is, we conjecture that some form of ill-posedness must occur, when one considers solutions to (2) with $\beta = 1/2$.

We also have the following corollary. Consider

$$(5) \quad \begin{cases} z_t + iz_{xx} = \langle \nabla \rangle^\beta z \langle \nabla \rangle^\beta z & (t, x) \in \mathbf{R}_+^1 \times \mathbf{R}^1 \\ z(0, x) = z_0 \end{cases}$$

Setting $u = \langle \nabla \rangle^\beta z$ yields the equation

$$(6) \quad u_t + iu_{xx} = \langle \nabla \rangle^\beta [u^2],$$

for u . By Theorem 1, we conclude that (6) is well-posed in $H^{-\alpha}$, for all $\frac{1}{2} < \alpha < 1 - \beta$. Therefore, in terms of z , we have well-posedness in $H^{-\alpha+\beta}$.

Corollary 1. *Let $\beta \in [0, 1/2)$ and $0 < \gamma < 1 - 2\beta$. The equation (5) is well posed in $H^{-\gamma}$.*

The paper is organized as follows. In Section 2, we introduce the $X^{s,b}$ spaces, as well as Tao's theory of bilinear estimates with $X^{s,b}$ entries, [21]. In Section 3, we construct the normal forms and set the function spaces that will be used. In Section 3.2, we provide the basic estimates for the normal form transformation. In Section 3.3, we state and prove the bilinear and trilinear estimates, needed for the fixed point argument. This is where the main technical difficulties are present. In Section 3.4, we conclude the proof, by reducing it to the bilinear and trilinear estimates proved in Section 2 and Section 3.3. In Section 4, we give some ideas on how to approach the question for local well-posedness for the problem with nonlinearities of the form $\langle \nabla \rangle^\beta [u\bar{u}]$ and $\langle \nabla \rangle^\beta [\bar{u}^2]$.

2. SOME NOTATIONS AND PRELIMINARIES

2.1. Littlewood-Paley projections and Paraproducts. Introduce the Fourier transform and its inverse via

$$\begin{aligned} \hat{h}(\xi) &= \int_{\mathbf{R}^1} h(x) e^{-ix\xi} dx, \\ h(x) &= \frac{1}{2\pi} \int_{\mathbf{R}^1} \hat{h}(\xi) e^{ix\xi} d\xi. \end{aligned}$$

Let $\Phi : \mathbf{R}^1 \rightarrow \mathbf{R}^1$ be a positive, smooth even function supported in $\{\xi : |\xi| \leq 2\}$, and $\Phi(\xi) = 1$ for all $|\xi| \leq 1$. Define $\varphi(\xi) = \Phi(\xi) - \Phi(2\xi)$, which is supported in the annulus $1/2 \leq |\xi| \leq 2$. Clearly $\Phi(\xi) + \sum_{k \in \mathbf{Z}^+} \varphi(2^{-k}\xi) = 1$ for all ξ .

The k^{th} Littlewood-Paley projection is $\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi) \widehat{f}(\xi)$ for $k \in \mathbf{Z}^+$. Similarly, $\widehat{P_{\leq 0} f}(\xi) = \Phi(\xi) \widehat{f}(\xi)$, and for any subset $A \subseteq \mathbf{Z}$, we denote $\widehat{P_A f}(\xi) = \sum_{k \in A} \varphi(2^{-k}\xi) \widehat{f}(\xi)$ with the summation replaced by $\Phi(\xi)$ when necessary. The kernels of $P_k, P_{\leq 0}$ are uniformly integrable and thus $P_k, P_{\leq 0} : L^p \rightarrow L^p$ for $1 \leq p \leq \infty$ with the bound equivalent to $\|\widehat{\Phi}(\xi)\|_{L^1}$ (independent of k). We will often use the notation g_k in place of $P_k g$.

Define the operator $\langle \nabla \rangle^\alpha : H^s \rightarrow H^{s-\alpha}$ by $\widehat{\langle \nabla \rangle^\alpha g}(\xi) = (1 + \xi^2)^{1/2} \widehat{g}(\xi)$. Then $\|g\|_{H^s} = \|\langle \nabla \rangle^s g\|_{L^2}$, so $\|g_k\|_{H^s} \sim 2^{ks} \|g\|_{L^2}$ and $\|g_{\leq 0}\|_{H^s} \sim \|g\|_{L^2}$.

Note that in order to simplify the exposition, we will use the notation somewhat loosely in the sense that g_k will always denote a function so that $\widehat{g_k}(\xi) = \psi(2^{-k}\xi) \widehat{g}(\xi)$, where $\psi \in C_0^\infty([1/2, 2])$ (but the function ψ may change from line to line).

Next, we introduce the following basic decomposition from the theory of the paraproducts. For any two Schwartz functions f, g and $k \in \mathbf{Z}$,

$$P_k(fg) = P_k\left(\sum_{l,m} f_l g_m\right) = P_k\left(\sum_{|l-m|\leq 3} f_l g_m\right) + P_k\left(\sum_{|l-m|>3} f_l g_m\right).$$

Furthermore, in the first sum, we have the restriction $\min(l, m) > k - 5$; and in the second sum, we have $|\max(l, m) - k| \leq 3$. Otherwise the $\widehat{f_l g_m}$ will be away from $\{\xi : |\xi| \sim 2^k\}$, and thus $P_k(f_l g_m) = 0$. We refer to the first summand as "high-high interaction" terms, and the second summand as "high-low interaction" terms.

2.2. $X^{s,b}$ spaces and bilinear L^2 multipliers. For $s, b \in \mathbf{R}^1$ and a function h of one variable, we define the (inhomogeneous) $X_{\tau=h(\xi)}^{s,b}$ spaces to be the completion of $\mathcal{S}(\mathbf{R} \times \mathbf{R})$ with respect to the norm³

$$(7) \quad \|u(t, x)\|_{X_{\tau=h(\xi)}^{s,b}} = \left(\int_{\mathbf{R}^1 \times \mathbf{R}^1} (1 + \xi^2)^s (1 + |\tau - h(\xi)|)^{2b} |\tilde{u}(\tau, \xi)|^2 d\tau d\xi \right)^{\frac{1}{2}}$$

where $\tilde{u}(\tau, \xi) = \int_{\mathbf{R}^1 \times \mathbf{R}^1} u(t, x) e^{-i(t\tau + x\xi)} dt dx$. Note that $X_{\tau=h(\xi)}^{s,0} = L_t^2 H_x^s$.

For Schrödinger's equations, we use $h(\xi) = \pm \xi^2$. For convenience, we will shorten the notation $X^{s,b} := X_{\tau=\xi^2}^{s,b}$. Sometimes it will be necessary to use the space $X_{\tau=-\xi^2}^{s,b}$, and it will be distinguished properly.

The usefulness of $X^{s,b}$ spaces come from the fact that it measures smoothness of a function not only in the classical Sobolev sense, but also in terms of the interaction between space and time frequencies. More specifically, for $f \in L^2$, the free solution $e^{-it\partial_x^2} f \in L^2$ has no additional smoothness in the Sobolev scale than the initial data, but it is indeed very smooth in the weight $\tau - \xi^2$ (and in fact lives, on the Fourier side, exactly on the parabola $\tau = \xi^2$).

$X^{s,b}$ spaces can be used to study local-in-time solutions. This is usually done by multiplying the solution by a smooth cutoff function. There is the following classical estimate

Proposition 1. *Let Φ be a smooth cutoff adapted to $(-2, 2)$ as defined as in Section 2.1. If $u(t, x)$ solves $(\partial_t + i\partial_x^2)u = F$ and $u(0, x) = u_0(x)$, then for $T > 0$, $s \in \mathbf{R}^1$ and $\delta > 0$, there exists $C_{T,\delta}$, so that*

$$\|\Phi(t/T)u\|_{X^{s, \frac{1}{2}+\delta}} \leq C_{T,\delta}(\|u_0\|_{H^s} + \|F\|_{X^{s, -\frac{1}{2}+\delta}}).$$

We give some additional properties of $X^{s,b}$ norms.

³All of this makes perfect sense in higher spatial dimensions, but in this paper, we will confine our attention to the case $x \in \mathbf{R}^1$

Using $\widetilde{u}(\tau, \xi) = \overline{\widetilde{u}(-\tau, -\xi)}$, we obtain

$$(8) \quad \|\overline{u}(t, x)\|_{X_{\tau=\xi^2}^{s,b}} = \int_{\mathbf{R}^1 \times \mathbf{R}^1} (1 + \xi^2)^s (1 + |\tau + \xi^2|)^{2b} |\widetilde{u}(\tau, \xi)|^2 d\tau d\xi = \|u\|_{X_{\tau=-\xi^2}^{s,b}}.$$

Also we have the duality relation: $(X_{\tau=\xi^2}^{s,b})^* = X_{\tau=-\xi^2}^{-s,-b}$, which also allows us to define the equivalent norm

$$(9) \quad \|u(t, x)\|_{X_{\tau=\xi^2}^{s,b}} = \sup_{\|v\|_{X_{\tau=-\xi^2}^{-s,-b}}=1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u(t, x) v(t, x) dt dx \right|.$$

Note that using (8), we can also write

$$(10) \quad \|u(t, x)\|_{X_{\tau=\xi^2}^{s,b}} = \sup_{\|v\|_{X_{\tau=-\xi^2}^{-s,-b}}=1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u(t, x) \overline{v}(t, x) dt dx \right|.$$

Observe that for $u, v \in X_{\tau=\xi^2}^{s,b}$,

$$\begin{aligned} \|uv\|_{X_{\tau=\xi^2}^{s,b}} &= \sup_{\|w\|_{X_{\tau=-\xi^2}^{-s,-b}}=1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u(t, x) v(t, x) w(t, x) dt dx \right| \\ &= \sup_{\|v\|_{X_{\tau=-\xi^2}^{-s,-b}}=1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} \widehat{uv}(\tau, \xi) \widehat{w}(-\tau, -\xi) dt dx \right| \\ (11) \quad &= \sup_{\|w\|_{X_{\tau=-\xi^2}^{-s,-b}}=1} \left| \int_{\substack{\tau_1 + \tau_2 + \tau_3 = 0; \\ \xi_1 + \xi_2 + \xi_3 = 0}} \widehat{u}(\tau_1, \xi_1) \widehat{v}(\tau_2, \xi_2) \widehat{w}(\tau_3, \xi_3) d\sigma \right| \end{aligned}$$

where $d\sigma$ is the measure on the given hyperplane inherited from $\mathbf{R}^3 \times \mathbf{R}^3$.

Next, we point out an useful relationship between the mixed Lebesgue spaces and the $X^{s,b}$ spaces, which is based on the Strichartz estimates for the free Schrödinger equation. Namely, since we have $\|e^{-it\partial_x^2} f\|_{L_t^q L_x^r} \leq C_{Str} \|f\|_{L^2}$, for all pairs $(q, r) : 2 \leq q, r \leq \infty$, $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$, we may conclude the following estimate

$$(12) \quad \|u\|_{L_t^q L_x^r} \leq C_{Str, \delta} \|u\|_{X^{0, \frac{1}{2} + \delta}}.$$

for all Strichartz pairs $(q, r) : \frac{2}{q} + \frac{1}{r} = \frac{1}{2}$ and for all $\delta > 0$.

Motivated from the expression (11), we introduce the bilinear L^2 multiplier norm. For the rest of the section, we generally follow Tao's setup, [21].

Let $\tau = (\tau_1, \tau_2, \tau_3) \in \mathbf{R}^3$, $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$. Let Γ be a hyperplane in $\mathbf{R}^3 \times \mathbf{R}^3$ such that $\tau_1 + \tau_2 + \tau_3 = 0$ and $\xi_1 + \xi_2 + \xi_3 = 0$, and let $d\sigma$ be the measure on Γ inherited from $\mathbf{R}^3 \times \mathbf{R}^3$.

Given a function $m(\tau, \xi)$ defined on Γ , we define $\|m\|_{\mathcal{M}}$ to be the smallest constant C such that the following inequality holds:

$$(13) \quad \left| \int_{\Gamma} m(\tau, \xi) u(\tau_1, \xi_1) v(\tau_2, \xi_2) w(\tau_3, \xi_3) d\sigma \right| \leq C \|u\|_{L^2_{\tau_1, \xi_1}} \|v\|_{L^2_{\tau_2, \xi_2}} \|w\|_{L^2_{\tau_3, \xi_3}}.$$

From the definition (13), we claim the Comparison Principle. That is, if $|m(\tau, \xi)| \leq M(\tau, \xi)$ for all $(\tau, \xi) \in \Gamma$, then $\|m\|_{\mathcal{M}} \leq \|M\|_{\mathcal{M}}$.

2.3. Estimates on bilinear L^2 multiplier norms from [21]. We introduce new notations which will be useful for working with these multipliers. Let

$$\widetilde{u_{N,L}}(\tau, \xi) = \chi_{[N, 2N]}(|\xi|) \chi_{[L, 2L]}(|\tau - \xi^2|) \widetilde{u}(\tau, \xi),$$

By using capitalized letters N, L , we will always assume that N, L are dyadic numbers, i.e. $N = 2^j, L = 2^l$ for some $j, l \in \mathbf{Z}$.

For $\varepsilon_j = \pm 1$ for $j = 1, 2, 3$, following multipliers will be used often to decompose bilinear operators with respect to Fourier frequencies of each term:

$$\chi^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} := \chi_{[H, 2H]}(|\varepsilon_1 \xi_1^2 + \varepsilon_2 \xi_2^2 + \varepsilon_3 \xi_3^2|) \prod_{j=1}^3 \chi_{[N_j, 2N_j]}(|\xi_j|) \chi_{[L_j, 2L_j]}(|\tau - \varepsilon_j \xi_j^2|).$$

Note that $\chi^{(\varepsilon_1, \varepsilon_2, \varepsilon_3)}$ depends on $H, N_1, N_2, N_3, L_1, L_2, L_3$. The following statement is Proposition 11.1 in [21].

Proposition 2 ((+, +, +) case). *Let $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ satisfy $N_{\max} \sim N_{\text{med}}$, $H \sim N_{\max}^2$ and $L_{\max} \sim \max(H, L_{\text{med}})$. Then we have the following estimates.*

(1) *In the exceptional case where $N_{\max} \sim N_{\min}$ and $L_{\max} \sim H$,*

$$(14) \quad \|\chi^{(+, +, +)}\|_{\mathcal{M}} \leq C L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{4}}$$

(2) *Otherwise (i.e. $N_{\min} \ll N_{\max}$ or $L_{\max} \asymp H$), there is an absolute constant C , so that*

$$(15) \quad \|\chi^{(+, +, +)}\|_{\mathcal{M}} \leq C \frac{L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}}$$

The following Proposition covers the other important case - namely, when not all $\varepsilon_j, j = 1, 2, 3$ match. Note that while almost all the cases below already appear in the work of Tao, [21], there is the estimate (17), which is not stated explicitly⁴ (and shall be crucial for us in the sequel).

Proposition 3 ((+, +, -) case). *Let $H, N_1, N_2, N_3, L_1, L_2, L_3 > 0$ satisfy $N_{\max} \sim N_{\text{med}}$, $H \sim N_1 N_2$ and $L_{\max} \sim \max(H, L_{\text{med}})$. Then we have the following estimates.*

(1) *There is an absolute constant C so that*

$$(16) \quad \|\chi^{(+, +, -)}\|_{\mathcal{M}} \leq C L_{\min}^{\frac{1}{2}} N_{\min}^{\frac{1}{2}}$$

⁴although it is implicit in the arguments in [21]

(2) *There is an absolute constant C so that*

$$(17) \quad \|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C \min \left(\frac{L_1 L_3}{N_2}, \frac{L_2 L_3}{N_1} \right)^{\frac{1}{2}}$$

(3) *If $L_3 = L_{\max}$*

(a) *and $N_1 \sim N_2 \sim N_3$ holds, then there is an absolute constant C so that*

$$(18) \quad \|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{4}}.$$

(b) *and $N_1 \sim N_2 \sim N_3$ does not hold, then there is an absolute constant C so that*

$$(19) \quad \|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C \frac{L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}}$$

Proof. We will only prove (17). For the others, we refer to [21].

Define the following sets.

$$\begin{aligned} A_1 &= \{(\tau, \xi) \in \mathbf{R}^2 : \xi \sim N_1, \tau - \xi^2 \sim L_1\} \\ A_2 &= \{(\tau, \xi) \in \mathbf{R}^2 : \xi \sim N_2, \tau - \xi^2 \sim L_2\} \\ A_3 &= \{(\tau, \xi) \in \mathbf{R}^2 : \xi \sim N_3, \tau + \xi^2 \sim L_3\} \end{aligned}$$

To prove (17), let $N_j = N_{\min}$. Define $R = \{(\tau, \xi) \in \mathbf{R}^2 : |\xi| \leq \varepsilon N_{\min}\}$. Then we can find $m = O(1/\varepsilon)$ numbers $\xi_j^0 \sim N_j$, so that the sets of type $(0, \xi_j^0) + R$ covers the set A . Then we can apply the Box Localization (Corollary 3.13 of [21]) so that

$$\|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C \left\| \prod_{k=1}^3 \chi_{A_k \cap [(0, \xi_k^0) + R]}(\tau_k, \xi_k) \right\|_{\mathcal{M}}$$

for some $\xi_k^0 \in A_k$ so that $\xi_1^0 + \xi_2^0 + \xi_3^0 \leq \varepsilon N_{\min}$. Denote $A_k^0 = A_k \cap [(0, \xi_k^0) + R]$. Now by comparison principle and Corollary 3.10 of [21],

$$\begin{aligned} \|\chi^{(+,+, -)}\|_{\mathcal{M}} &\leq C \left\| \prod_{k=2}^3 \chi_{A_k^0}(\tau_k, \xi_k) \right\|_{\mathcal{M}} \\ &\leq C |\{(\tau_2, \xi_2) \in A_2^0 : (\tau, \xi) - (\tau_2, \xi_2) \in A_3^0\}|^{\frac{1}{2}} \end{aligned}$$

for some $(\tau, \xi) \in A_1 + 2R$. Note that $\xi \sim N_1$ for $\varepsilon > 0$ small.

We have $\tau_2 = \xi_2^2 + O(L_2)$ and $\tau - \tau_2 = -(\xi - \xi_2)^2 + O(L_3)$. First we can remove τ_2 by restricting it to an interval of length at most $O(\min(L_2, L_3))$ for a fixed ξ_2 . Furthermore, these restrictions give $\xi_2^2 - (\xi - \xi_2)^2 = \tau + O(\max(L_2, L_3))$; that is $2\xi\xi_2 = \tau + \xi^2 + O(\max(L_2, L_3))$. So

$$\|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C [\min(L_2, L_3) |\{\xi_2 \sim N_2 : \xi_2 = \frac{\tau + \xi^2}{2\xi} + O(\max(L_2, L_3)/\xi)\}|]^{\frac{1}{2}}.$$

Clearly, ξ_2 above is contained in an interval of length at most $O(\max(L_2, L_3)/N_1)$. So we get the estimate

$$\|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C \left(\frac{L_2 L_3}{N_1} \right)^{\frac{1}{2}}.$$

By reversing the role of A_1, A_2 and following the same arguments, we also obtain

$$\|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C \left(\frac{L_1 L_3}{N_2} \right)^{\frac{1}{2}}.$$

This proves (17). \square

We now apply Proposition 2 and Proposition 3 to deduce some important bilinear estimates, which will be one of the main tools in the sequel.

Lemma 1. *Let $T > 0$ and $u, v \in \mathcal{S}((-T, T) \times \mathbf{R}^1)$. Then for $\delta > 0$, $k > 0$, and some $C = C(\delta, T)$;*

$$(20) \quad \|(u_k v_{\ll k})_{\sim k}\|_{L_{t,x}^2} \leq C 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}$$

$$(21) \quad \|(u_k v_{\sim k})_{\ll k}\|_{L_{t,x}^2} \leq C 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}$$

$$(22) \quad \|u_k v\|_{L_{t,x}^2} \leq C \|u\|_{X_T^{0, \frac{1}{2}+\delta}} \|v\|_{X_T^{0, \frac{1}{2}+\delta}}$$

In addition, there are the following estimates concerning the bilinear form $(u, v) \rightarrow u\bar{v}$

$$(23) \quad \|(u_k \bar{v}_k)_k\|_{L_{tx}^2} \leq C 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}$$

$$(24) \quad \|(u_k \bar{v}_{\ll k})_k\|_{L_{tx}^2} \leq C 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}$$

$$(25) \quad \|u_k \bar{v}\|_{L_{tx}^2} \leq C \|u\|_{X_T^{0, \frac{1}{2}+\delta}} \|v\|_{X_T^{0, \frac{1}{2}+\delta}}$$

Remark: It is easy to see from the arguments below that constants on the right side of (20), (21), (23) and (24) can be replaced by $2^{(-1/2+\varepsilon)k} C_{T, \delta, \varepsilon}$ for any $\varepsilon > 0$. But we will not take advantage of this in the sequel, thus we have allowed the constants to depend on $\delta > 0$ only to keep the involved parameters to the minimum.

Proof. We first dispense with the easy estimates (22) and (25). Indeed, taking into account the boundedness of $P_{\sim k}$ and $P_{\ll k}$ on all L^p spaces, we estimate both expressions by Hölder's and (12)

$$CT^{1/4} \|u\|_{L_T^8 L_x^4} \|v\|_{L_T^8 L_x^4} \leq C_\delta T^{1/4} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}},$$

since $q = 8, r = 4$ is a Strichartz pair.

For the estimates (20) and (21), we use Proposition 3. We use the partition of unity $\chi^{(+,+, +)}$, where N_1, N_2, N_3 indicates the respective frequencies of u, v, uv . Denote by \sum , summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$. Note that $N_{\max} \sim 2^k$ and the relation $L_{\max} \sim \max(L_{\text{med}}, N_{\max}^2)$, which holds trivially by the constraints, see [21].

For (20), we apply (15) to obtain

$$\begin{aligned}
\|(u_k v_{\ll k})_{\sim k}\|_{L_{t,x}^2} &= \sup_{\|w\|_{L_{t,x}^2}=1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u_k(t, x) v_{\ll k}(t, x) w_{\sim k}(t, x) dt dx \right| \\
&= \sup_{\|w\|_{L_{t,x}^2}=1} \left| \int_{\Gamma} \sum \chi^{(+,+,+)} \tilde{u}_k(\tau_1, \xi_1) \widetilde{v_{\ll k}}(\tau_2, \xi_2) \widetilde{w_{\sim k}}(\tau_3, \xi_3) d\sigma \right| \\
&\leq C \sum \frac{1}{L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} \frac{L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}} \\
&\leq C \sum \frac{1}{L_{\text{med}}^{\delta} N_{\max}^{\frac{1}{2}}} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}} \\
&\leq C_{\delta} 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}.
\end{aligned}$$

(21) is estimated exactly the same way as (20).

To prove (23) and (24), we use Proposition 3. We use the partition of unity $\chi^{(+,+,+)}$, where N_1, N_2, N_3 indicates the respective frequencies of $u, u\bar{v}, \bar{v}$. Denote by \sum , summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$. Note $L_{\max} \sim \max(L_{\text{med}}, N_1 N_2)$ and $N_1 \sim 2^k$.

For both (23) and (24), $N_2 \sim N_{\max} \sim 2^k$. Since the calculations will be almost identical, we will only prove (23) here. We apply (17) to obtain

$$\begin{aligned}
\|(u_k \overline{v_{\sim k}})_{\sim k}\|_{L^2} &= \sup_{\|w\|_{L^2}=1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u_k(t, x) \overline{v_{\sim k}}(t, x) w_{\sim k}(t, x) dt dx \right| \\
&= \sup_{\|w\|_{L^2}=1} \left| \int_{\Gamma} \sum \chi^{(+,+,+)} \tilde{u}_k(\tau_1, \xi_1) \widetilde{\overline{v_{\sim k}}}(\tau_3, \xi_3) \widetilde{w_{\sim k}}(\tau_2, \xi_2) d\sigma \right| \\
&\leq C \sum \frac{1}{L_1^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}+\delta}} \frac{L_1^{\frac{1}{2}} L_3^{\frac{1}{2}}}{N_2^{\frac{1}{2}}} \|u_k\|_{L_{t,x}^2} \|\overline{v_{\sim k}}\|_{L_{t,x}^2} \\
&\leq C_{\delta} 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}
\end{aligned}$$

□

We now provide a technical corollary, which allows us to put $\|v\|_{X^{0, \frac{1}{2}-\delta}}$ norms on the right hand sides of (23), (24) and (25) at the expense of slightly less gain in 2^k .

Corollary 2. *With the assumptions in Lemma 1, we have*

$$(26) \quad \|(u_k \overline{v_k})_k\|_{L_{tx}^2} \leq C 2^{(-\frac{1}{2}+5\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}-\delta}}$$

$$(27) \quad \|(u_k \overline{v_{\ll k}})_k\|_{L_{tx}^2} \leq C 2^{(-\frac{1}{2}+5\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}-\delta}}$$

$$(28) \quad \|u_k \bar{v}\|_{L_{tx}^2} \leq C 2^{2\delta k} \|u\|_{X_T^{0, \frac{1}{2}+\delta}} \|v\|_{X_T^{0, \frac{1}{2}-\delta}}$$

Proof. We will show only (26), the others follow similar route. Indeed, we use a combination of Hölders with the Sobolev embedding $\|u_k\|_{L_x^\infty} \leq C 2^{k/2} \|u_k\|_{L_x^2}$ to obtain the

following estimate

$$\|(u_k \overline{v_k})_k\|_{L_{tx}^2} \leq \|u_k\|_{L_{tx}^\infty} \|v_k\|_{L_{tx}^2} \leq C 2^{k/2} \|u_k\|_{L_x^\infty L_x^2} \|v\|_{X^{0,0}} \leq C_\delta 2^{k/2} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0,0}}.$$

For a fixed function u , we are set to use complex interpolation between this and (23). Noting that $[X^{0, \frac{1}{2}+\delta}, X^{0,0}]_{4\delta} = X^{0, \frac{1}{2}-\delta-4\delta^2}$, we conclude

$$\|(u_k \overline{v_k})_k\|_{L_{tx}^2} \lesssim 2^{-(\frac{1}{2}-4\delta-\varepsilon+4\delta\varepsilon)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}-\delta-4\delta^2}} \lesssim 2^{-k(\frac{1}{2}-5\delta)} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}-\delta}}$$

where we have let $0 < \varepsilon \leq \delta$ in the last inequality.

For the proof of (28), we interpolate between (25) and the estimate

$$\|u_k \bar{v}\|_{L_{tx}^2} \leq C \|u_k\|_{L_{tx}^\infty} \|v\|_{L_{tx}^2} \leq C_\delta 2^{k/2} \|u_k\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0,0}}$$

□

The next lemma is new and addresses one situation in the (generally unfavorable) case (22), where one still can get a gain of almost half derivative. For a smooth function $u \in \mathcal{S}(\mathbb{R}^{1+1})$, we define

$$\widehat{u}^+(t, \xi) := \widehat{u}(t, \xi) \chi_{(0, \infty)}(\xi), \quad \widehat{u}^-(t, \xi) := \widehat{u}(t, \xi) \chi_{(-\infty, 0)}(\xi).$$

Lemma 2. *For all $\delta > 0$ small and $k > 0$, there exists a constant $C = C_\delta > 0$ so that*

$$(29) \quad \|(u_{\sim k}^+ v_{\sim k}^-)_{\sim k}\|_{L^2} \leq C 2^{(-\frac{1}{2}+\delta)k} \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}.$$

Proof. We present the argument for $\|(u_k^+ v_k^-)_k\|_{L^2}$. The proof for the other case $\|(u_k^+ v_k^-)_k^+\|_{L^2}$ is analogous.

First we define the following sets.

$$A := \{(\tau, \xi) | \xi > 0, \xi \sim 2^k, |\tau - \xi^2| \sim L_1\}$$

$$B := \{(\tau, \xi) | \xi < 0, \xi \sim 2^k, |\tau - \xi^2| \sim L_2\}$$

$$C := \{(\tau, \xi) | \xi < 0, \xi \sim 2^k\}$$

Then we need to show

$$(30) \quad \|\chi_A(\tau_1, \xi_1) \chi_B(\tau_2, \xi_2) \chi_C(\tau_1 + \tau_2, \xi_1 + \xi_2)\|_{[3; \mathbb{R}^{1+1}]} \lesssim \frac{L_1^{\frac{1}{2}} L_2^{\frac{1}{2}}}{2^{\frac{k}{2}}}.$$

Note that if $\max(L_1, L_2) \gtrsim 2^{2k}$, then (14) gives us the desired statement. Otherwise, we have $\max(L_1, L_2) \ll 2^{2k}$.

For some $\varepsilon > 0$ small, we partition A (similarly B) into $m = O(1/\varepsilon)$ subsets A_1, \dots, A_m so that the diameter of A_j (similarly B_j) is less than $\varepsilon 2^k$ for all $1 \leq j \leq m$. Then, removing the terms when $\chi_{A_i} \chi_{B_j} \chi_C = 0$, we can omit χ_C from the expression (30).

We have the following general estimate

$$(31) \quad \|\chi_A(\xi_1) \chi_B(\xi_2)\|_{[3; \mathbb{Z}]} \leq |\{\xi_1 \in A : \xi - \xi_1 \in B\}|^{\frac{1}{2}}.$$

for some $\xi \in \mathbb{Z}$.

So the left side of (30) is bounded by

$$(32) \quad \sum_{i,j=1}^m |\{(\xi_1, \tau_1) \in A_i : (\tau, \xi) - (\tau_1, \xi_1) \in B_j\}|^{\frac{1}{2}}$$

where τ, ξ are fixed. Note that $(\xi, \tau) \in C$, so $\xi \sim -2^k$. Writing out the condition of the set gives $\tau_1 = \xi_1^2 + O(L_1)$ and $\tau - \tau_1 = (\xi - \xi_1)^2 + O(L_2)$. So, for a fixed ξ_1 , the τ_1 must be in an interval of length $O(\min(L_1, L_2))$. Then (32) is bounded by

$$\sum_{i,j=1}^m \min(L_1, L_2)^{\frac{1}{2}} |\{\xi_1 > 0, \xi_1 \sim 2^k : \xi_1^2 + (\xi - \xi_1)^2 = \tau + O(\max(L_1, L_2))\}|^{\frac{1}{2}}$$

We can write $\xi_1^2 + (\xi - \xi_1)^2 = \frac{\xi^2 + (2\xi_1 - \xi)^2}{2}$. So the condition given above can be written as

$$(\xi_1 - \frac{\xi}{2})^2 = C_{\tau, \xi} + O(\max(L_1, L_2))$$

where $C_{\tau, \xi} := \frac{2\tau - \xi^2}{4}$. Since ξ_1 and ξ have the opposite sign, the left hand side of the above is $\sim 2^{2k}$. On the other hand, $\max(L_1, L_2) \ll 2^{2k}$, so $C_{\tau, \xi} \sim 2^{2k}$. Then we have

$$\begin{aligned} |\xi_1 - (\frac{\xi}{2} + \sqrt{C_{\tau, \xi}})| &= \sqrt{C_{\tau, \xi} + O(\max(L_1, L_2))} - \sqrt{C_{\tau, \xi}} \\ &= \frac{O(\max(L_1, L_2))}{\sqrt{C_{\tau, \xi} + O(\max(L_1, L_2))} + \sqrt{C_{\tau, \xi}}} \\ &\lesssim \frac{O(\max(L_1, L_2))}{2^k}. \end{aligned}$$

So ξ_1 must be contained in an interval of length $\frac{O(\max(L_1, L_2))}{2^k}$. Using this interval in (32) gives the desired estimate (30). \square

We remark that interpolation can be applied to the above Lemma as in the proof of Corollary 2 to replace $X^{0, \frac{1}{2} + \delta}$ norm on the right side of (29) with $X^{0, \frac{1}{2} - \delta}$.

3. PROOF OF THEOREM 1

We first perform a change of variables, which transforms the problem (2) into a problem with data in L^2 . Namely, let $v : u = \langle \nabla \rangle^\alpha v$. A quick calculation then shows that (2) becomes

$$(33) \quad \begin{cases} v_t + i\partial_x^2 v = \langle \nabla \rangle^{\beta - \alpha} [\langle \nabla \rangle^\alpha v \langle \nabla \rangle^\alpha v] : & (t, x) \in \mathbf{R}^1 \times \mathbf{R}^1 \\ v(0, x) = \langle \nabla \rangle^{-\alpha} g =: f \in L^2(\mathbf{R}^1). \end{cases}$$

Thus, we need to study the well-posedness of (33) in the L^2 setting.

3.1. Setting up the normal forms. Introduce $G(u, v) := \langle \nabla \rangle^{\beta-\alpha} [\langle \nabla \rangle^\alpha u \langle \nabla \rangle^\alpha v]$, so that the nonlinearity in (33) is of the form $G(v, v)$, note $G(u, v) = G(v, u)$. Observe that the bilinear form G may be written as follows

$$G(u, v)(x) = \frac{1}{4\pi^2} \int \frac{\langle \xi \rangle^\alpha \langle \eta \rangle^\alpha}{\langle \xi + \eta \rangle^{\alpha-\beta}} \widehat{u}(\xi) \widehat{v}(\eta) e^{i(\xi+\eta)x} d\xi d\eta.$$

We now decompose the form $G(v, v)$ as follows

$$\begin{aligned} G(v, v) &= G(v_{\leq 0}, v) + G(v_{> 0}, v) = G(v_{\leq 0}, v) + G(v_{> 0}, v_{\leq 0}) + G(v_{> 0}, v_{> 0}) = \\ &= G(v_{\leq 0}, (Id + P_{> 0})v) + G(v_{> 0}, v_{> 0}). \end{aligned}$$

Next, we perform a change of variables $v \rightarrow z$, $v = e^{-it\partial_x^2} f + z$. Clearly, $z(0) = 0$ and

$$\begin{aligned} z_t + i\partial_x^2 z &= G([e^{-it\partial_x^2} f + z]_{\leq 0}, (Id + P_{> 0})[e^{-it\partial_x^2} f + z]) + \\ &\quad + G([e^{-it\partial_x^2} f + z]_{> 0}, [e^{-it\partial_x^2} f + z]_{> 0}) \end{aligned}$$

Clearly, a lot of terms are generated by this transformation. We comment now on the form of various terms (especially the least favorable ones!), since this will influence our normal form analysis.

Heuristically, if we expect the z term to be smoother, then the least smooth term is expected to be $G(e^{-it\partial_x^2} f_{> 0}, e^{-it\partial_x^2} f_{> 0})$. Indeed, there are α derivatives acting on each of the two entries (which free solutions and hence, in general, no better than L_x^2 smooth) and $\beta - \alpha$ derivatives acting on the product itself⁵. Thus, if we manage to build a smoother function h , which solves

$$(34) \quad (\partial_t + i\partial_x^2)h = G(e^{-it\partial_x^2} f_{> 0}, e^{-it\partial_x^2} f_{> 0}),$$

one would be compelled to change variables again, $z \rightarrow w$, where $z = h + w$. Define a bilinear operator T

$$T(u, v)(x) = \frac{1}{8\pi^2 i} \int \frac{\langle \xi \rangle^\alpha \langle \eta \rangle^\alpha}{\langle \xi + \eta \rangle^{\alpha-\beta}} \frac{1}{\xi \eta} \widehat{u}_{> 0}(\xi) \widehat{v}_{> 0}(\eta) e^{i(\xi+\eta)x} d\xi d\eta.$$

It is easy to check that for a pair of functions $u(t, x), v(t, x) \in \mathcal{S}(\mathbf{R}^1 \times \mathbf{R}^1)$,

$$(35) \quad (\partial_t + i\partial_x^2)T(u, v) = T((\partial_t + i\partial_x^2)u, v) + T(u, (\partial_t + i\partial_x^2)v) + G(u_{> 0}, v_{> 0}).$$

This last identity tells us that

$$(\partial_t + i\partial_x^2)T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f) = G(e^{-it\partial_x^2} f_{> 0}, e^{-it\partial_x^2} f_{> 0})$$

which provides an explicit solution⁶ of (34). Hence, set

$$h := T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f).$$

⁵In fact this $\beta - \alpha$ derivatives on the product may not be of much help in “high-high to low interaction scenario”

⁶Note that while the solution $h(t)$ of the Schrödinger equation (34) is not unique, it is completely determined by its value $h(0)$

We now change variables $z = h + w = T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f) + w$, whence we get the following equation for w

$$(36) \quad w_t + i\partial_x^2 w = G([e^{-it\partial_x^2}f + h + w]_{\leq 0}, (Id + P_{>0})[e^{-it\partial_x^2}f + h + w]) + 2G((h + w)_{>0}, e^{-it\partial_x^2}f_{>0}) + G((h + w)_{>0}, (h + w)_{>0})$$

Note also that since $z(0) = 0$, it follows that the Schrödinger equation for w is supplemented by the following initial condition: $w(0) = -h(0) = -T(f, f)$. We have now prepared ourselves to close the argument in the w variable. More precisely, the proof of Theorem 1 reduces to establishing the local well-posedness of the Schrödinger equation (36) in an appropriate function space.

Fix $1 > \delta > 0$. Consider the spaces

$$\begin{aligned} \mathcal{X} &= X^{\alpha - \frac{1}{2}, \frac{1}{2} + \delta}, \\ \mathcal{H} &= L_t^\infty H_x^{\frac{1}{2}} \cap X^{1 - \delta, \delta}, \end{aligned}$$

Our strategy will be to show that the fixed point argument for w closes in the space \mathcal{X} , given that $f \in L^2$, $h \in \mathcal{H}$ and where we will occasionally need to use the particular form $h = T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)$.

3.2. Estimates on the normal form. In this section, we show the required smoothness of the normal form h , namely $h \in \mathcal{H}$. This will be done in two steps - in Lemma 3 and Lemma 4.

Lemma 3. $T : L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow H^{\frac{1}{2}}(\mathbb{R})$ continuously.

Proof. Let $u, v \in \mathcal{S}(\mathbb{R})$. Then

$$(37) \quad \|T(u, v)\|_{H^{\frac{1}{2}}} \leq \sum_{k \geq l+3} \|T(u_k, v_l)\|_{H^{\frac{1}{2}}} + \sum_{|k-l| < 3} \|T(u_k, v_l)\|_{H^{\frac{1}{2}}} = I_1 + I_2$$

where $k, l > 0$. Regarding the first sum in (37), we apply Hölder's and then the Sobolev embedding $\|u_l\|_{L^\infty} \leq C2^{l/2}\|u_l\|_{L^2}$. We get

$$\begin{aligned} I_1 &\leq C \sum_{l>0} \sum_{k \geq l} 2^{\frac{k}{2}} \frac{2^{\alpha k + \alpha l}}{2^{(\alpha-\beta)k + k + l}} \|u_k v_l\|_{L^2} \leq C \sum_{l>0} \sum_{k \geq l} 2^{(\beta-\frac{1}{2})k + (\alpha-1/2)l} \|u_k\|_{L^2} \|v_l\|_{L^2} \\ &\leq C \|u\|_{L^2} \|v\|_{L^2} \sum_{l>0} 2^{(\alpha+\beta-1)l} \leq C \|u\|_{L^2} \|v\|_{L^2} \end{aligned}$$

Regarding the second sum in (37),

$$\begin{aligned} I_2 &\leq C \sum_{k>0} \sum_{m \leq k+2} 2^{\frac{1}{2}m} \frac{2^{2\alpha k}}{2^{(\alpha-\beta)m + 2k}} \|P_m(u_k v_k)\|_{L^2} \leq C \sum_{k>0} \sum_{m \leq k+2} 2^{(\frac{1}{2}-\alpha+\beta)m + (2\alpha-2)k} 2^{\frac{m}{2}} \|u_k v_k\|_{L^1} \\ &\leq C \|u\|_{L^2} \|v\|_{L^2} \sum_{k>0} 2^{(\alpha+\beta-1)k} \leq C \|u\|_{L^2} \|v\|_{L^2}. \end{aligned}$$

□

The next lemma provides a different type of estimate, namely that if we measure $T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)$ in averages L_{tx}^2 sense, we actually get a full spatial derivative gain. More precisely,

Lemma 4. *Let $u, v \in X_{\tau=\xi^2}^{0, \frac{1}{2}+\delta}$, then for $0 \leq \delta < 1 - \alpha - \beta$,*

$$\|T(u, v)\|_{X^{1-\delta, \delta}} \leq C_\delta \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}.$$

Proof. By using the partition of unity $\chi^{(+, +, -)}$, we can localize spacial and time frequencies to their respective indices. (Here we localize u, v, uv respectively to N_1, N_2, N_3 .) Also denote by the symbol \sum to be the summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$. We have

$$\begin{aligned} \|T(u, v)\|_{X^{1-\delta, \delta}} &\leq C \sum \frac{N_3^{1-\alpha+\beta-\delta}}{N_1^{1-\alpha} N_2^{1-\alpha}} \|uv\|_{X^{0, \delta}} \\ &\leq C \sum \frac{N_3^{1-\alpha+\beta-\delta}}{N_1^{1-\alpha} N_2^{1-\alpha}} \sup_{\|w\|_{X^{0, -\delta}_{\tau=-\xi^2}}=1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u(t, x) v(t, x) w(t, x) dt dx \right| \\ (38) \quad &\leq C \sup_{\|w\|_{X^{0, -\delta}_{\tau=-\xi^2}}=1} \sum \frac{N_3^{1-\alpha+\beta-\delta}}{N_1^{1-\alpha} N_2^{1-\alpha}} \|\chi^{(+, +, -)}\|_{\mathcal{M}} \|u\|_{L^2_{t,x}} \|v\|_{L^2_{t,x}} \|w\|_{L^2_{t,x}} \end{aligned}$$

$$(39) \quad \leq C \|u\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}} \sum \frac{N_3^{1-\alpha+\beta-\delta} L_3^\delta}{N_1^{1-\alpha} N_2^{1-\alpha} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} \|\chi^{(+, +, -)}\|_{\mathcal{M}}$$

We refer to Proposition 3. If $\max(L_1, L_2) = L_{\max}$, we use that $\max(N_1, N_2) \gtrsim N_3$ to simplify (39) and then apply the multiplier bound (16). We estimate the sum in (39) by

$$\sum \frac{L_3^\delta N_3^{\beta-\delta}}{L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} L_{\min}^{\frac{1}{2}} N_{\min}^{\frac{1}{2}} \leq C \sum \frac{N_{\min}^{\frac{1}{2}-\delta} N_{\max}^\beta}{L_{\max}^{\frac{1}{2}}} \leq C_\delta$$

If $L_3 = L_{\max}$ and $N_1 \sim N_2 \sim N_3 \sim N$, then we can assume $L_3 \sim N^2$, so applying (18) yields the estimate

$$\sum \frac{N^{1-\alpha+\beta-\delta} L_3^\delta}{N^{1-\alpha} N^{1-\alpha} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{4}} \leq C \sum \frac{1}{N^{1-\alpha-\beta-\delta}} \leq C_\delta,$$

provided $0 < \delta < 1 - \alpha - \beta$.

The remaining case is when $L_3 = L_{\max} \sim N_1 N_2$ with the bound (19). We have

$$\sum \frac{N_3^{1-\alpha+\beta-\delta} L_3^\delta}{N_1^{1-\alpha} N_2^{1-\alpha} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta}} \frac{L_{\min}^{\frac{1}{2}} L_{\text{med}}^{\frac{1}{2}}}{N_{\max}^{\frac{1}{2}}} \leq C \sum N_{\max}^{\beta-\frac{1}{2}+\delta} \leq C.$$

□

3.3. Some bilinear and trilinear estimates involving typical right-hand sides. Let us start with few words regarding strategy. All the terms (with an exception of one single term) in the right hand-side of (36), which contain at least one smooth term (i.e. in the form $u_{\leq 0}$) will be dealt with by relatively simple arguments, mainly based on Lemma 1. For all other terms, we shall need specific (bilinear and trilinear) estimates, which handle different type of configurations (i.e. h and w , h and $e^{-i\partial_x^2} f$) on the right-hand side of (36). These multilinear estimates are presented below. We also attempt to indicate the relevancy of each such estimate, before the statement of each Lemma.

The next lemma is useful, when one deals with terms in the form $G(w, e^{-i\partial_x^2} f)$ and $G(w, w)$ on the right hand side of (36).

Lemma 5. *Let z satisfy $(\partial_t + i\partial_x^2)z = G(u_{>0}, v_{>0})$ with $z(0, x) \equiv 0$, and let $0 < 10\delta < 1 - \alpha - \beta$. Then*

$$\|z\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \leq C_{T,\delta} \|u\|_{X^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}.$$

Proof. By using the partition of unity $\chi^{(+,+, -)}$, we can localize spacial and time frequencies to their respective indices. (Here we localize u, v, uv respectively to N_1, N_2, N_3 .) Also denote by the symbol \sum to be the summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$.

Applying Proposition 1, we obtain

$$\begin{aligned} \|z\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} &\leq C_{T,\delta} \|G(u_{>0}, v_{>0})\|_{X^{\alpha-\frac{1}{2}, -\frac{1}{2}+\delta}} \\ &\leq C_{T,\delta} \sum \frac{N_1^\alpha N_2^\alpha}{N_3^{\frac{1}{2}-\beta}} \|uv\|_{X^{0, -\frac{1}{2}+\delta}} \\ &\leq C_{T,\delta} \sum \frac{N_1^\alpha N_2^\alpha}{N_3^{\frac{1}{2}-\beta}} \sup_{\substack{\|w\|_{X^{0, \frac{1}{2}-\delta}}=1 \\ \tau=-\xi^2}} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u(t, x) v(t, x) w(t, x) dt dx \right| \\ &\leq C_{T,\delta} \sum \frac{N_1^\alpha N_2^\alpha}{N_3^{\frac{1}{2}-\beta}} \sup_{\substack{\|w\|_{X^{0, \frac{1}{2}-\delta}}=1 \\ \tau=-\xi^2}} \|\chi^{(+,+, -)}\|_{\mathcal{M}} \|u\|_{L_{t,x}^2} \|v\|_{L_{t,x}^2} \|w\|_{L_{t,x}^2} \\ (40) \quad &\leq C_{T,\delta} \|u\|_{X^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}} \sum \frac{N_1^{\frac{1}{2}} N_2^\alpha}{N_3^{\frac{1}{2}-\beta} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}-\delta}} \|\chi^{(+,+, -)}\|_{\mathcal{M}} \end{aligned}$$

Now we refer to Proposition 3 for appropriate bounds. If we have $L_{\max} \sim L_{\text{med}} \gg N_1 N_2$, we apply (16). Then the sum in (40) is estimated by

$$\sum \frac{N_1^{\frac{1}{2}} N_2^\alpha}{N_3^{\frac{1}{2}-\beta} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}-\delta}} N_{\min}^{\frac{1}{2}} L_{\min}^{\frac{1}{2}} \leq \sum \frac{N_1^{\frac{1}{2}} N_2^\alpha N_{\min}^\beta}{L_{\max}} \leq \sum \frac{1}{L_{\max}^{1-\alpha-\beta}} \leq C_{\alpha,\beta}$$

Otherwise, we may assume that $L_{\max} \sim N_1 N_2$.

If $L_1 = L_{\max}$, we apply $\|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C \frac{L_2^{\frac{1}{2}} L_3^{\frac{1}{2}}}{N_1^{\frac{1}{2}}}$. If $L_2 = L_{\max}$, then we can use

$$\|\chi^{(+,+, -)}\|_{\mathcal{M}} \leq C \frac{L_1^{\frac{1}{2}} L_3^{\frac{1}{2}}}{N_2^{\frac{1}{2}}}.$$

Both cases will work out similarly. For instance, if $L_1 = L_{\max}$, then the sum in (40) is estimated

$$\sum \frac{N_1^{\frac{1}{2}} N_2^\alpha}{N_3^{\frac{1}{2}-\beta} L_1^{\frac{1}{2}+\delta} L_2^{\frac{1}{2}+\delta} L_3^{\frac{1}{2}-\delta}} \frac{L_2^{\frac{1}{2}} L_3^{\frac{1}{2}}}{N_1^{\frac{1}{2}}} \leq \sum \frac{N_2^\alpha}{N_3^{\frac{1}{2}-\beta} L_{\max}^{\frac{1}{2}+\delta}} \leq \sum \frac{1}{L_{\max}^\delta} = C_\delta.$$

If $L_3 = L_{\max}$ and $N_1 \sim N_2 \sim N_3 \sim N$, then we can estimate the sum in (40) via (18).

$$\sum \frac{N^{\alpha+\beta}}{L_{\max}^{\frac{1}{2}-\delta}} \leq \sum N^{\alpha+\beta-1+2\delta} \leq C_{\alpha,\beta,\delta}.$$

Otherwise, (19) can be used to estimate the sum in (40)

$$\sum \frac{N_1^{\frac{1}{2}} N_2^{\alpha-\frac{1}{2}}}{N_3^{\frac{1}{2}-\beta} L_{\max}^{\frac{1}{2}-\delta}} \leq \sum \frac{1}{L_{\max}^{\delta}} \leq C_{\delta}.$$

□

The next lemma deals with right-hand sides of the form $G(h, h)$.

Lemma 6. *Let z satisfy $(\partial_t + i\partial_x^2)z = G(u_{>0}, v_{>0})$ with $z(0, x) \equiv 0$. Then for $0 < \delta < 1 - \alpha - \beta$,*

$$\|z\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \leq C_{T,\delta} \|u\|_{X^{1-\delta,\delta}} \|v\|_{X^{1-\delta,\delta}}.$$

Proof. By using the partition of unity $\chi_{(+,+, -)}$, we can localize spacial and time frequencies to their respective indices. (Here we localize u, v, uv respectively to N_1, N_2, N_3 .) Also denote by the symbol \sum to be the summation over $N_1, N_2, N_3, L_1, L_2, L_3 \geq 1$.

Applying Proposition 1, we obtain

$$\begin{aligned} \|z\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} &\leq C_{T,\delta} \|G(u_{>0}, v_{>0})\|_{X^{\alpha-\frac{1}{2}, -\frac{1}{2}+\delta}} \\ &\leq C_{T,\delta} \sum \frac{N_1^{\alpha} N_2^{\alpha}}{N_3^{\frac{1}{2}-\beta}} \|uv\|_{X^{0, -\frac{1}{2}+\delta}} \\ &\leq C_{T,\delta} \sum \frac{N_1^{\alpha} N_2^{\alpha}}{N_3^{\frac{1}{2}-\beta}} \sup_{\substack{X^{0, \frac{1}{2}-\delta} \\ \tau=-\xi^2}} \|w\| \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} u(t, x) v(t, x) w(t, x) dt dx \right| \\ &\leq C_{T,\delta} \sum \frac{N_1^{\alpha} N_2^{\alpha}}{N_3^{\frac{1}{2}-\beta}} \sup_{\substack{X^{0, \frac{1}{2}-\delta} \\ \tau=-\xi^2}} \|\chi^{(+,+, -)}\|_{\mathcal{M}} \|u\|_{L_{t,x}^2} \|v\|_{L_{t,x}^2} \|w\|_{L_{t,x}^2} \\ (41) \quad &\leq C_{T,\delta} \|u\|_{X^{1-\delta,\delta}} \|v\|_{X^{1-\delta,\delta}} \sum \frac{1}{N_1^{1-\alpha} N_2^{1-\alpha} N_3^{\frac{1}{2}-\beta} L_1^{\delta} L_2^{\delta} L_3^{\frac{1}{2}-\delta}} \|\chi^{(+,+, -)}\|_{\mathcal{M}} \end{aligned}$$

We refer to Proposition 3 for appropriate bounds. We apply the bound (16) to the multiplier to estimate the sum in (41) by

$$\sum \frac{N_{\min}^{\frac{1}{2}}}{N_1^{1-\alpha} N_2^{1-\alpha} N_3^{\frac{1}{2}-\beta} L_{\max}^{\delta}} \leq \sum \frac{1}{L_{\max}^{\delta}} \leq C_{\delta}.$$

□

Our next lemma treats all the terms on the right-hand side of (36) in the form $G(h, e^{-it\partial_x^2} f) = G(T(e^{-it\partial_x^2} f, e^{-it\partial_x^2} f), e^{-it\partial_x^2} f)$. Unfortunately, we cannot control such terms only with the *a posteriori* information $h \in \mathcal{H}, e^{-it\partial_x^2} f \in X^{0, \frac{1}{2}+\delta}$. Instead, we must treat the whole expression as a trilinear one, which then yields the desired control.

Lemma 7. *Let z satisfy $(\partial_t + i\partial_x^2)z = G(T(f, g), v_{>0})$ with $z(0, x) \equiv 0$, where $f, g \in X^{0, \frac{1}{2}+\delta}$. Then for $0 < 10\delta < 1 - \alpha - \beta$,*

$$\|z\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \leq C_{T,\delta} \|f\|_{X^{0, \frac{1}{2}+\delta}} \|g\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}.$$

Remark: In addition, we have similar control for z_1, z_2 , so that $(\partial_t + i\partial_x^2)z_1 = G(T(f, g)_{>0}, v_{>0})$, $(\partial_t + i\partial_x^2)z_2 = G(T(f, g)_{\leq 0}, v_{>0})$. Namely

$$(42) \quad \|z_1\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} + \|z_2\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \leq C_{T,\delta} \|f\|_{X^{0, \frac{1}{2}+\delta}} \|g\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}}.$$

Proof. We handle the term z_1 first, after which, we quickly indicate how to treat the general case for z (and $z_2 = z - z_1$ is subsequently controlled).

Noting $T(f, g) = C\langle \nabla \rangle^{\beta-\alpha} \left[\frac{\langle \nabla \rangle^\alpha}{\nabla} f_{>0} \cdot \frac{\langle \nabla \rangle^\alpha}{\nabla} g_{>0} \right]$, we can write

$$G(T(f, g), v) = C\langle \nabla \rangle^{\beta-\alpha} \left[\langle \nabla \rangle^\beta \left[\frac{\langle \nabla \rangle^\alpha}{\nabla} f_{>0} \frac{\langle \nabla \rangle^\alpha}{\nabla} g_{>0} \right] \cdot \langle \nabla \rangle^\alpha v \right]$$

Applying Proposition 1 and the duality relation (10),

$$\begin{aligned} \|z_1\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} &\leq C_{T,\delta} \left\| \langle \nabla \rangle^{\beta-\alpha} \left[\langle \nabla \rangle^\beta \left[\frac{\langle \nabla \rangle^\alpha}{\nabla} f_{>0} \frac{\langle \nabla \rangle^\alpha}{\nabla} g_{>0} \right] \cdot \langle \nabla \rangle^\alpha v \right] \right\|_{X^{\alpha-\frac{1}{2}, -\frac{1}{2}+\delta}} \\ &\leq C_{T,\delta} \sum_{k,l>0} 2^{\beta k + \alpha l} \sum_{k_1, k_2 > 0} 2^{(\alpha-1)(k_1+k_2)} \|(f_{k_1} g_{k_2})_k v_l\|_{X^{\alpha-\frac{1}{2}, -\frac{1}{2}+\delta}} \\ &\leq C_{T,\delta} \sum_{\substack{k,l>0 \\ k_1, k_2 > 0}} 2^{(\alpha-1)(k_1+k_2) + \beta k + \alpha l} \sup_{\substack{\|w\|_{X^{\frac{1}{2}-\beta, \frac{1}{2}-\delta}=1} \\ \tau=\xi^2}} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (f_{k_1} g_{k_2})_k v_l \bar{w} \, dt \, dx \right|. \end{aligned}$$

Note that unless $k-3 \leq \max(k_1, k_2)$, otherwise the integral above vanishes. We split the last sum $\sum_{k,l>0}$ into three parts

$$\sum_{l \leq k-3} \cdot + \sum_{k \leq l-3} \cdot + \sum_{k \sim l} \cdot$$

where $k, l > 0$. We denote the corresponding terms by $I_1 + I_2 + I_3$. On each summand, we will apply Lemma 1 to obtain the desired estimate. We need to estimate the integral

$$(43) \quad 2^{(\alpha-1)(k_1+k_2) + \beta k + \alpha l} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (f_{k_1} g_{k_1})_k v_l \bar{w} \, dt \, dx \right|.$$

For I_1 , we have high-low interaction between $(f_{k_1} g_{k_2})_k$ and v_l , so $(f_{k_1} g_{k_2})_k v_l = P_{\sim k}[(f_{k_1} g_{k_2})_k v_l]$. Hence it suffices to control

$$2^{(\alpha+\beta)k + (\alpha-1)(k_1+k_2)} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (f_{k_1} g_{k_2})_k v_l \bar{w}_{\sim k} \, dt \, dx \right|.$$

Thus by the Cauchy-Swartz inequality,

$$(43) \leq C 2^{(\alpha+\beta)k + (\alpha-1)(k_1+k_2)} \|(f_{k_1} g_{k_2})_k\|_{L_{t,x}^2} \|v_l \bar{w}_{\sim k}\|_{L_{t,x}^2}.$$

Here we need to consider two cases: first when $2^{k_1} \sim 2^{k_2} \sim 2^k$, and second when this does not take place. In the first case, we apply (22) and (27)

$$\begin{aligned} (43) &\leq C 2^{(3\alpha+\beta-2)k} \|(f_{\sim k} g_{\sim k})_k\|_{L_{t,x}^2} \|v_l \bar{w}_{\sim k}\|_{L_{t,x}^2} \\ &\leq C_{\alpha,\delta,T} 2^{(3\alpha+2\beta-3+5\delta)k} \|f\|_{X^{0,\frac{1}{2}+\delta}} \|g\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{\frac{1}{2}-\beta,\frac{1}{2}-\delta}}. \end{aligned}$$

On the other hand, if $2^{k_1} \sim 2^{k_2} \sim 2^k$ does not take place, then we can gain $\frac{1}{2}$ derivative from $\|(f_{k_1} g_{k_2})_k\|_{L^2}$ via (20) or (21). Thus the estimate follows

$$\begin{aligned} (43) &\leq C 2^{(\alpha+\beta)k+(\alpha-1)(k_1+k_2)} \|(f_{k_1} g_{k_2})_k\|_{L_{t,x}^2} \|v_l \bar{w}_{\sim k}\|_{L_{t,x}^2} \\ &\leq C_{\alpha,\delta,T} 2^{(\alpha+\beta-1+6\delta)k+(\alpha-1)(k_1+k_2)} \|f\|_{X^{0,\frac{1}{2}+\delta}} \|g\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{0,\frac{1}{2}-\delta}}. \end{aligned}$$

Clearly in both cases, the sums in $k > 0$ goes through since $0 < 10\delta < 1 - \alpha - \beta$ and hence the bound (42) for this portion of the sum.

For I_2 , note that now $(f_{k_1} g_{k_2})_k v_l = P_{\sim l}[(f_{k_1} g_{k_2})_k v_l]$ and hence

$$(43) \leq C 2^{(\alpha-1)(k_1+k_2)+(\alpha+\beta)l} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (f_{k_1} g_{k_1})_k v_l \bar{w}_{\sim l} dt dx \right|$$

If $2^{k_1} \sim 2^{k_2} \sim 2^l$, then applying Cauchy-Schwartz inequality, (21) and (25)

$$\begin{aligned} (43) &\leq C 2^{(3\alpha+\beta-2)l} \|(f_{\sim l} g_{\sim l})_k\|_{L_{t,x}^2} \|v_l \bar{w}_{\sim l}\|_{L_{t,x}^2} \\ &\leq C_{T,\delta} 2^{(3\alpha+2\beta-3+3\delta)l} \|f\|_{X^{0,\frac{1}{2}+\delta}} \|g\|_{X^{0,\frac{1}{2}+\delta}} \|v\|_{X^{0,\frac{1}{2}+\delta}} \|w\|_{X^{\frac{1}{2}-\beta,\frac{1}{2}-\delta}}. \end{aligned}$$

If $2^{k_1} \sim 2^{k_2} \sim 2^l$ does not hold, say $|k_1 - l| > 3$. Then we would like to estimate the last integral by something close to $\|f_{k_1} v_l\|_{L_{t,x}^2} \|g_{k_2} \bar{w}_{\sim l}\|_{L_{t,x}^2}$ which would give us at least a $1/2$ derivative gain in l . However, we cannot quite do that, since the integral in (43) is not a pointwise product, but rather the operator P_k acting on $f_{k_1} g_{k_2}$, which then is multiplied by $v_l \bar{w}_l$.

The following calculation however provides a substitute for this, namely by Plancherel's and triangle inequality

$$\begin{aligned} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (f_{k_1} g_{k_2})_k v_l \bar{w}_{\sim l} dt dx \right| &\leq \int_{\mathbf{R}^1 \times \mathbf{R}^1} |\widehat{f_{k_1}} * \widehat{g_{k_2}}(\xi)| |\varphi(2^{-k}\xi)| |\widehat{v_l} * \widehat{\bar{w}_{\sim l}}(-\xi)| d\xi dt \\ &\leq \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} Q[f_{k_1}] Q[g_{k_2}] Q[v_l] Q[\bar{w}_{\sim l}] dx dt \right|, \end{aligned}$$

where $Q[h] := \mathcal{F}^{-1}[\widehat{h}]$. Note that $Q[h]_k = Q[h_k]$ and $\|Q[h]\|_{X^{s,b}} = \|h\|_{X^{s,b}}$, by the definition of $\|\cdot\|_{X^{s,b}}$. In other words, we have managed to remove the Littlewood-Paley operator P_k (and to reduce to an expression as an integral of pointwise product of four functions), at the expense of introducing the operators Q , which do not really affect the $X^{s,b}$ norms of the entries. With that last reduction in mind, we continue our estimation of (43).

By (20) and (28) respectively

$$\begin{aligned}
 (43) &\leq C 2^{\alpha l + (\alpha + \beta - 1 + \delta)k + (\alpha - 1)(k_1 + k_2)} \|Q[f_{k_1}]Q[v_l]\|_{L_{t,x}^2} \|Q[g_{k_2}]Q[\bar{w}_l]\|_{L_{t,x}^2} \\
 &\leq C_{T,\delta} 2^{(\alpha + \beta - 1 + 3\delta)l + (\alpha - 1)(k_1 + k_2)} \|f\|_{X^{0, \frac{1}{2} + \delta}} \|v\|_{X^{0, \frac{1}{2} + \delta}} \|g\|_{X^{0, \frac{1}{2} + \delta}} \|w\|_{X^{\frac{1}{2} - \beta, \frac{1}{2} - \delta}}.
 \end{aligned}$$

This clearly sums in l , provided $0 < 10\delta < 1 - \alpha - \beta$.

For I_3 , we have

$$(43) \leq C 2^{(\alpha + \beta)k + (\alpha - 1)(k_1 + k_2)} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (f_{k_1} g_{k_2})_k (v_{\sim k} \bar{w})_{\sim k} dt dx \right|.$$

At this point, let us discuss the frequency localization for \bar{w} . Clearly,

$$(44) \quad (v_{\sim k} \bar{w})_{\sim k} = (v_{\sim k} \bar{w}_{< k+3})_{\sim k} = (v_{\sim k} \bar{w}_{\sim k})_{\sim k} + (v_{\sim k} \bar{w}_{\ll k})_{\sim k},$$

In particular, \bar{w} may not be high frequency.

If $|k_1 - k| \geq 3$ or $|k_2 - k| \geq 3$, then by (20) or (21) (applied to $\|(f_{k_1} g_{k_2})_k\|_{L_{t,x}^2}$) and either (26) or (27) (applied to $\|(v_{\sim k} \bar{w})_{\sim k}\|_{L^2}$)

$$\begin{aligned}
 (43) &\leq C 2^{(\alpha + \beta)k + (\alpha - 1)(k_1 + k_2)} \|(f_{k_1} g_{k_2})_k\|_{L_{t,x}^2} (\|(v_{\sim k} \bar{w}_{\sim k})_{\sim k}\|_{L^2} + \|(v_{\sim k} \bar{w}_{\ll k})_{\sim k}\|_{L^2}) \\
 &\leq C_\delta 2^{(\alpha + \beta - 1 + 6\delta)k + (\alpha - 1)(k_1 + k_2)} \|f\|_{X^{0, \frac{1}{2} + \delta}} \|g\|_{X^{0, \frac{1}{2} + \delta}} \|v\|_{X^{0, \frac{1}{2} + \delta}} \|w\|_{X^{0, \frac{1}{2} - \delta}}.
 \end{aligned}$$

Summing in k yields a bound, provided $0 < 10\delta < 1 - \alpha - \beta$.

Otherwise, $2^{k_1} \sim 2^{k_2} \sim 2^k$. In this case, we first handle the $w_{\sim k}$ term, which is easier due to the gain of $\frac{1}{2} - \beta$ derivatives in k . Applying Cauchy-Swartz inequality, (22) and (26),

$$\begin{aligned}
 (43) &\leq C 2^{(3\alpha + \beta - 2)k} \|(f_{\sim k} g_{\sim k})_k\|_{L_{t,x}^2} \|(v_{\sim k} \bar{w}_{\sim k})_{\sim k}\|_{L_{t,x}^2} \\
 &\leq C 2^{(3\alpha + 2\beta - 3 + 5\delta)k} \|f\|_{X^{0, \frac{1}{2} + \delta}} \|g\|_{X^{0, \frac{1}{2} + \delta}} \|v\|_{X^{0, \frac{1}{2} + \delta}} \|w\|_{X^{\frac{1}{2} - \beta, \frac{1}{2} - \delta}}.
 \end{aligned}$$

For the term with $w_{\ll k}$, we need a more refined analysis, which is possible thanks to the estimate (29). We can write

$$(45) \quad (f_{\sim k} g_{\sim k})_k = (f_{\sim k}^+ g_{\sim k}^-)_k + (f_{\sim k}^- g_{\sim k}^+)_k + (f_{\sim k}^+ g_{\sim k}^+)_k^+ + (f_{\sim k}^- g_{\sim k}^-)_k^-.$$

For the first two terms, due to (29) and (27), we obtain

$$\begin{aligned}
 (43) &\leq C 2^{(3\alpha + \beta - 2)k} \|(f_{\sim k}^+ g_{\sim k}^-)_k\|_{L_{t,x}^2} \|(v_{\sim k} \bar{w}_{\ll k})_{\sim k}\|_{L_{t,x}^2} \\
 &\leq C_\delta 2^{(3\alpha + \beta - 3 + 6\delta)k} \|f\|_{X^{0, \frac{1}{2} + \delta}} \|g\|_{X^{0, \frac{1}{2} + \delta}} \|v\|_{X^{0, \frac{1}{2} + \delta}} \|w\|_{X^{0, \frac{1}{2} - \delta}}.
 \end{aligned}$$

To deal with the next two terms in (45), note that the integral in (43) is in the form $\int (f_k^+ g_k^+)_k^+ (v_k^- \bar{w}_{\ll k})_k^-$ or $\int (f_k^- g_k^-)_k^- (v_k^+ \bar{w}_{\ll k})_k^+$. We estimate the first one, the second one

being symmetrically equivalent to the first. We need to once again apply the bounds involving the operator Q .

$$\begin{aligned}
(43) &\leq C2^{(3\alpha+\beta-2)k} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (f_{\sim k}^+ g_{\sim k}^+)_k (v_{\sim k}^- \overline{w}_{\ll k})_{\sim k}^- dt dx \right| \\
&\leq C2^{(3\alpha+\beta-2)k} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} Q[f_{\sim k}^+] Q[g_{\sim k}^+] Q[v_{\sim k}^-] Q[\overline{w}_{\ll k}] dt dx \right| \\
&\leq C2^{(3\alpha+\beta-2)k} \|Q[f_{\sim k}^+] Q[v_{\sim k}^-]\|_{L_{t,x}^2} \|Q[g_{\sim k}^+] Q[\overline{w}_{\ll k}]\|_{L_{t,x}^2} \\
&\leq C_\delta 2^{(3\alpha+\beta-3+6\delta)k} \|f\|_{X^{0, \frac{1}{2}+\delta}} \|g\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}} \|w\|_{X^{0, \frac{1}{2}-\delta}}
\end{aligned}$$

where we have applied (29) for $\|Q[f_{\sim k}^+] Q[v_{\sim k}^-]\|_{L_{t,x}^2}$ and (27) for $\|Q[g_{\sim k}^+] Q[\overline{w}_{\ll k}]\|_{L_{t,x}^2}$.

Now to prove the same estimate for z_2 , it is clear that the sums I_1 and I_3 do not appear (or is finite), since $k < 0$ and $l > 0$. So we need to regard the sum of type I_2 . Note that we can consider $l \gg 1$, otherwise there is nothing to prove. We will use a projection $P_{\leq 0}$ instead of P_k to represent this case. With the restriction $P_{\leq 0}(f_{k_1} g_{k_2})$, we must have $|k_1 - k_2| \leq 3$.

$$(43) \leq C2^{\alpha l + (2\alpha-2)k_1} \left| \int_{\mathbf{R}^1 \times \mathbf{R}^1} (f_{k_1} g_{\sim k_1})_{\leq 0} v_l \overline{w}_{\sim l} dt dx \right|$$

If $|k_1 - l| \leq 3$, then by (21) and (28)

$$\begin{aligned}
(43) &\leq C2^{(3\alpha-2)l} \|(f_{\sim l} g_{\sim l})_{\leq 0}\|_{L_{t,x}^2} \|v_l \overline{w}_{\sim l}\|_{L_{t,x}^2} \\
&\leq C_{T,\delta} 2^{(3\alpha+\beta-3+3\delta)l} \|f\|_{X^{0, \frac{1}{2}+\delta}} \|g\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}} \|w\|_{X^{\frac{1}{2}-\beta, \frac{1}{2}-\delta}}.
\end{aligned}$$

Otherwise, we can assume that $|k_1 - l| > 3$, thus by (20) and (28)

$$\begin{aligned}
(43) &\leq C2^{\alpha l + (2\alpha-2)k_1} \|Q[f_{k_1}] Q[v_l]\|_{L_{t,x}^2} \|Q[g_{\sim k_1}] Q[\overline{w}_l]\|_{L_{t,x}^2} \\
&\leq C_{T,\delta} 2^{(\alpha+\beta-1+6\delta)l - (2\alpha-2)k_1} \|f\|_{X^{0, \frac{1}{2}+\delta}} \|v\|_{X^{0, \frac{1}{2}+\delta}} \|g\|_{X^{0, \frac{1}{2}+\delta}} \|w\|_{X^{\frac{1}{2}-\beta, \frac{1}{2}-\delta}}.
\end{aligned}$$

Both of these cases are summable in $l > 0$, so we are done. \square

3.4. Conclusion of the proof of Theorem 1. Now that we have the needed multilinear estimates, we perform a fixed point argument for the solution w of (36) in the space \mathcal{X} . For

simplicity, we group the terms on the right-hand side of (36) as follows⁷

$$\begin{aligned}
\mathcal{N}_1 &= G([e^{-it\partial_x^2}f + w]_{\leq 0}, (Id + P_{>0})[e^{-it\partial_x^2}f + h + w]); \\
\mathcal{N}_2 &= G(h_{\leq 0}, (Id + P_{>0})[h + w]) = \\
&= G(h_{\leq 0}, (Id + P_{>0})[h]) + G(T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)_{\leq 0}, (Id + P_{>0})[w]); \\
\mathcal{N}_3 &= G(h_{\leq 0}, e^{-it\partial_x^2}(Id + P_{>0})f) = G(T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)_{\leq 0}, e^{-it\partial_x^2}(Id + P_{>0})f); \\
\mathcal{N}_4 &= 2G(e^{-it\partial_x^2}f_{>0}, h_{>0}) = 2G(T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)_{>0}, e^{-it\partial_x^2}f_{>0}); \\
\mathcal{N}_5 &= 2G(e^{-it\partial_x^2}f_{>0}, w_{>0}); \\
\mathcal{N}_6 &= G(h_{>0}, h_{>0}); \mathcal{N}_7 = G(h_{>0}, w_{>0}) = G(T(e^{-it\partial_x^2}f, e^{-it\partial_x^2}f)_{>0}, w_{>0}); \\
\mathcal{N}_8 &= G(w_{>0}, w_{>0}).
\end{aligned}$$

In order to finish the proof, we need to show that $w^0 := e^{-it\partial_x^2}[-T(f, f)]$ and $\mathbf{w}^j : (\partial_t + i\partial_x^2)\mathbf{w}^j = \mathcal{N}_j, \mathbf{w}^j(0) = 0, j = 1, \dots, 8$, we have

$$(46) \quad \sum_{j=0}^8 \|\mathbf{w}^j\|_{\mathcal{X}} \leq C_{T,\delta}(\|f\|_{L^2} + \|h\|_{\mathcal{H}} + \|w\|_{\mathcal{X}})^2(1 + \|f\|_{L^2} + \|h\|_{\mathcal{H}} + \|w\|_{\mathcal{X}}).$$

3.4.1. *Estimates for \mathbf{w}^0 .* We have by Proposition 1 and Lemma 3 (with $s = \alpha - \frac{1}{2} < \frac{1}{2}$)

$$\|\mathbf{w}^0\|_{\mathcal{X}} = \|e^{-it\partial_x^2}[T(f, f)]\|_{X_T^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \lesssim \|T(f, f)\|_{H^{\alpha-\frac{1}{2}}} \lesssim \|f\|_{L^2}^2.$$

3.4.2. *Estimates for \mathbf{w}^1 .* Let us note first that for any two functions μ, ν , $G(\mu_{\leq 0}, \nu) = \langle \nabla \rangle^{\beta-\alpha}(\langle \nabla \rangle^{\alpha}\mu_{\leq 0}\langle \nabla \rangle^{\alpha}\nu)$ behaves for all practical purposes like $\mu_{\leq 0}\langle \nabla \rangle^{\beta}\nu$. Thus,

$$\begin{aligned}
\|\mathbf{w}^1\|_{\mathcal{X}} &\leq C_T \|G([e^{-it\partial_x^2}f + w]_{\leq 0}, (Id + P_{>0})[e^{-it\partial_x^2}f + h + w])\|_{L_t^2 H_x^{\alpha-\frac{1}{2}}} \\
&\leq C_T \|[e^{-it\partial_x^2}f + w]_{\leq 0} \cdot \langle \nabla \rangle^{\beta}(Id + P_{>0})[e^{-it\partial_x^2}f + h + w]\|_{L_t^2 H_x^{\alpha-\frac{1}{2}}}.
\end{aligned}$$

By Hölder's inequality, we have

$$\|[e^{-it\partial_x^2}f + w]_{\leq 0} \cdot \langle \nabla \rangle^{\beta}(Id + P_{>0})[h]\|_{L_t^2 H_x^{\alpha-\frac{1}{2}}} \leq \|[e^{-it\partial_x^2}f + w]_{\leq 0}\|_{L_{t,x}^{\infty}} \|h\|_{L_t^2 H_x^{\alpha+\beta-\frac{1}{2}}}.$$

By the definition of \mathcal{H} however, $\mathcal{H} \hookrightarrow X^{1-\delta, \delta} \hookrightarrow L_t^2 H_x^{\alpha+\beta-\frac{1}{2}}$. Thus $\|h\|_{L_t^2 H_x^{\alpha+\beta-\frac{1}{2}}} \leq C\|h\|_{\mathcal{H}}$ and by Sobolev embedding and (12)

$$\|[e^{-it\partial_x^2}f + w]_{\leq 0}\|_{L_{t,x}^{\infty}} \leq C\|e^{-it\partial_x^2}f + w\|_{L_t^{\infty} L_x^2} \leq C(\|f\|_{L^2} + \|w\|_{\mathcal{X}})$$

Regarding the remaining term in \mathcal{N}_1 , we can again split in two terms

$(Id + P_{>0})[e^{-it\partial_x^2}f + w] = [e^{-it\partial_x^2}f + w]_{\leq 0} + 2[e^{-it\partial_x^2}f + w]_{>0}$. The low frequency term is

⁷Recall that G is a bilinear form

easy to deal with (by the argument above for h), whereas for the high-frequency term, we have by Lemma 1 (more specifically (20)) that

$$\begin{aligned} & \| [e^{-it\partial_x^2} f + w]_{\leq 0} \cdot \langle \nabla \rangle^\beta [e^{-it\partial_x^2} f + w]_{> 0} \|_{L_t^2 H_x^{\alpha-\frac{1}{2}}} \\ & \leq \| [e^{-it\partial_x^2} f + w]_{\leq 0} \|_{X^{0, \frac{1}{2}+\delta}} \sum_{k>0} 2^{-(\frac{1}{2}-\delta)k} 2^{\alpha+\beta-\frac{1}{2}k} \| [e^{-it\partial_x^2} f + w]_k \|_{X^{0, \frac{1}{2}+\delta}} \\ & \leq C(\|f\|_{L^2} + \|w\|_{X^{0, \frac{1}{2}+\delta}})^2. \end{aligned}$$

3.4.3. *Estimates for \mathbf{w}^2 .* Write $\mathbf{w}^2 = \mathbf{w}_1^2 + \mathbf{w}_2^2$, where \mathbf{w}_1^2 is the solution corresponding from the first term in \mathcal{N}_2 . Then,

$$\begin{aligned} \|\mathbf{w}_1^2\|_{\mathcal{X}} & \leq C_T \|G(h_{\leq 0}, (Id + P_{>0})[h])\|_{L_t^2 H_x^{\alpha-\frac{1}{2}}} \leq C \|h_{\leq 0}\|_{L_{t,x}^\infty} \|h\|_{L_t^2 H_x^{\alpha-\frac{1}{2}}} \\ & \leq C \|h_{\leq 0}\|_{L_t^\infty H_x^{\frac{1}{2}}} \|h\|_{\mathcal{H}} \leq C \|h\|_{\mathcal{H}}^2. \end{aligned}$$

since $\mathcal{H} \hookrightarrow L_t^2 H_x^{\alpha-\frac{1}{2}} \cap L_t^\infty H_x^{\frac{1}{2}}$.

As far as \mathbf{w}_2^2 is concerned, we apply Lemma 7 (more precisely (42) for z_2), which yields

$$\|\mathbf{w}_2^2\|_{\mathcal{X}} \leq C_T \|e^{it\partial_x^2} f\|_{X^{0, \frac{1}{2}+\delta}}^2 \|(Id + P_{>0})w\|_{X^{0, \frac{1}{2}+\delta}} \leq C \|f\|_{L^2}^2 \|w\|_{\mathcal{X}}.$$

3.4.4. *Estimates for \mathbf{w}^3 .* The estimate for \mathbf{w}^3 is pretty similar to the one for \mathbf{w}_2^2 . Also, the low frequency term $G(h_{\leq 0}, e^{it\partial_x^2} f_{\leq 0})$ is already estimated in Section 3.4.2, so here we estimate only the non-linearity $G(h_{\leq 0}, e^{it\partial_x^2} f_{>0})$. By (42), applied for z_2 ,

$$\|\mathbf{w}^3\|_{\mathcal{X}} \leq C_T \|e^{-it\partial_x^2} f\|_{X^{0, \frac{1}{2}+\delta}}^2 \|e^{-it\partial_x^2} f\|_{X^{0, \frac{1}{2}+\delta}} \leq C \|f\|_{L^2}^3.$$

3.4.5. *Estimates for \mathbf{w}^4 .* We have by (42), applied for z_1 , that

$$\|\mathbf{w}^4\|_{\mathcal{X}} \leq C \|e^{-it\partial_x^2} f_{>0}\|_{X^{0, \frac{1}{2}+\delta}}^3 \leq C \|f\|_{L^2}^3.$$

3.4.6. *Estimates for \mathbf{w}^5 .* For \mathbf{w}^5 , we apply Lemma 5, whence

$$\|\mathbf{w}^5\|_{\mathcal{X}} \leq C \|w\|_{X^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \|e^{-it\partial_x^2} f\|_{X^{0, \frac{1}{2}+\delta}} \leq C \|w\|_{\mathcal{X}} \|f\|_{L^2}.$$

3.4.7. *Estimates for \mathbf{w}^6 .* The estimate for \mathbf{w}^6 follows from Lemma 6,

$$\|\mathbf{w}^6\|_{\mathcal{X}} \leq C \|h\|_{X^{1-\delta, \delta}}^2 \leq C \|h\|_{\mathcal{H}}^2.$$

3.4.8. *Estimates for \mathbf{w}^7 .* This term is in fact simpler than \mathbf{w}^4 (since w in the second component is in fact smoother than the free solution in \mathcal{N}_4). We deal with it in the same way. Namely, by (42), applied to z_1 , we have

$$\|\mathbf{w}^7\|_{\mathcal{X}} \leq C \|e^{-it\partial_x^2} f\|_{X^{0, \frac{1}{2}+\delta}}^2 \|w\|_{X^{0, \frac{1}{2}+\delta}} \leq C \|f\|_{L^2}^2 \|w\|_{\mathcal{X}}.$$

3.4.9. *Estimates for \mathbf{w}^8 .* Finally, the estimate for w^8 follows from Lemma 5. We have

$$\|\mathbf{w}^8\|_{\mathcal{X}} \leq C \|w\|_{X^{\alpha-\frac{1}{2}, \frac{1}{2}+\delta}} \|w\|_{X^{0, \frac{1}{2}+\delta}} \leq C \|w\|_{\mathcal{X}}^2$$

4. REGARDING L.W.P. FOR NONLINEARITIES OF THE FORM $\langle \nabla \rangle^\beta [u\bar{u}]$ AND $\langle \nabla \rangle^\beta [\bar{u}^2]$

We will just briefly sketch the analysis that one needs to undertake, in order to pursue well-posedness of the problem

$$u_t + iu_{xx} = \langle \nabla \rangle^\beta [u\bar{u}].$$

As a byproduct of this discussion, we will hopefully be able to shed some light on the issue with low regularity, which is present in this particular case, [14].

Following the ideas of Section 3, we need to construct T , so that (35) is satisfied, where of course $G(u, v) = \langle \nabla \rangle^{\beta-\alpha} [\langle \nabla \rangle^\alpha v \langle \nabla \rangle^\alpha \bar{v}]$. It is easy to see that the needed T is in the form

$$(47) \quad T(u, v)(x) = \frac{1}{8\pi^2 i} \int \frac{\langle \xi \rangle^\alpha \langle \eta \rangle^\alpha}{\langle \xi + \eta \rangle^{\alpha-\beta}} \frac{1}{\xi(\xi + \eta)} \widehat{u}_{>0}(\xi) \widehat{\bar{v}}(\eta) e^{i(\xi+\eta)x} d\xi d\eta.$$

Note that this transformation may be performed only when the output function $T(u, v)$ is Fourier localized, so that its frequency satisfies $\gtrsim 1$, so that we do not run into trouble with the term $(\xi + \eta)^{-1}$ inside the symbol of T . This is the reason why, in general (and unless we impose some homogeneous Sobolev norms in small frequencies, as is done in [14]), we cannot do better than $H^{-\frac{1}{4}+}$ l.w.p.

It is also clear from the form (47), that in the case of “high-high” interactions, the (generally smoothing) term $(\xi + \eta)^{-1}$ is not of much help to achieve better smoothness of $T(u, v)$. Therefore, performing this transformation would be advantageous, only if $2\alpha < 1$. This is a simple (if a little naive) way to see the optimality of restriction $\alpha < 1/2$ in the results of [14].

For the nonlinearities of the form $\langle \nabla \rangle^\beta [\bar{u}^2]$, following the same ideas, we come up with the following normal form

$$(48) \quad T(u, v)(x) = \frac{1}{8\pi^2 i} \int \frac{\langle \xi \rangle^\alpha \langle \eta \rangle^\alpha}{\langle \xi + \eta \rangle^{\alpha-\beta}} \frac{1}{2(\xi^2 + \eta^2 + \xi\eta)} \widehat{u}(\xi) \widehat{\bar{v}}(\eta) e^{i(\xi+\eta)x} d\xi d\eta.$$

Clearly, this normal form gains a derivative in each variable (very similar to the case $\langle \nabla \rangle^\beta [u^2]$ and hence, one may expect to get an identical result to Theorem 1 for this nonlinearity as well. We will not pursue these issues here.

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