6 Sequences and series

AREAS OF STUDY

- Sequences and series as maps between the natural numbers and the real numbers, and the use of technology to generate sequences and series and their graphs
- Sequences generated by recursion: arithmetic ($t_{n+1} = t_n + d$), geometric ($t_{n+1} = rt_n$) and fixed point iteration (for example, $t_1 = 2$, $t_{n+1} = {t_n}^2$, $t_1 = 0.5$, $t_{n+1} = 0.8t_n(1 - t_n)$)
- Practical applications of sequences and series, such as financial arithmetic, population modelling and musical scales

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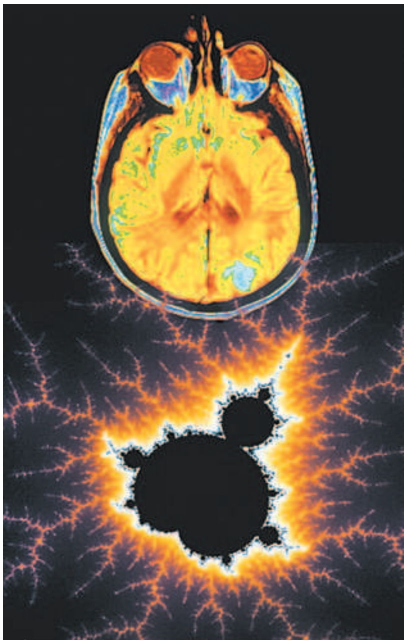
10 Quick Questions

6A Describing sequences

Sequences of numbers play an important part in our everyday life. For example, the following sequence:

$$2.25, 2.37, 2.58, 2.57, 2.63, \ldots$$



gives the end-of-day trading price (for 5 consecutive days) of a share in an electronics company. It looks like the price is on the rise, but is it possible to accurately predict the future price per share of the company?

The following sequence is more predictable:

$$10\,000, 9000, 8100, \ldots$$

This is the estimated number of radioactive decays of a medical compound each minute after administration to a patient. The compound is used to diagnose tumours. In the first minute, 10 000 radioactive decays are predicted; during the second minute, 9000, and so on. Can you predict the next number in the sequence? You're correct if you said 7290. Each successive term here is 90% of, or 0.90 times, the previous term.

Sequences are strings of numbers. They may be finite in number or infinite. Number sequences may follow an easily recognisable pattern or they may not. A great deal of recent mathematical work has gone into deciding whether certain strings follow a pattern (in which case subsequent terms could be predicted) or whether they are random (in which case subsequent terms cannot be predicted). This work forms the basis of chaos theory, speech recognition software for computers, weather prediction and stock market forecasting, to name but a few uses. The list is almost endless.

Sequences which follow a pattern can be described in a number of different ways. They may be listed in sequential order, they may be described as a functional definition, or they may be described in an iterative definition.

1 Listing in sequential order

Consider the sequence of numbers t: {5, 7, 9, . . .}. The numbers in sequential order are firstly 5 then 7 and 9 with the indication that there are more numbers to follow. The symbol t is the name of the sequence and the first three terms in the sequence shown are $t_1 = 5$, $t_2 = 7$ and $t_3 = 9$. The fourth term, t_4 if the pattern were to continue, would be the number 11. In general, t_n is the nth term in the sequence. In this example, the next term is simply the previous term with the number 2 added to it, with the first term being the number 5.

Another possible sequence is t: {5, 10, 20, 40, . . .}. In this case it appears that the next term is twice the previous term. The fifth term here, if the pattern continued, would be $t_5 = 80$. It can be difficult to determine whether or not a pattern exists in some sequences. Can you find the next term in the following sequence?

$$t: \{1, 1, 2, 3, 5, 8, \ldots\}$$

Here the next term is the sum of the previous *two terms*, hence the next term would be 5 + 8 which is equal to 13, and so on. This sequence is called the Fibonacci sequence and is named after its discoverer, Leonardo Fibonacci, a thirteenth century mathematician.

Here is another sequence; can you find the next term here?

$$t: \{7, 11, 16, 22, 29, \ldots\}$$

In this sequence the difference between successive terms increases by 1 for each pair. The first difference is 4, the next difference is 5 and so on. The sixth term is thus 37 which is 8 more than 29.

2 Functional definition

A functional definition is expressed in the form:

$$t_n = 2n - 7, n \in \{1, 2, 3, 4, \ldots\}$$

Using this definition the nth term can be readily calculated. For this example $t_1 = 2 \times 1 - 7 = {}^{-}5$, $t_2 = 2 \times 2 - 7 = {}^{-}3$, $t_3 = 2 \times 3 - 7 = {}^{-}1$ and so on. We can readily calculate the 100th term, $t_{100} = 2 \times 100 - 7 = 193$, simply by substituting the value $n = 100$ into the expression for t_n.

Look at the following example:

$$d_n = 4.9n^2, n \in \{1, 2, 3, \ldots\}$$

For this example, in which the sequence is given the name d, $d_1 = 4.9 \times 1^2 = 4.9$, $d_2 = 4.9 \times 2^2 = 19.6$. Listing the sequence would yield d: {4.9, 19.6, 44.1, 78.4, . . .}. The 10th term would be $4.9 \times 10^2 = 490$.

Here is another example:

$$c_n = \cos(n\pi) + 1, n \in \{1, 2, 3, \ldots\}$$

Here the sequence would be c: {0, 2, 0, 2, . . .}.

3 Iterative definition

An iterative definition is expressed in the form:

$$t_{n+1} = 3t_n - 2;\ t_1 = 6$$

This definition looks complicated, but is actually straightforward. You may have already come across this idea on a spreadsheet. The word *iteration* means the calculation of the next term from the previous term using the same procedure. The symbol t_{n+1} simply means the next term after the term t_n. In the above example the first term, t_1, is 6 (this is given in the definition) and so the next term, t_2, is $3 \times 6 - 2 = 16$, and the following term is $3 \times 16 - 2 = 46$. In each and all cases the next term is found by multiplying the previous term by 3 and then subtracting 2. We could write the sequence out as a table:

n	t_n	Comment
	$t_1 = 6$	Given in the definition
1	$t_2 = 3t_1 - 2$ $= 3 \times 6 - 2$ $= 16$	Using t_1 to find the next term, t_2
2	$t_3 = 3t_2 - 2$ $= 3 \times 16 - 2$ $= 46$	Using t_2 to find the next term, t_3
3	$t_4 = 3t_3 - 2$ $= 3 \times 46 - 2$ $= 136$	Using t_3 to find the next term, t_4

An example of this sequence using notation found in a spreadsheet would be:
A1 = 6 (the first term is equal to 6)
A2 = 3 × A1 − 2 (the next term is 3 times the previous term minus 2).

You could then apply the **Fill Down** option in the **Edit** menu of the spreadsheet from cell A2 downwards to generate as many terms in the sequence as required. This would result in the next cell down being three times the previous cell, less 2. The iterative definition finds a natural use in a spreadsheet environment and consequently much use is made of it. A drawback is that you cannot find the *n*th term directly as in the functional definition, but the advantage is that more complicated systems can be successfully modelled using iterative descriptions and hence are more interesting and relevant.

WORKED EXAMPLE 1

a **Find the next three terms in the sequence, b: $\{14, 7, \frac{7}{2}, \ldots\}$.**
b **Find the 4th, 8th and 12th terms in the following sequence: $e_n = n^2 - 3n, n \in \{1, 2, 3, \ldots\}$.**
c **Find the 2nd, 3rd and 5th terms for the following sequence: $k_{n+1} = 2k_n + 1, k_1 = {}^-0.50$.**

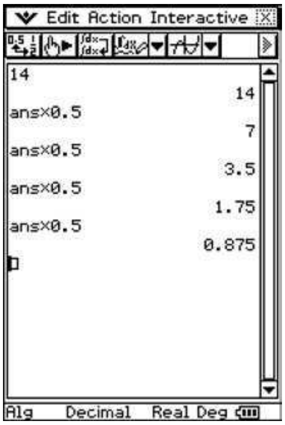
THINK

a 1 In this example the sequence is listed and a simple pattern is evident. From inspection, the next term is half the previous term and so the sequence would be $14, 7, \frac{7}{2}, \frac{7}{4}, \frac{7}{8}, \frac{7}{16}$.

2 On the Main screen, complete the entry lines as:
14
ans × 0.5
Press EXE repeatedly to generate the sequence.

WRITE

a The next three terms are $\frac{7}{4}, \frac{7}{8}, \frac{7}{16}$.



b **1** This is an example of a functional definition. The nth term of the sequence is found simply by substitution into the expression $e_n = n^2 - 3n$.

b $e_n = n^2 - 3n$

2 Find the 4th term by substituting $n = 4$.

$e_4 = 4^2 - 3 \times 4$
$= 4$

3 Find the 8th term by substituting $n = 8$.

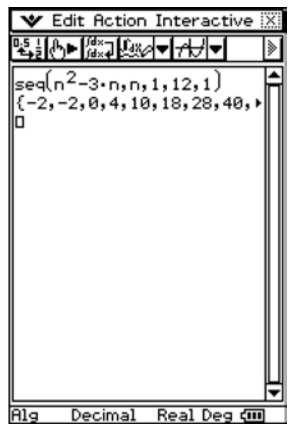
$e_8 = 8^2 - 3 \times 8$
$= 40$

4 Find the 12th term by substituting $n = 12$.

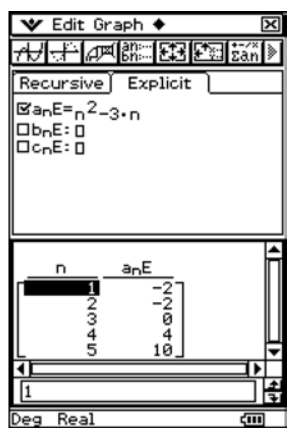
$e_{12} = 12^2 - 3 \times 12$
$= 108$

5 On the Main screen, tap:
- **Action**
- **List – Create**
- **seq**

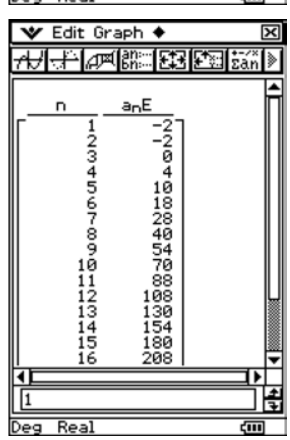
Complete the entry line as:
seq($n^2 - 3n$, n, 1, 12, 1)
Then press EXE.
Note: Scroll through the numbers in the sequence to find the 4th, 8th and 12th terms.



An alternative method to the one above for generating a sequence is shown.
On the Sequence screen, complete the sequence as shown.
To create the table, tap the table icon.



6 To find the required terms, tap Resize.



c **1** This is an example of an iterative definition. We can find the 2nd, 3rd and 5th terms for the sequence $k_{n+1} = 2k_n + 1$, $k_1 = {}^-0.50$ by iteration.

c $k_{n+1} = 2k_n + 1,$
$k_1 = {}^-0.50$

2 Substitute $k_1 = {}^-0.50$ into the formula to find k_2.

$$\begin{aligned} k_2 &= 2 \times -0.50 + 1 \\ &= 0 \end{aligned}$$

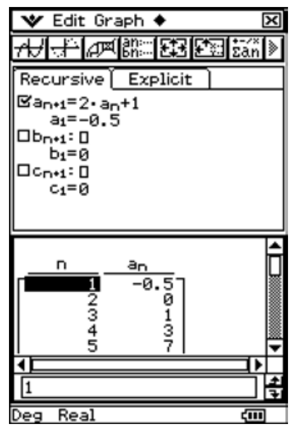
3 Continue the process until the value of k_5 is found.

$$\begin{aligned} k_3 &= 2 \times 0 + 1 \\ &= 1 \\ k_4 &= 2 \times 1 + 1 \\ &= 3 \\ k_5 &= 2 \times 3 + 1 \\ &= 7 \end{aligned}$$

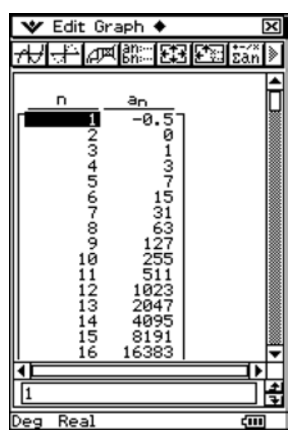
4 Write the answer.

Thus $k_2 = 0$, $k_3 = 1$ and $k_5 = 7$.

5 Again, on the Sequence screen, complete the sequence as shown, tapping [n,an] to type the sequence. To create the table, tap [table icon].



To enlarge the table, tap [Resize].



Logistic equation

The *logistic equation* is a model of population growth. It gives the rule for determining the population in any year, based on the population in the previous year. Since we need the previous term in order to be able to generate the next term of the sequence, then the logistic equation is an example of an iterative definition. It is of the general form:

$$t_{n+1} = at_n(1 - t_n),$$

where $0 < t_0 < 1$ and a is a constant.

Depending on the value of a, sequences generated by use of the logistic equation could be convergent, divergent, or oscillating. A string of numbers that converges to (settles at) a certain fixed value is called a *convergent* sequence. Sequence t_n can converge to only one possible number, x, called the *limit* of the sequence. This can be written as $t_n \to x$. (The symbol $\to$ is read as 'tends to', or 'approaches'.) A sequence whose terms grow further and further apart is called *divergent*. That is, a sequence is divergent if $t_n \to \infty$, or $t_n \to {}^{-}\infty$ as $n \to \infty$. Finally, a sequence whose terms tend to fluctuate between two (or more) values is called *oscillating*. An oscillating sequence is neither convergent nor divergent.

WORKED EXAMPLE 2

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Worked example 2

Given that $a = 2$ and $t_0 = 0.7$, use the logistic equation to generate a sequence of 6 terms, and state whether the sequence is convergent, divergent, or oscillating. If the sequence is convergent, state its limit.

THINK | **WRITE**

Method 1: Using the rule

1. Write the logistic equation, replacing a with its given value (that is, 2).

$$t_{n+1} = at_n(1 - t_n)$$
$$= 2t_n(1 - t_n)$$

2. To find t_1, substitute the value of t_0 (that is, 0.7) in place of t_n and evaluate.

$$t_1 = 2t_0(1 - t_0)$$
$$= 2 \times 0.7 \times (1 - 0.7) = 0.42$$

3. To find the next term, t_2, substitute the value of t_1 (that is, 0.42) in place of t_n and evaluate.

$$t_2 = 2t_1(1 - t_1)$$
$$= 2 \times 0.42 \times (1 - 0.42)$$
$$= 0.4872$$

4. Continue the iterative process four more times, each time substituting the value of the previous term into the logistic equation to find the next term.

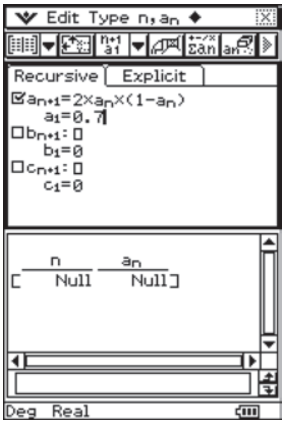
$$t_3 = 2t_2(1 - t_2)$$
$$= 2 \times 0.4872 \times (1 - 0.4872)$$
$$= 0.499\,672\,3$$
$$t_4 = 2t_3(1 - t_3)$$
$$= 2 \times 0.499\,672\,3 \times (1 - 0.499\,672\,3)$$
$$= 0.499\,999\,8$$
$$t_5 = 2t_4(1 - t_4)$$
$$= 2 \times 0.499\,999\,8 \times (1 - 0.499\,999\,8) = 0.5$$
$$t_6 = 2t_5(1 - t_5)$$
$$= 2 \times 0.5 \times (1 - 0.5) = 0.5$$

5. The terms of the sequence are growing closer and closer to each other, finally settling at 0.5.

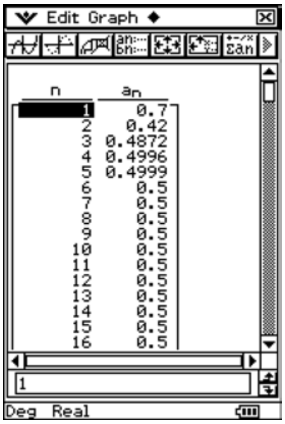
The sequence is convergent; the limit of the sequence is 0.5.

Method 2: Using technology

1. On the Sequence screen, complete the sequence as shown, tapping $\boxed{n, a_n}$ to type the sequence.



2 To create the table, tap .
To find the required terms, tap Resize.
Note: The first term is when $n = 2$, thus the terms are $n - 1$.



Note that instead of saying ‘the limit of the sequence is 0.5’ in the previous example, we could simply write $t_n \to 0.5$.

REMEMBER

1. A sequence is a string of numbers or expressions which may follow a recognisable pattern.
2. A sequence can be described in a number of ways.
   (a) As a list — for example: t_n: $\{1, 7, 2, 6, \ldots\}$
   (*Note:* $t_3 = 2$)
   (b) As a function — for example: $t_n = 2n - n^2$, $n \in \{1, 2, 3, \ldots\}$
   (*Note:* $t_5 = 2 \times 5 - 5^2 = -15$)
   (c) As a recursive or iterative formula — for example: $t_{n+1} = 2t_n - 3$, $t_1 = 6$
   (*Note:* $t_2 = 2 \times 6 - 3 = 9$)
3. The *logistic equation* is a model of population growth and is an example of an iterative definition. It is of the general form:
$$t_{n+1} = at_n(1 - t_n),$$
   where $0 < t_0 < 1$ and a is a constant.
4. A sequence that converges to (settles at) a certain fixed number, x (the *limit* of the sequence) is called *convergent*. This can be written as $t_n \to x$. A sequence whose terms grow further and further apart is called *divergent*. That is, a sequence is divergent if $t_n \to \infty$, or $t_n \to {}^{-}\infty$ as $n \to \infty$. A sequence whose terms fluctuate between two (or more) values is called *oscillating*. An oscillating sequence is neither convergent nor divergent.

EXERCISE 6A Describing sequences

1 WE1a For each of the following sequences, write a rule for obtaining the next term in the sequence and hence evaluate the next three terms.

a $\{1, 4, 7, \ldots\}$
b $\{1, 0, {}^{-}1, {}^{-}2, \ldots\}$
c $\{1, 4, 16, 64, \ldots\}$
d $\{3, \frac{3}{2}, \frac{3}{4}, \ldots\}$
e $\{2, {}^{-}5, 8, {}^{-}11, 14, \ldots\}$
f $\{2, 5, 9, 14, 20, \ldots\}$
g $\{3, 4, 7, 11, 18, \ldots\}$
h $\{2a - 5b, a - 2b, b, {}^{-}a + 4b, \ldots\}$
i $\{1, 0, {}^{-}1, 0, 1, \ldots\}$
j $\{1.0, 1.1, 1.11, \ldots\}$
k $\{1024, {}^{-}512, 256, {}^{-}128, \ldots\}$

2 WE1b Find the first, fifth and tenth terms in the following sequences.

a $t_n = 2n - 5,\ n \in \{1, 2, 3, \ldots\}$
b $t_n = 4 \times 3^{n-2},\ n \in \{1, 2, 3, \ldots\}$
c $t_n = \dfrac{n}{n+1},\ n \in \{1, 2, 3, \ldots\}$
d $t_n = 17 - 3.7n,\ n \in \{1, 2, 3, \ldots\}$
e $t_n = 5 \times \left(\frac{1}{2}\right)^n,\ n \in \{1, 2, 3, \ldots\}$
f $t_n = 5 \times \left(\frac{1}{2}\right)^{(3-n)},\ n \in \{1, 2, 3, \ldots\}$
g $t_n = (^{-}1)^n + n,\ n \in \{1, 2, 3, \ldots\}$
h $t_n = 3^n 2^{^{-}n},\ n \in \{1, 2, 3, \ldots\}$
i $t_n = n^2 - n + 41,\ n \in \{1, 2, 3, \ldots\}$
j $t_n = a + (n-1)d,\ n \in \{1, 2, 3, \ldots\}$
k $t_n = ar^{n-1},\ n \in \{1, 2, 3, \ldots\}$

3 WE1c Using a CAS calculator, or other method, find the third, eighth and tenth terms in the following sequences.

a $u_{n+1} = u_n + 2,\ u_1 = 3$
b $u_{n+1} = u_n - 2,\ u_1 = \frac{1}{2}$
c $u_{n+1} = 3u_n,\ u_1 = 0.85$
d $u_{n+1} = {}^{-}2u_n,\ u_1 = {}^{-}3$
e $u_{n+1} = \frac{3}{4}u_n,\ u_1 = \frac{4}{3}$
f $u_{n+1} = u_n - 7,\ u_1 = 14$
g $u_{n+1} = {}^{-}u_n + 2,\ u_1 = 3$
h $u_{n+1} = u_n + (^{-}1)^n u_n,\ u_1 = 3$
i $u_n = 2u_{n-1},\ u_1 = \frac{-1}{4}$
j $u_{n+1} = au_n + a,\ u_1 = a$
k $u_{n+2} = u_{n+1} + u_n,\ u_1 = 1,\ u_2 = 1$
l $u_{n+1} = {}^{-}u_n{}^2 - 2,\ u_1 = 3$

4 WE2 Given the following values of a and t_0, use the logistic equation to generate a sequence of six terms. State whether the sequence is convergent, divergent, or oscillating. If the sequence is convergent, state its limit.

a $a = 0.8,\ t_0 = 0.5$
b $a = 0.4,\ t_0 = 0.6$
c $a = 1.1,\ t_0 = 0.9$
d $a = 1.9,\ t_0 = 0.4$
e $a = 2.1,\ t_0 = 0.5$
f $a = 2.5,\ t_0 = 0.3$
g $a = 3,\ t_0 = 0.2$
h $a = 3.4,\ t_0 = 0.7$
i $a = 4.2,\ t_0 = 0.1$
j $a = 4.5,\ t_0 = 0.8$

5 Study the pattern in each of the following sequences and where possible write the next two terms in the sequence, describing the pattern that you use.

a 5, 6, 8, 11, . . .
b 4, 9, 12, 13, 12, 9, . . .
c 9, 8, 9, 0, . . .
d 6, 12, 12, 6, $1\frac{1}{2}$, . . .
e 5, 8, 13, 21, . . .
f 1, 3, 7, 15, . . .
g 1, 3, 2, 4, 3, . . .

6 MC **a** Which of the following functional definitions could be used to describe a sequence $\{3, 1, {}^{-}1, \ldots\}$?

A $t_n = n - 2,\ n \in \{1, 2, 3, \ldots\}$
B $t_n = 2n - 5,\ n \in \{1, 2, 3, \ldots\}$
C $t_n = 5n - 2,\ n \in \{1, 2, 3, \ldots\}$
D $t_n = 5 - 2n,\ n \in \{1, 2, 3, \ldots\}$
E $t_n = 2(5 - n),\ n \in \{1, 2, 3, \ldots\}$

b Which of the following recursive definitions could be used to describe a sequence $\{20, {}^{-}10, 5, \ldots\}$?

A $t_{n+1} = t_n - 30,\ t_1 = 20$
B $t_{n+1} = \dfrac{t_n}{2},\ t_1 = {}^{-}20$
C $t_{n+1} = t_n - \dfrac{t_n}{2},\ t_1 = 20$
D $t_{n+1} = t_n - 10,\ t_1 = 20$
E $t_{n+1} = \dfrac{^{-}t_n}{2},\ t_1 = 20$

c Which of the following sequences is generated by the definition $t_n = \dfrac{6n^2 - 12}{2}$, $n \in \{1, 2, 3, \ldots\}$?

A $\{^{-}3, 6, 15, \ldots\}$
B $\{^{-}3, 6, {}^{-}12, \ldots\}$
C $\{^{-}3, 6, 21, \ldots\}$
D $\{^{-}3, 6, 12, \ldots\}$
E $\{^{-}3, 6, 18, \ldots\}$

7 Write the iterative definition for each of the following sequences.

a $\{7, 5, 3, 1, {}^{-}1, \ldots\}$
b $\{12, 6, 3, 1.5, \ldots\}$
c $\{12, 12.6, 13.2, \ldots\}$
d $\{2, 11, 56, 281, \ldots\}$
e $\{4, {}^{-}12, 36, \ldots\}$
f $\{2, 4, 16, 256, \ldots\}$

8 In the township of Grizabella, the population of stray cats in any given year is given as p_{n+1}. This can be calculated using the formula $p_{n+1} = 1.3p_n(1 - p_n)$, where p_n is the number of cats (in hundreds) in the preceding year. If in 2005 there were 28 stray cats in Grizabella township, calculate:

a the expected number of stray cats for 2006 and 2007

b the limiting number of stray cats that Grizabella township can sustain.

9 In the neighbouring township of Macavity, the size of the population of stray cats follows the logistic equation

$$p_{n+1} = 0.3p_n(1 - p_n),$$

where p_{n+1} and p_n refer to the population size (in hundreds) in any given year and in the preceding year respectively. It is known that in 2005, there were 62 stray cats in the township. By generating and examining the sequence of numbers using the above equation, decide what will happen in the long run to the population size of stray cats in Macavity township. (That is, will the population of cats keep increasing, decreasing, or settle at a particular value?)

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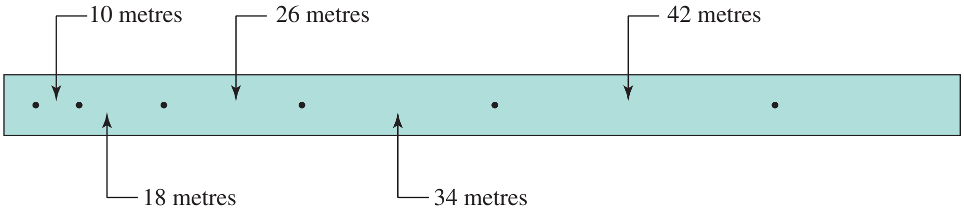
6B Arithmetic sequences

At a racetrack a new prototype racing car unfortunately develops an oil leak. Each second, a drop of oil hits the road. The driver of the car puts her foot on the accelerator and the car increases speed at a steady rate as it hurtles down the straight. The diagram below shows the pattern of oil drops on the road with the distances between the drops labelled.



The sequence of distances travelled in metres each second is {10, 18, 26, 34, 42, . . .}. The first term in the sequence, t_1, is 10 and as you can see, each subsequent term is 8 more than the previous term. This type of sequence is given a special name — an *arithmetic sequence*.

An arithmetic sequence is a sequence where there is a common difference between any two successive terms.

We can list the sequence in a table as in table A. From this table we can see that it is possible to write a *functional definition* for the sequence in terms of the first term, 10, and the common difference, 8, and thus:

$$\begin{aligned} t_n &= 10 + (n-1) \times 8 \\ &= 2 + 8n,\ n \in \{1, 2, 3, \ldots\} \end{aligned}$$

We can readily get a general formula for the nth term of an arithmetic sequence whose first term is a and whose common difference is d (see table B).

Table A

n	t_n	t_n
1	$10 + 0 \times 8$	10
2	$10 + 1 \times 8$	18
3	$10 + 2 \times 8$	26
4	$10 + 3 \times 8$	34
n	$10 + (n-1) \times 8$ $= 10 + 8n - 8$ $= 2 + 8n$	$2 + 8n$

Table B

n	t_n
1	$a + 0 \times d$
2	$a + 1 \times d$
3	$a + 2 \times d$
4	$a + 3 \times d$
n	$a + (n-1) \times d$ $= (a - d) + dn$

In general then:

The nth term of an arithmetic sequence is given by

$$t_n = a + (n-1) \times d = (a - d) + nd,\ n \in \{1, 2, 3, \ldots\}$$

where a is the first term and d is the common difference.

If we consider three successive terms in an arithmetic sequence, namely x, y and z, then since $y - x =$ the common difference, d, and $z - y = d$, it follows that:

$$y - x = z - y \Rightarrow y = \frac{z + x}{2}$$

The middle term of any three consecutive terms in an arithmetic sequence is called an *arithmetic mean* and is the average of the outer two.

That is, $y = \dfrac{z + x}{2}$ for any 3 consecutive terms, x, y, z of the arithmetic sequence.

WORKED EXAMPLE 3

Show that the following sequences are arithmetic.

a $\frac{-7}{4}, \frac{-11}{8}, {}^{-}1, \ldots$ **b** $x^2 - 4x, 3x^2 - 7x, 5x^2 - 10x, \ldots$

THINK	WRITE
a 1 To show that a sequence is arithmetic you need to show that the difference between any two successive terms is a constant. Find the difference between the first and the second terms.	**a** $t_2 - t_1$ $= \frac{-11}{8} - \left(\frac{-7}{4}\right)$ $= \frac{-11 + 14}{8}$ $= \frac{3}{8}$

2. Find the difference between the second and the third terms.

$t_3 - t_2$
$= {}^{-}1 - \left(\frac{-11}{8}\right)$
$= \frac{3}{8}$

3. Compare the differences and draw your conclusion.

$t_2 - t_1 = t_3 - t_2 = \frac{3}{8}$
$\therefore$ The sequence is arithmetic.

b 1. Find the difference between the first two terms.

b $t_2 - t_1$
$= 3x^2 - 7x - (x^2 - 4x)$
$= 2x^2 - 3x$

2. Find the difference between the second and the third terms.

$t_3 - t_2$
$= 5x^2 - 10x - (3x^2 - 7x)$
$= 2x^2 - 3x$

3. Compare the differences and draw your conclusion.

$t_2 - t_1 = t_3 - t_2 = 2x^2 - 3x$
$\therefore$ The sequence is arithmetic.

WORKED EXAMPLE 4

State which of the following are arithmetic sequences by finding the difference between successive terms. For those which are arithmetic, find the next term in the sequence, t_4, and consequently find the *functional definition* for the nth term for the sequence, t_n.

a t: {4, 9, 15, . . .}
b t: {${}^{-}2$, 1, 4, . . .}

THINK / WRITE

a 1. To check that a sequence is arithmetic, see if a common difference exists.

a $9 - 4 = 5$
$15 - 9 = 6$

2. There is no common difference, as $5 \neq 6$.

Since there is no common difference the sequence is not arithmetic.

b 1. To check that a sequence is arithmetic, see if a common difference exists.

b $1 - {}^{-}2 = 3$
$4 - 1 = 3$

2. The common difference is 3.

The sequence is arithmetic with the common difference $d = 3$.

3. The next term in the sequence, t_4, can be found by adding 3 to the previous term, t_3.

$t_4 = t_3 + 3$
$= 4 + 3$
$= 7$

4. To find the functional definition, write the formula for the nth term of the arithmetic sequence.

$t_n = a + (n - 1) \times d$
$= (a - d) + nd$

5. Identify the values of a and d.

$a = -2$ and $d = 3$

6. Substitute $a = {}^{-}2$ and $d = 3$ into the formula and simplify.

$t_n = ({}^{-}2 - 3) + n \times 3$
$t_n = 3n - 5$

WORKED EXAMPLE 5

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Worked example 5

Find the missing terms in this arithmetic sequence: {41, *a*, 55, *b*, . . .}.

THINK	WRITE
1 The first three successive terms are 41, *a*, 55. Write the rule for the middle term of the three successive terms of an arithmetic sequence.	For x, y, z: $y = \frac{x+z}{2}$
2 Identify the variables.	$x = 41$; $y = a$; $z = 55$
3 Substitute the values of x, y and z into the formula in step 1 and evaluate.	$a = \frac{41+55}{2}$ $= 48$
4 Find the common difference. (The second term is now known.)	$d = t_2 - t_1$ $= 48 - 41$ $= 7$
5 Find the value of b by adding the common difference to the preceding term.	$b = 55 + 7$ $= 62$
6 State your answer.	So $a = 48$, $b = 62$

WORKED EXAMPLE 6

Find the 16th and *n*th terms in an arithmetic sequence with the 4th term 15 and 8th term 37.

THINK	WRITE
Method 1: Using the rule	
1 Write the formula for the *n*th term of the arithmetic sequence.	$t_n = a + (n - 1) \times d$
2 Substitute $n = 4$ and $t_4 = 15$ into the formula and label it equation [1].	t_4: $a + 3d = 15$ [1]
3 Substitute $n = 8$ and $t_8 = 37$ into the formula and label it equation [2].	t_8: $a + 7d = 37$ [2]
4 Solve the simultaneous equations: subtract equation [1] from equation [2] to eliminate a.	[2] − [1]: $a + 7d - a - 3d = 37 - 15$ $4d = 22$
5 Divide both sides by 4.	$d = \frac{22}{4}$ $= 5\frac{1}{2}$
6 Substitute $d = 5\frac{1}{2}$ into equation [1], and solve for a.	Substituting $d = 5\frac{1}{2}$ into [1]: $a + 3 \times 5\frac{1}{2} = 15$ $a = -1\frac{1}{2}$

7 To find the nth term of the arithmetic sequence, substitute the values of a and d into the general formula and simplify.

$$t_n = {}^{-}1\tfrac{1}{2} + (n-1) \times 5\tfrac{1}{2}$$
$$= \frac{-3}{2} + (n-1)\frac{11}{2}$$
$$= \frac{{}^{-}3 + 11n - 11}{2}$$

8 To find the 16th term, substitute $n = 16$ into the formula, established in the previous step and evaluate.

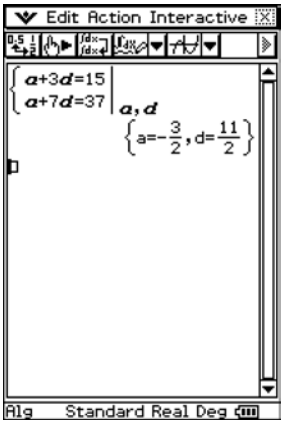
$$t_n = \frac{11n - 14}{2},\ n \in \{1, 2, 3, \ldots\}$$
$$\text{If } n = 16,\quad t_{16} = \frac{11 \times 16 - 14}{2}$$
$$= 81$$

Method 2: Using technology

1 Write the two equations that represent t_4 and t_8.

t_4: $a + 3d = 15$ [1]
t_8: $a + 7d = 37$ [2]

2 To solve equations [1] and [2] simultaneously, on the Main screen complete the entry line as shown.
Then press EXE.



3 Write the answer.

$$\text{If } a = \frac{-3}{2} \text{ and } d = \frac{11}{2},$$
$$t_n = \frac{11n - 14}{2}$$
$$t_{16} = 81$$

REMEMBER

1. An arithmetic sequence is one where successive terms have a common difference. This common difference is given the symbol d. Thus $t_{n+1} - t_n = d$ for all values of n. The first term in the sequence is given the symbol a.
2. If x, y, z are successive terms in an arithmetic sequence then y is called an arithmetic mean and is given by $y = \dfrac{x+z}{2}$.
   That is, the middle term is the average of the outer two terms.
3. An arithmetic sequence can be written as a, $a + d$, $a + 2d$, . . . and so the nth term t_n is:
   $t_n = a + (n - 1)d$ using the function notation, or
   $t_{n+1} = t_n + d$, $t_1 = a$ using the iterative notation.

EXERCISE
6B Arithmetic sequences

1 WE3 Show that the following sequences are arithmetic.

a $\{^-12, ^-7, ^-2, \ldots\}$

b $\{^-0.12, 3.48, 7.08, \ldots\}$

c $\{\frac{^-3}{8}, \frac{3}{8}, \frac{9}{8}, \ldots\}$

d $\{2.3, ^-1.7, ^-5.7, \ldots\}$

e $\{\frac{5}{9}, \frac{^-1}{9}, \frac{^-7}{9}, \ldots\}$

f $\{18, ^-18, ^-54, \ldots\}$

g $\{5\frac{2}{3}, 7\frac{4}{15}, 8\frac{13}{15}, \ldots\}$

h $\{x + 9, 2x + 7, 3x + 5, \ldots\}$

i $\{3x^2 - 4x, 5x^2 - 2x, 7x^2, \ldots\}$

j $\{3(2 - x), 2(2 - x), 2 - x, \ldots\}$

2 WE4 State which of the following are arithmetic sequences by finding the difference between successive terms. For those which are arithmetic, find the next term in the sequence, t_4, and consequently find the *functional definition* for the nth term for the sequence, t_n.

a t_n: $\{3, 5, 7, \ldots\}$

b t_n: $\{4, 7, 11, \ldots\}$

c t_n: $\{3, 6, 12, \ldots\}$

d t_n: $\{^-3, 0, 3, \ldots\}$

e t_n: $\{^-2, ^-6, ^-10, \ldots\}$

f t_n: $\{\frac{2}{7}, \frac{11}{14}, \frac{9}{7}, \ldots\}$

g t_n: $\{\frac{3}{4}, \frac{3}{2}, \frac{3}{1}, \ldots\}$

h t_n: $\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, \ldots\}$

i t_n: $\{\frac{1}{4}, \frac{^-3}{2}, -\frac{13}{4}, \ldots\}$

j t_n: $\{2\pi + 3, 4\pi + 1, 6\pi - 1, \ldots\}$

3 Find the term given in brackets for each of the following arithmetic sequences.

a $\{4, 9, 14, \ldots\}, (t_{21})$

b $\{^-2, 10, 22, \ldots\}, (t_{58})$

c $\{^-27, ^-12, 3, \ldots\}\ (t_{100})$

d $\{2, ^-11, ^-24, \ldots\}\ (t_{2025})$

4 Find the functional definition for the nth term of the following arithmetic sequences:

a where the first term is 5 and the common difference is $^-3$

b where the first term is 2.5 and the common difference is $\frac{1}{2}$

c where the first term is $^-3$ and the common difference is 3

d where the first term is $2x$ and the common difference is $5x$.

5 Find the nth term in the arithmetic sequence where the first term is 6 and the third term is 10.

6 Find the nth term in the arithmetic sequence where the first term is 3 and the third term is 13.

7 WE5 Find the missing terms in this arithmetic sequence: $\{16, m, 27, n\}$

8 Find the missing terms in the arithmetic sequence below.

$x - 3y$, ________________, $^-3x + 5y$, ________________, . . .

9 WE6 Find the 4th term and nth term in the arithmetic sequence whose first term is 6 and whose 7th term is $^-10$.

10 If $t_{10} = 100$ and $t_{15} = 175$, find the first term, the common difference and hence the nth term for the arithmetic sequence.

11 If $t_{10} = \frac{^-1}{2}$ and $t_{13} = \frac{3}{4}$, find the first term, the common difference and hence the nth term for the arithmetic sequence.

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Using elimination to solve simultaneous equations

12 Insert four evenly spaced numbers between 8 and 36.

13 For the arithmetic sequence $\{22, m, n, 37, \ldots\}$, find the values for m and n.

14 For the following arithmetic sequences, find the *iterative definition* and use it in a CAS calculator to generate the first 50 numbers in the sequence.

a t_n: $\{3, 7, 11, \ldots\}$

b t_n: $\{^-3, 0, 3, \ldots\}$

c t_n: $\{^-2, ^-6, ^-10 \ldots\}$

d t_n: $\{\frac{2}{7}, \frac{11}{14}, \frac{9}{7}, \ldots\}$

e t_n: $\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, \ldots\}$

f t_n: $\{\frac{1}{4}, \frac{^-3}{2}, \frac{^-13}{4}, \ldots\}$

g t_n: $\{2\pi + 3, 4\pi + 1, 6\pi - 1, \ldots\}$

15 The first three terms in an arithmetic sequence are 37, 32, 27 and the kth term is $^{-}3$. Find the value for k.

16 Find the value of x such that the following forms an arithmetic progression:
$\ldots x, 3x + 4, 10x - 7 \ldots$

17 MC For the following sequence t: $\{4, 11, 18, \ldots\}$, the difference between the 4th and the 10th term is:
A 35 B 42 C 49 D 56 E 63

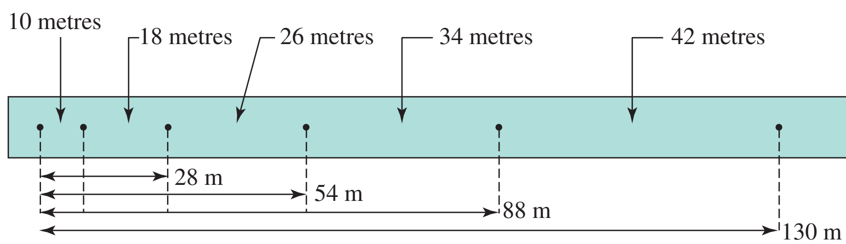
18 MC The tenth term in an arithmetic sequence is 12 and the third term is $^{-}2$. The first term in the sequence is:
A $^{-}7$ B $^{-}3$ C $^{-}5$ D $^{-}8$ E $^{-}6$

19 The ratio between the first term and the second term in an arithmetic sequence is $\frac{3}{4}$. The ratio between the second term and the third term is $\frac{4}{5}$.
a Calculate the ratio of the third term to the fourth term.
b Find the ratio of the nth and the nth + 1 term in the sequence.

6C Arithmetic series

Often we have a sequence of numbers and we wish to know their sum. For an example, we return to the oil drops on the racetrack from the start of the previous section on arithmetic sequences. The distance covered by the car each second illustrated the concept of an arithmetic sequence.

The total distance covered by the car is the sum of the individual distances covered each second. So after one second the car has travelled 10 m, after 2 seconds the car has travelled 10 + 18 m = 28 m, after three seconds the car has travelled a total distance of 10 + 18 + 26 m = 54 m, and so on.



A series, S_n, is the sum of a sequence of n terms $t_1 + t_2 + t_3 + \ldots + t_n$.

Thus:

$$S_1 = t_1$$
$$S_2 = t_1 + t_2$$
$$S_3 = t_1 + t_2 + t_3$$
$$S_n = t_1 + t_2 + t_3 + \ldots + t_{n-2} + t_{n-1} + t_n.$$

For an arithmetic sequence, the sum of the first n terms, S_n, can be written in two ways:

1. The first term in the arithmetic sequence is a, the common difference is d, and the last term — that is, the nth term — in the sequence is l.

$$\begin{aligned} S_n &= a + (a + d) + (a + 2d) + \ldots (a + (n-3)d) + (a + (n-2)d + a + (n-1)d \\ &= a + (a + d) + (a + 2d) + \ldots (l - 3d) + (l - 2d) + (l - d) + l \qquad [1] \end{aligned}$$

2. We can write the sum S_n in reverse order starting with the nth term and summing back to the first term a:

$$S_n = l + (l - d) + (l - 2d) + \ldots + (a + 2d) + (a + d) + a. \qquad [2]$$

If we add equation [1] and equation [2] together and recognise that there are n terms each of which equal $(a + l)$ we get:

$$2S_n = (a + l) + (a + l) + \ldots n \text{ times}$$
$$= n(a + l)$$

and so: $$S_n = \frac{n}{2}(a + l)$$

or since l is the nth term, $l = a + (n - 1)d$, so $S_n = \frac{n}{2}[a + a + (n - 1)d]$

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

The sum of the first n terms in the arithmetic sequence is given by

$$S_n = \frac{n}{2}(a + l)$$

where a is the first term and l is the last term; or alternatively, since $l = a + (n - 1)d$, by

$$S_n = \frac{n}{2}(2a + (n - 1)d)$$

where a is the first term and d is the common difference.

If we know the first term, a, the common difference, d, and the number of terms, n, that we wish to add together we can calculate the sum directly without having to add up all the individual terms.

It is worthwhile also to note that $S_{n+1} = S_n + t_{n+1}$. This tells us that the next term in the series S_{n+1} is the present sum S_n plus the next term in the sequence t_{n+1}. This result is useful in spreadsheets where one column gives the sequence and an adjacent column is used to give the series.

WORKED EXAMPLE 7

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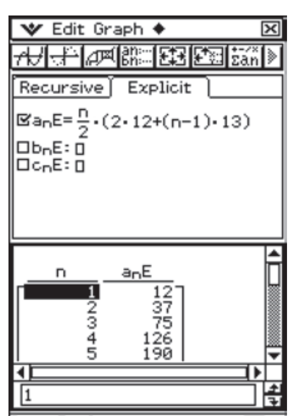
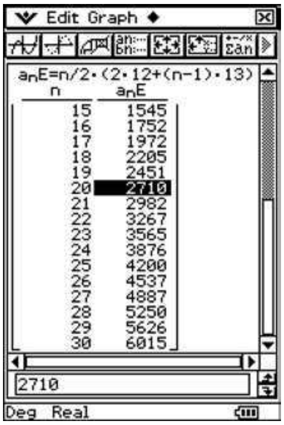
Worked example 7

Find the sum of the first 20 terms in the sequence t_n: {12, 25, 38, . . .}.

THINK	WRITE
Method 1: Using the rule	
1 Write the formula for the sum of the first n terms in the arithmetic sequence.	$S_n = \frac{n}{2}(2a + (n - 1)d)$
2 Identify the variables.	$a = 12$, $d = 25 - 12 = 13$, $n = 20$
3 Substitute values of a, d and n into the formula and evaluate.	$S_{20} = \frac{20}{2}(2 \times 12 + 19 \times 13)$ $S_{20} = 2710$
Method 2: Using technology	
1 On the Sequence screen, complete the sequence as shown, tapping [n,an] to type the sequence. To create the table, tap [table icon].	

2 To find the required terms, tap .



3 Write the answer. $S_{20} = 2710$

REMEMBER

1. The sum of the first n terms of an arithmetic sequence is $S_n = \frac{n}{2}(2a + (n-1)d)$ and so to find S_n the values for a and d need to be found for the sequence whose series is required.
2. In general, $t_{n+1} = S_{n+1} - S_n$.

EXERCISE 6C Arithmetic series

1 WE7 Consider the following sequences and find the sums of the terms as indicated.
- **a** t_n: {1, 2, 3, . . .}. Find S_{10}, S_{50}, S_{100}.
- **b** t_n: {1, 3, 5, . . .}. Find S_5, S_{10}, S_{20}.
- **c** $t_n = 3n + 7$, $n \in \{1, 2, 3, \ldots\}$. Find S_5, S_{10}, S_n.
- **d** $t_n = {}^-4n + 15$, $n \in \{1, 2, 3, \ldots\}$. Find S_5, S_{10}, S_n.
- **e** $t_{n+1} = t_n + 5.5$, $t_1 = 2.5$. Find S_5, S_{10}, S_{20}.
- **f** $t_{n+1} = t_n + \pi$, $t_1 = 2\pi$. Find S_5, S_{10}, S_{20}.
- **g** The first term is 4 and the common difference is 3. Find S_4, S_{16}, S_{64}.
- **h** The first term is 14 and the common difference is ${}^-3\frac{1}{2}$. Find S_4, S_9, S_{14}.
- **i** The first term is 50 and the 10th term is ${}^-40$. Find S_{10}.
- **j** The 5th term is 10 and the 8th term is 16. Find S_5, S_{50}, S_{500}.

2 **a** Find the sum of the first 50 positive integers.
- **b** Find the sum of the first 100 positive integers.

3 **a** Find the sum of all the half-integers between 0 and 100.
Note: The sequence of half-integers is $\{\frac{1}{2}, 1\frac{1}{2}, 2\frac{1}{2}, 3\frac{1}{2}, \ldots\}$
- **b** Compare your answer with that for question **2b**.

4 Find the sum of the first 12 terms of an arithmetic sequence in which the second term is 8 and thirteenth term is 41.

5 A sequence of numbers is defined by t_n: {15, 9, 3, ${}^-3$, . . .}.
- **a** Find the sum of the first 13, 16 and 19 terms in the sequence.
- **b** Find the sum of all the terms between and including t_{10} and t_{15}.

6 A sequence of numbers is defined by $t_n = 2n - 7, n \in \{1, 2, 3, \ldots\}$. Find
 a the sum of the first 20 terms
 b the sum of all the terms between and including t_{21} and t_{40}
 c the average of the first 40 terms.
 Hint: You need to find the sum first.

7 Find the equation that gives the sum of the first n positive integers.

8 a Show that the sum of the first n odd integers is equal to the perfect square n^2.
 b Show that the sum of the first n even integers is equal to $n^2 + n$.

9 A sequence is 5, 7, 9, 11, . . . How many consecutive terms need to be added to obtain 357?

10 Consider the sum of the first n integers. For what value of n will the sum first exceed 1000?

11 a Find the sum of all integers divisible by 3 which lie between 200 and 400.
 b Find the sum of all integers divisible by 6 which lie between 200 and 400.

12 The first term in an arithmetic sequence is 5 and the sum of the first 20 terms is 1240. Find the common difference, d.

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WorkSHEET 6.1

13 The sum of the first four terms of an arithmetic sequence is 58, and the sum of the next four terms is twice that number. Find the sum of the following four terms.

14 The sum of a series is given by $S_n = 4n^2 + 3n$. Use the result that $t_{n+1} = S_{n+1} - S_n$ to prove that the sequence of numbers, t_n, whose series is $S_n = 4n^2 + 3n$ is arithmetic. Find both the functional and iterative equations for the sequence, t_n.

6D Geometric sequences

A farmer is breeding worms which he hopes to sell to local shire councils for use in the decomposition of waste at rubbish dumps. Worms reproduce readily and the farmer expects a 10% increase per week in the mass of worms that he is farming. A 10% increase per week would mean that the mass of worms would increase by a constant factor of $(1 + \frac{10}{100})$ or 1.1.

He starts off with 10 kg of worms. By the beginning of the second week he will expect $10 \times 1.1 = 11$ kg of worms, by the start of the third week he would expect $11 \times 1.1 = 10 \times (1.1)^2 = 12.1$ kg of worms, and so on. This is an example of a *geometric sequence.*

A geometric sequence is the sequence where each term is obtained by multiplying the preceding term by a certain constant factor.

The first term is 10 and the common factor here is 1.10 which represents a 10% increase on the previous term. We can put the results of the above example into a table.

n	t_n	t_n
1	$10 \times (1.1)^0$	10
2	$10 \times (1.1)^1$	11
3	$10 \times (1.1)^2$	12.1
4	$10 \times (1.1)^3$	13.31
n	$10 \times (1.1)^{n-1}$	$10 \times (1.1)^{n-1}$

From this table we can see that

$$t_2 = 1.1 \times t_1,\ t_3 = 1.1 \times t_2$$

and so on. In general:

$$t_{n+1} = 1.1 \times t_n$$

The common factor or common ratio whose value is 1.1 for this example can be found by dividing any two successive terms: $\dfrac{t_{n+1}}{t_n}$.

A geometric sequence, t, can be written in terms of the first term, a, and the common ratio, r. Thus:

$$t: \{a, ar, ar^2, ar^3, \ldots, ar^{n-1}, \ldots\}$$

The first term $t_1 = a$, the second term $t_2 = ar$, the third term $t_3 = ar^2$ and consequently the nth term, t_n is ar^{n-1}.

For a geometric sequence:

$$t_n = ar^{n-1}$$

where a is the first term and r the common ratio, given by

$$r = \frac{t_{n+1}}{t_n}$$

If we consider three consecutive terms in a geometric sequence, x, y and z, then

$$\frac{y}{x} = r = \frac{z}{y}$$

where r is the common factor.

Thus the middle term, y, called the *geometric mean*, can be calculated in terms of the outer two terms, x and z.

For a geometric sequence . . . , $x, y, z,$ . . . :

$$y^2 = xz$$

WORKED EXAMPLE 8

State whether the sequence is geometric by finding the ratio of successive terms: t_n: {2, 6, 18, . . .}. If it is geometric, find the next term in the sequence, t_4, and the nth term for the sequence, t_n.

THINK	WRITE
1 Find the ratio $\dfrac{t_2}{t_1}$.	$\dfrac{t_2}{t_1} = \dfrac{6}{2}$ $= 3$
2 Find the ratio $\dfrac{t_3}{t_2}$.	$\dfrac{t_3}{t_2} = \dfrac{18}{6}$ $= 3$
3 Compare the ratios and make your conclusion.	Since $\dfrac{t_2}{t_1} = \dfrac{t_3}{t_2} = 3$, the sequence is geometric with the common ratio $r = 3$.
4 Since the sequence is geometric, to find the fourth term, multiply the preceding (third) term by the common ratio.	$t_4 = t_3 \times r$ $= 18 \times 3$ $= 54$
5 Write the general formula for the nth term of the geometric sequence.	$t_n = ar^{n-1}$

6 Identify the values of a and r.	$a = 2; r = 3$
7 Substitute the values of a and r into the general formula.	$t_n = 2 \times 3^{n-1}$

WORKED EXAMPLE 9

Find the nth term and the 10th term in the geometric sequence, where the first term is 3 and the third term is 12.

THINK	WRITE
1 Write the general formula for the nth term in the geometric sequence.	$t_n = ar^{n-1}$
2 State the value of a (the first term in the sequence) and the value of the third term.	$a = 3; t_3 = 12$
3 Substitute all known values into the general formula.	$12 = 3 \times r^{3-1}$ $= 3 \times r^2$
4 Solve for r (note that there are two possible solutions).	$r^2 = \frac{12}{3}$ $= 4$ $r = ^{\pm}\sqrt{4}$ $= ^{\pm}2$
5 Substitute the values of a and r into the general equation. Since there are two possible values for r, you must show both expressions for the nth term of the sequence.	So $t_n = 3 \times 2^{n-1}$, or $t_n = 3 \times (^{-}2)^{n-1}$
6 Find the 10th term by substituting $n = 10$ into each of the two expressions for the nth term.	When $n = 10$, $t_{10} = 3 \times 2^{10-1}$ (using $r = 2$) $= 3 \times 2^9$ $= 1536$ or $t_{10} = 3 \times (^{-}2)^{10-1}$ (using $r = ^{-}2$) $= 3 \times (^{-}2)^9$ $= ^{-}1536$

WORKED EXAMPLE 10

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Worked example 10

The fifth term in a geometric sequence is 14 and the seventh term is 0.56. Find the common ratio, r, the first term, a, and the nth term for the sequence.

THINK	WRITE
1 Write the general rule for the nth term of the geometric sequence.	$t_n = ar^{n-1}$
2 Use the information about the 5th term to form an equation. Label it [1].	When $n = 5$, $t_n = 14$ $14 = a \times r^{5-1}$ $14 = a \times r^4$ [1]
3 Similarly, use information about the 7th term to form an equation. Label it [2].	When $n = 7$, $t_n = 0.56$ $0.56 = a \times r^{7-1}$ $0.56 = a \times r^6$ [2]

4. Solve equations simultaneously: Divide equation [2] by equation [1] to eliminate a.

$\frac{[2]}{[1]}$ gives $\frac{ar^6}{ar^4} = \frac{0.56}{14}$

5. Solve for r.

$r^2 = 0.04$

$r = {}^{\pm}\sqrt{0.04}$

$= {}^{\pm}0.2$

6. Since there are two solutions, we have to perform two sets of computations. Consider the positive value of r first. Substitute the value of r into either of the two equations, say equation [1], and solve for a.

If $r = 0.2$

Substitute r into [1]:

$a \times (0.2)^4 = 14$

$0.0016a = 14$

$a = 14 \div 0.0016$

$= 8750$

7. Substitute the values of r and a into the general equation to find the expression for the nth term.

The nth term is:

$t_n = 8750 \times (0.2)^{n-1}$

8. Now consider the negative value of r.

If $r = {}^{-}0.2$

9. Substitute the value of r into either of the two equations, say equation [1], and solve for a. (Note that the value of a is the same for both values of r.)

Substitute r into [1]

$a = ({}^{-}0.2)^4 = 14$

$0.0016a = 14$

$a = 14 \div 0.0016$

$= 8750$

10. Substitute the values of r and a into the general formula to find the second expression for the nth term of the sequence.

The nth term is:

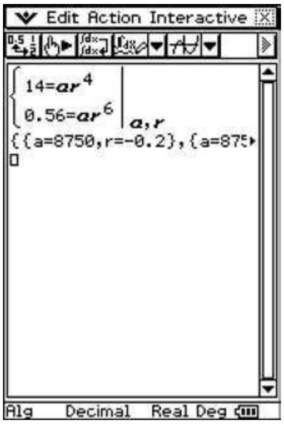
$t_n = 8750 \times ({}^{-}0.2)^{n-1}$

11. Write the two equations that represent t_5 and t_7.

t_5: $14 = a \times r^4$ [1]

t_7: $0.56 = a \times r^6$ [2]

12. To solve equations [1] and [2] simultaneously, on the Main screen complete the entry line as shown.
Then press EXE.



13. Write the answer.

When $r = {}^{-}0.2$ and $a = 8750$,

$t_n = 8750 \times ({}^{-}0.2)^{n-1}$

When $r = 0.2$ and $a = 8750$,

$t_n = 8750 \times (0.2)^{n-1}$

REMEMBER

1. A geometric sequence is one where each successive term is obtained by multiplying the preceding term by the constant number. This number is called the *common ratio* and is given the symbol r. Thus $\frac{t_{n+1}}{t_n} = r$ for all values of n.
   The first term in the sequence is given the symbol a.
2. If x, y, z are successive terms in the geometric sequence then y is called a *geometric mean* and is given by $y^2 = xz$.
3. A geometric sequence can be written as $a, ar, ar^2, \ldots$ and so the nth term t_n is:
   $t_n = ar^{n-1}$ using the function notation, or
   $t_{n+1} = rt_n,\ t_1 = a$ using the iterative notation.

EXERCISE

6D Geometric sequences

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Digital doc
Spreadsheet 036
Fibonacci sequences

1 WE8 State which of the following are geometric sequences by finding the ratio of successive terms. For those which are geometric, find the next term in the sequence, t_4 and the nth term for the sequence, t_n.

a t_n: $\{3, 6, 9, \ldots\}$ **b** t_n: $\{4, 12, 36, \ldots\}$
c t_n: $\{3, 6, 12, \ldots\}$ **d** t_n: $\{4, 6, 9, \ldots\}$ **e** t_n: $\{^-3, 1, \frac{^-1}{3}, \ldots\}$
f t_n: $\{2, ^-6, 18, \ldots\}$ **g** t_n: $\{\frac{2}{7}, \frac{6}{14}, \frac{9}{14}, \ldots\}$ **h** t_n: $\{\frac{3}{4}, \frac{3}{2}, \frac{3}{1}, \ldots\}$
i t_n: $\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, \ldots\}$ **j** t_n: $\{\frac{1}{4}, \frac{^-3}{2}, 9, \ldots\}$ **k** t_n: $\{2\pi, 4\pi^2, 8\pi^3, \ldots\}$

2 For each of the following:
  i show that the sequence is geometric
  ii find the nth term and consequently the 6th and the 10th terms.

a t: $\{5, 10, 20, \ldots\}$ **b** t: $\{2, 5, 12.5, \ldots\}$
c t: $\{1, ^-3, 9, \ldots\}$ **d** t: $\{2, ^-4, 8, \ldots\}$
e t: $\{2.3, 3.45, 5.175, \ldots\}$ **f** t: $\{\frac{1}{2}, 1, 2, \ldots\}$
g t: $\{\frac{1}{3}, \frac{1}{12}, \frac{1}{48}, \ldots\}$ **h** t: $\{\frac{3}{5}, \frac{^-1}{5}, \frac{1}{15}, \ldots\}$
i t: $\{x, 3x^4, 9x^7, \ldots\}$ **j** t: $\{\frac{1}{x}, \frac{2}{x^2}, \frac{4}{x^3}, \ldots\}$

3 WE9 Find the nth term and the 10th term in the geometric sequence where:
a the first term is 2 and the third term is 18 (Why are there two possible answers?)
b the first term is 1 and the third term is 4 (Why are there two possible answers?)
c the first term is 5 and the fourth term is 40
d the first term is $^-1$ and the second term is 2
e the first term is 9 and the third term is $\frac{1}{81}$. (Why are there two possible answers?)

4 Find the 4th term in the geometric sequence where the first term is 6 and the 7th term is $\frac{3}{32}$.

5 Find the nth term in the geometric sequence where the first term is 3 and the fourth term is $6\sqrt{2}$.

6 For the geometric sequence $3, m, n, 192, \ldots$, find the values for m and n.

7 Consider the geometric sequence t: $\{16, m, 81, n, \ldots\}$. Find the values of m and n, if it is known that both are positive numbers.

8 For the geometric sequence a, 15, b, 0.0375, . . . , find the values of a and b, given that they are positive numbers.

9 WE10 The third term in a geometric sequence is 100 and the fifth term is 400. Find the common ratio, r, the first term, a, and the nth term for the sequence.

10 If $t_2 = \frac{1}{2}$ and $t_5 = \frac{27}{16}$, find the first term, a, the common factor, r, and hence the nth term for the geometric sequence.

11 Find the value of x such that the following sequence forms a geometric progression: $x - 1, 3x + 4, 6x + 8$.

12 Insert three terms in between 8, _, _, _, $\frac{1}{32}$ such that the sequence of numbers is geometric.

13 The difference between the first term and the second term in a geometric sequence is 6. The difference between the second term and the third term is 3.

a Calculate the difference between the third term and the fourth term.

b Find the nth term in the sequence.

14 The first two terms in a geometric sequence are 120, 24, and the kth term is 0.0384. Find the value for k.

6E Geometric series

When we add up or sum the terms in a sequence we get the series for that sequence. If we look at the geometric sequence {2, 6, 18, 54, . . .} where the first term $t_1 = a = 2$ and the common ratio is 3 we can quickly calculate the first few terms in the series of this sequence.

$$S_1 = t_1 = 2$$
$$S_2 = t_1 + t_2 = 2 + 6 = 8$$
$$S_3 = t_1 + t_2 + t_3 = 2 + 6 + 18 = 26$$
$$S_4 = t_1 + t_2 + t_3 + t_4 = 2 + 6 + 18 + 54 = 80$$

In general the sum of the first n terms is:

$$S_n = t_1 + t_2 + t_3 + \ldots + t_{n-2} + t_{n-1} + t_n.$$

For a geometric sequence the first term is a, the second term is ar, the third term is ar^2 and so on up to the nth term which is ar^{n-1}. Thus:

$$S_n = a + ar + ar^2 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \qquad [1]$$

If we multiply equation [1] by r we get:

$$rS_n = ar + ar^2 + ar^3 + \ldots ar^{n-2} + ar^{n-1} + ar^n \qquad [2]$$

Note that on the right-hand side of equations [1] and [2] all but two terms are common, namely the first term in equation [1], a, and the last term in equation [2], ar^n. If we take the difference between equation [2] and equation [1] we get:

$$rS_n - S_n = ar^n - a \qquad [2] - [1]$$
$$\therefore \quad (r-1)S_n = a(r^n - 1)$$
$$\therefore \quad S_n = \frac{a(r^n - 1)}{r - 1};\ r \neq 1 \qquad (r \text{ cannot equal } 1)$$

We now have an equation which allows us to calculate the sum of the first n terms of a geometric sequence.

The sum of the first n terms of a geometric sequence is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1};\ r \neq 1$$

where a is the first term of the sequence and r is the common ratio.

WORKED EXAMPLE 11

Find the sum of the first 5 terms (S_5) of these geometric sequences.

a t_n: {1, 4, 16, . . .} **b** $t_n = 2\,(2)^{n-1}, n \in \{1, 2, 3, \ldots\}$ **c** $t_{n+1} = \frac{1}{4}t_n, t_1 = \frac{-1}{2}$

THINK | **WRITE**

Method 1: Using the rule

a 1 Write the general formula for the sum of the first n terms of the geometric sequence.

a $S_n = \dfrac{a(r^n - 1)}{r - 1}$

2 Write the sequence.

t_n: {1, 4, 16, . . .}

3 Identify the variables: a is the first term; r can be established by finding the ratio; n is known from the question.

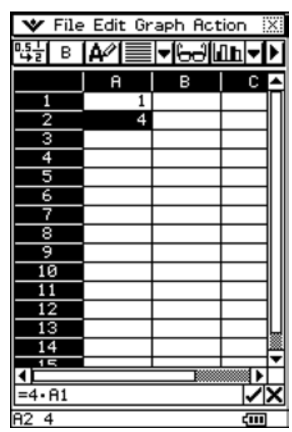
$a = 1; r = \frac{4}{1} = 4; n = 5$

4 Substitute the values of a, r and n into the formula and evaluate.

$$S_5 = \frac{1(4^5 - 1)}{4 - 1}$$
$$= \frac{1024 - 1}{3}$$
$$= 341$$

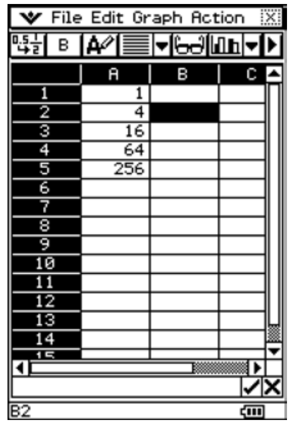
Method 2: Using technology

a 1 On the Spreadsheet screen, type the initial value of 1 in cell A1. Complete the entry line in cell A2 as:
= 4 × A1
Then press EXE.

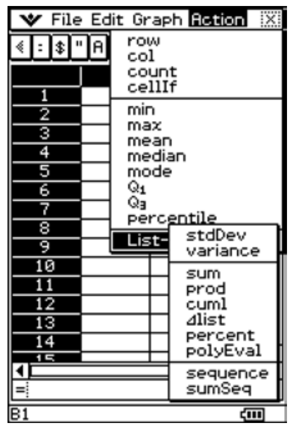


2 To find the terms in the sequence highlight A2 to A5 and tap:
- **Edit**
- **Fill Range**

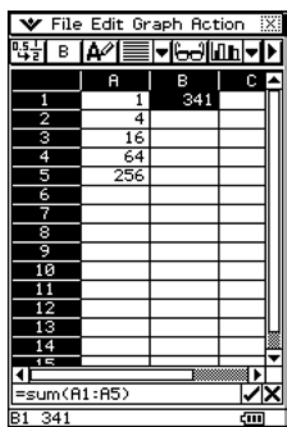
3 Tap **OK**.



4 To sum the sequence, in cell B1, tap:
- **Action**
- **List-Calculation**
- **sum**



5 Highlight cells A1 to A5 and then press (EXE).
The answer will appear in cell B1.



6 Write the answer. If t_n: $\{1, 4, 16, \ldots\}$ then, $S_5 = 341$.

b 1 Write the sequence. **b** $t_n = 2(2)^{n-1}$, $n \in \{1, 2, 3, \ldots\}$

2 Compare the given rule with the general formula for the *n*th term of the geometric sequence $t_n = ar^{n-1}$ and identify values of *a* and *r*; the value of *n* is known from the question. $a = 2$; $r = 2$; $n = 5$

3 Substitute values of *a*, *r* and *n* into the general formula for the sum and evaluate.

$$S_5 = \frac{2(2^5 - 1)}{2 - 1}$$

$$= \frac{2(32 - 1)}{1}$$

$$= 62$$

c 1. Write the sequence.

c $t_{n+1} = \frac{1}{4}t_n,\ t_1 = \frac{^-1}{2}$

2. This is an iterative formula, so the coefficient of t_n is our r; $a = t_1$; n is known from the question.

$r = \frac{1}{4};\ a = \frac{^-1}{2};\ n = 5$

3. Substitute values of a, r and n into the general formula for the sum and evaluate.

$$S_5 = \frac{\frac{^-1}{2}\left[\left(\frac{1}{4}\right)^5 - 1\right]}{\frac{1}{4} - 1}$$

$$= \frac{\frac{^-1}{2} \times \left(\frac{1}{1024} - 1\right)}{\frac{^-3}{4}}$$

$$= \frac{^-341}{512}$$

The infinite sum of a geometric sequence where $r < 1$

When the constant ratio, r, is less than 1 or greater than $^-1$, that is, $\{r: {}^-1 < r < 1\}$, each successive term in the sequence gets closer to zero. This can readily be shown with the following two examples.

$$g: \{2, {}^-1, \tfrac{1}{2}, \tfrac{^-1}{4}, \ldots\} \text{ where } a = 2 \text{ and } r = \frac{^-1}{2}$$

$$h: \{40, \tfrac{1}{2}, \tfrac{1}{160}, \ldots\} \text{ where } a = 40 \text{ and } r = \frac{1}{80}$$

In both the examples, successive terms approach zero as n increases. In the second case the approach is more rapid than in the first and the first sequence alternates positive and negative. A simple investigation with a spreadsheet will quickly reveal that for geometric sequences with the size or magnitude of $r < 1$ the series eventually settles down to a near constant value. We say that the series converges to a value S_∞ which is the sum to infinity of all terms in the geometric sequence. We can find the value S_∞ by recognising that as $n \to \infty$ the term $r^n \to 0$, provided r is between $^-1$ and 1. We write this technically as $^-1 < r < 1$ or $|r| < 1$. The symbol $|r|$ means the magnitude or size of r. Using our equation for the sum of the first n terms:

$$S_n = \frac{a(r^n - 1)}{r - 1};\ r \neq 1$$

Taking $^-1$ as a common factor from the numerator and denominator:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

As $n \to \infty$, $r^n \to 0$ and hence $1 - r^n \to 1$. Thus the top line or numerator will equal a when $n \to \infty$:

$$S_\infty = \frac{a}{1 - r};\quad |r| < 1$$

We now have an equation which allows us to calculate the sum to infinity, S_∞ of a geometric sequence.

The sum to infinity S_∞ of the geometric sequence is given by:

$$S_\infty = \frac{a}{1 - r};\quad |r| < 1$$

where a is the first term of the sequence and r is the common ratio whose magnitude is less than one.

WORKED EXAMPLE 12

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Tutorial
int-1055
Worked example 12

a **Find the sum to infinity for the sequence t_n: {10, 1, 0.1, . . .}.**
b **Find the fourth term in the geometric sequence whose first term is 6 and whose sum to infinity is 10.**

THINK	WRITE
a 1 Write the formula for the nth term of the geometric sequence.	**a** $t_n = ar^{n-1}$
2 From the question we know that the first term, a, is 10 and $r = 0.1$.	$a = 10, \quad r = 0.1$
3 Write the formula for the sum to infinity.	$S_\infty = \dfrac{a}{1-r}; \quad \lvert r\rvert < 1$
4 Substitute $a = 10$ and $r = 0.10$ into the formula and evaluate.	$S_\infty = \dfrac{10}{1-0.1}$ $S_\infty = \frac{10}{0.9} = \frac{100}{9} = 11\frac{1}{9}$
b 1 Write the formula for the sum to infinity.	**b** $S_\infty = \dfrac{a}{1-r}; \quad \lvert r\rvert < 1$
2 From the question it is known that the infinite sum is equal to 10 and that the first term a is 6. Write down this information.	$a = 6; S_\infty = 10$
3 Substitute known values into the formula.	$10 = \dfrac{6}{1-r}$
4 Solve for r.	$10(1 - r) = 6$ $10 - 10r = 6$ $10r = 4$ $r = 0.4$
5 Write the general formula for the nth term of the geometric sequence.	$t_n = ar^{n-1}$
6 To find the 4th term substitute $a = 6$, $n = 4$ and $r = 0.4$ into the formula and evaluate.	$t_4 = 6 \times (0.4)^3$ $= 0.384$

REMEMBER

1. The sum of the first n terms in a geometric sequence is:

$$S_n = \frac{a(1-r^n)}{1-r} \text{ with } r \neq 1$$

or

$$S_n = \frac{a(r^n-1)}{r-1} \text{ with } r \neq 1$$

2. When the magnitude of r is less than one, that is, $^-1 < r < 1$, the sum of a geometric sequence to infinity, S_∞ is given by:

$$S_\infty = \frac{a}{1-r}$$

EXERCISE

6E Geometric series

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Digital doc
Spreadsheet 036
Fibonacci series

1 WE11 Consider the following sequences and find the terms indicated.

a t_n: {1, 2, 4, . . .}. Find S_5, S_{10}, S_{20}.

b t_n: {1, 3, 9, . . .}. Find S_5, S_{10}, S_{20}.

c $t_n = 3(^-2)^{n-1}$, $n \in$ {1, 2, 3, . . .}. Find S_5, S_{10}, S_{20}.

d $t_n = {}^-4(1.2)^{n-1}$, $n \in$ {1, 2, 3, . . .}. Find S_1, S_{10}, S_{20}.

e $t_{n+1} = 2t_n$, $t_1 = \frac{3}{2}$. Find S_1, S_5, S_{10}.

f $t_{n+1} = \frac{1}{2}t_n$, $t_1 = \frac{^-2}{3}$. Find S_1, S_5, S_{10}.

g The first term is 3000 and the common ratio is 1.05. Find S_4, S_{16}, S_{64}.

h The first term is 1400 and the common ratio is $^-1.1$. Find S_4, S_9, S_{14}.

i The first term is 20; every other term is obtained by multiplying the preceding term by 5. Find S_5, S_{10}.

j The first term is $^-2$; every other term is obtained by multiplying the preceding term by $\frac{^-1}{2}$. Find S_5, S_{10}.

2 Consider the following geometric sequences and find the terms indicated.

a The first term is 440 and the 12th term is 880. Find S_6.

b The 5th term is 1 and the 8th term is 8. Find S_1, S_{10}, S_{20}.

3 Find the sum of the first 12 terms of a geometric sequence in which the second term is $\frac{8}{3}$ and the fifth term is 9.

4 What minimum number of terms of the series $2 + 3 + 4\frac{1}{2} + \ldots$ must be taken to give a sum in excess of 100?

5 The sum of the first four terms of a geometric sequence is 312, and the sum of the next four terms is 625 times that number. Find the sum of the following four terms.

6 Find the sum of all powers of 2 between 500 and 50 000.

7 Find the sum of all powers of 4 between 500 and 50 000.

8 WE12a Find the sum to infinity for the following geometric sequences.

a t_n: $\{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$

b t_n: $\{1, \frac{^-1}{2}, \frac{1}{4}, \frac{^-1}{8}, \ldots\}$

c t_n: $\{1, \frac{1}{3}, \frac{1}{9}, \ldots\}$

d t_n: $\{1, \frac{2}{3}, \frac{4}{9}, \ldots\}$

e t_n: $\{1, \frac{^-2}{3}, \frac{4}{9}, \frac{^-8}{9}, \ldots\}$

9 For the infinite geometric sequence $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$, find the sum to infinity. Consequently, find what proportion each of the first three terms contributes to this sum as a percentage.

10 For the infinite geometric sequence $\{1, \frac{1}{4}, \frac{1}{16}, \ldots\}$, find the sum to infinity. Consequently, find what proportion each of the first three terms contributes to this sum as a percentage.

11 For the infinite geometric sequence $\{1, \frac{3}{4}, \frac{9}{16}, \ldots\}$, find the sum to infinity. Consequently, find what proportion each of the first three terms contributes to this sum as a percentage.

12 A sequence of numbers is defined by $t_n = 3\left(\frac{1}{2}\right)^{n-1}$, $n \in$ {1, 2, 3, . . .}.

a Find the sum of the first 20 terms.

b Find the sum of all the terms between and including t_{21} and t_{40}.

c Find the sum to infinity, S_∞.

13 A sequence of numbers is defined by t_n: {9, $^-3$, 1, . . .}.

a Find the sum of the first 9 terms.

b Find the sum of all the terms between and including t_{10} and t_{15}.

c Find the sum to infinity, S_∞.

14 The first term of the geometric sequence is 5 and the fourth term is 0.078 125. Find the sum to infinity.

15 The sum of the first four terms of a geometric sequence is 30 and the sum to infinity is 32. Find the first three terms of the sequence.

16 For the geometric sequence $\sqrt{5}+\sqrt{3}, \sqrt{5}-\sqrt{3}, \ldots$, find the common factor, r, and the sum of the infinite series, S_∞.

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Digital doc
WorkSHEET 6.2

17 If $1 + 3x + 9x^2 + \ldots = \frac{2}{3}$, find the value of x.

18 WE12b The first term in a geometric sequence is 4 and $S_\infty = 6$. Find the common factor, r.

19 If the common ratio for a geometric sequence is 0.99 and the sum to infinity is 100, what is the value of the first and second terms in the sequence?

20 Show that $x^n - 1$ always has a factor $(x - 1)$ for $n \in \{1, 2, 3, \ldots\}$.

21 A student stands at one side of a road 10 metres wide, and walks half-way across. The student then walks half of the remaining distance across the road, then half the remaining distance again and so on.

a Will the student ever make it *past* the other side of the road?

b Does the width of the road affect your answer?

6F Applications of sequences and series

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Interactivity
int-0973
Applications of sequences and series
eLesson
eles-0080
The Fibonacci sequence

This section consists of a mixture of problems where the work covered in the first five exercises is applied to a variety of situations.

The following general guidelines can assist you in solving the problems.

1. Read the question carefully.
2. Decide whether the information suggests an arithmetic or geometric sequence. Check to see if there is a constant difference between successive terms or a constant ratio. If there is neither, look for a simple number pattern such as the difference between successive terms changing in a regular way.
3. Write the information from the problem using appropriate notation. For example, if you are told that the 5th term is 12, write $t_5 = 12$. If the sequence is arithmetic, you then have an equation to work with, namely: $a + 4d = 12$. If you know the sequence is geometric, then $ar^4 = 12$.
4. Define what you have to calculate and write an appropriate formula or formulas. For example, if you have to find the 10th number in a sequence which you know is geometric, you have an equation: $t_{10} = ar^9$. This can be calculated if a and r are known or can be established.
5. Use algebra to find what is required in the problem.

WORKED EXAMPLE 13

In 1970 the cost of 1 megabyte of computer memory was \$2025. In 1980 the cost for the same amount of memory had reduced to \$45 and by 1990 the cost had dropped to \$1.

a What was the cost of 1 megabyte of memory in the year 2000?

b How much memory, in megabytes, could you buy for \$10 in the year 2010 based on the current trend?

THINK

a

1. Present the given information in a table.
2. Study the table. The information suggests a geometric sequence for the cost at each ten-year interval. Verify this by checking for a constant ratio between successive terms.
3. To find the cost in the year 2000, find the fourth term in the sequence by multiplying the preceding (third) term by the common ratio.
4. Interpret the result and clearly answer the question.

b

1. If the cost of 1 megabyte can be found in the year 2010 then the amount of memory purchased for $10 can be determined. To find the predicted cost in the year 2010 the fifth term in the sequence needs to be determined.
2. Take the reciprocal of t_5 to get the amount of memory per dollar.
3. Find the amount of memory that can be purchased for $10.

WRITE

a

Year	1970	1980	1990	2000	2010
Cost ($)	2025	45	1	?	?

$45 \div 2025 = \frac{1}{45}$ and $1 \div 45 = \frac{1}{45}$ so the three terms form a geometric sequence with common ratio $r = \frac{1}{45}$.

$$t_4 = t_3 \times r$$
$$t_4 = 1 \times \frac{1}{45}$$
$$= \frac{1}{45}$$
$$= 0.022\ldots$$

In the year 2000 one would have paid about 2 cents for a megabyte of memory.

b

$$t_5 = t_4 \times r$$
$$= \frac{1}{45} \times \frac{1}{45}$$
$$= \frac{1}{2025} \text{ of a dollar per megabyte}$$

The amount of memory per dollar is 2025 megabytes.

So $10 would buy 10×2025 = 20 250 megabytes.

WORKED EXAMPLE 14

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Tutorial
int-1056
Worked example 14

Express the recurring decimal 0.131 313 13 . . . as a proper fraction.

THINK

1. Express the given number as a geometric series.
2. State the values of a and r.
3. Find the sum to infinity, S_∞. Write the formula for the sum to infinity.
4. Substitute values of a and r into the formula and simplify.
5. Multiply both numerator and denominator by 100 to get rid of the decimal point.

WRITE

$$0.131\,313\ldots = 0.13 + 0.001\,3 + 0.000\,013\ldots$$

$$a = 0.13 \text{ and } r = \frac{0.0013}{0.13} = 0.01$$

$$S_\infty = \frac{a}{1-r}$$
$$S_\infty = \frac{0.13}{1-0.01}$$
$$S_\infty = \frac{0.13}{0.99}$$
$$S_\infty = \frac{13}{99}$$

REMEMBER

To solve problems, use the following guidelines.
1. Identify the type of the sequence by checking whether there is a common difference, or a common ratio.
2. Translate given information into mathematical statements, using appropriate notation.
3. Define what you have to find and write appropriate formula(s).
4. Use algebra to find what is required.

EXERCISE
6F Applications of sequences and series

1 WE13 In 1970 the Smith family purchased a small house for \$60 000. Over the following years, the value of their property rose steadily. In 1975 the value of the house was \$69 000 and in 1980 it reached \$79 350.
- **a** Assuming that the pattern continues through the years, find (to the nearest dollar) the value of the Smiths' house in **i** 1985, **ii** 1995.
- **b** By what factor will the value of the house have increased by the year 2010, compared to the original value?

2 An accountant has been working with the same company for 15 years. She commenced on a salary of \$28 000 dollars and has received a \$2500 increase each year.
- **a** What type of sequence of numbers does her annual income follow?
- **b** How much did she earn in her 15th year of employment?
- **c** How much has she earned from the company altogether?
- **d** What was her percentage increase at the end of **i** her first and **ii** her fourteenth year of employment?

3 A chemist has been working with the same company for 15 years. He commenced on a salary of \$28 000 dollars and has received a 4% increase each year.
- **a** What type of sequence of numbers does his annual income follow?
- **b** How much did he earn in his 15th year of employment?
- **c** How much has he earned from the company altogether?
- **d** What was his increase in salary at the end of **i** his first and **ii** his fourteenth year of employment?

4 A biologist is growing a tissue culture in a Petri dish. The initial mass of the culture was 20 milligrams. By the end of the first day the culture was a mass of 28 milligrams.
- **a** Assuming that the daily growth is *arithmetic*, find the mass of the culture after the second, third, tenth and nth day.
- **b** On what day will the culture mass first exceed 200 milligrams?
- **c** Assuming that the daily growth is *geometric*, find the mass of the culture after the second, third, tenth and nth day.
- **d** On what day will the culture mass first exceed 200 milligrams?

5 Logs of wood can be stacked so that there is one more log on each descending layer than on the previous layer. The top row has 6 logs and there are 20 rows.
- **a** How many logs are in the stack altogether?

b The logs are to be separated into two equal piles. They are separated by removing logs from the top of the pile. How many rows down will workers take away before they remove half the stack?

6 *As I was going to St Ives*
I met a man with seven wives.
Every wife had seven sacks,
Every sack had seven cats,
Every cat had seven kits.
Kits, cats, sacks and wives,
How many were coming from St Ives?

Note: This is a variation on the original riddle, which asks 'How many were *going to* St Ives'.

7 Thoughtful Frank has 100 movie tickets to give away to people at a local shopping centre. He gives the first person one ticket, the next person two tickets, the third person three tickets and so on until he can no longer give the nth person n tickets. How many tickets did the last lucky person receive? How many tickets did Frank have left?

8 Kind-hearted Kate has 200 movie tickets to give away to people at the shopping centre. She gives the first person one ticket, the next person two tickets, the third person four tickets and so on following a geometric progression until she can no longer give the nth person $2^{(n-1)}$ tickets. How many tickets did the last lucky person receive? How many tickets did Kate have left?

9 The King of Persia, so the story goes, offered Xanadu any reward to secure the safety of his kingdom. As his reward, Xanadu requested a chessboard with one grain of rice on the first square, two grains on the second, four on the third and so on until the 64th square had its share of rice deposited.

a Find the total number of grains of rice that the king needed to supply.

b If each grain of rice weighs 0.10 grams, how many kilograms of rice does this represent? (*Note:* There are 10^3 grams in 1 kilogram.)

10 As legend has it, the King of Constantinople offered Xanadu's cousin Yittrius any reward to secure the safety of his city. This Yittrius accepted: she requested a chessboard with one grain of rice on the first square, three grains of rice on the second square, five grains of rice on the third square and so on until the 64th square had its share of rice deposited.

a Find the total number of grains of rice that the king needed to supply.

b If each grain of rice weighs 0.10 gram, how many kilograms of rice does this represent? (*Note:* There are 10^3 grams in 1 kilogram.)

11 A student is 3.0 m from the door to a classroom and decides that he will take a 1.0 m step followed by a step of half that distance, and half again and so on until he gets to the classroom door. Show that he will never get any closer than one metre from the door.

12 A hiker walks 36 km on the first day and $\frac{2}{3}$ that distance on the second. Every day thereafter she walks $\frac{2}{3}$ of the distance she walked on the day before. Will the hiker cover the distance of 100 km to complete the walk and on what day will she complete the task?

13 WE14 Recurring decimals can be expressed as rational numbers. Find the fraction equivalent of the following recurring decimal numbers by writing the decimal number as a sum of infinite terms.

a $0.1111\ldots = 0.1 + 0.01 + 0.001 + \ldots$

b $0.333\,333\,333\ldots$

c $0.5757\ldots$

d $2.343\,434\ldots$

e $3.142\,142\,142\ldots$

f $21.2121\ldots$

g $16.666\ldots$

14 In 1990, 100 students enrolled for a hypocorisma subject at a local university. Each subsequent year for the next decade the enrolment increased by 20%.

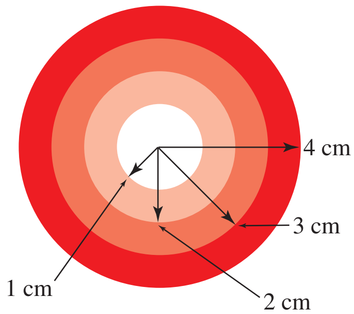
a Find the number of students enrolled in hypocorisma in 1995.

b Over the course of the decade find the total number of students who had enrolled in hypocorisma.

15 For tax purposes, the value of a computer used for a business depreciates by 8.5% of the initial cost each year. For economic reasons the business sells its computers when they first depreciate to less than half their initial value. After how many years will a computer used by this business be sold?

16 The side lengths of a right-angled triangle form the successive terms of an arithmetic sequence. The perimeter of the triangle is 72 m. What are the side lengths of the triangle?

17 A circular board is divided into a series of concentric circles of radius 1 cm, 2 cm, 3 cm and 4 cm as shown at right.



a Find the areas of each of the successive shaded regions and show that they form an arithmetic progression.

b A dart is fired at the board at random and hits the board. What is the probability of striking each of the four regions of the board?
(*Note:* The probability of striking a region = area of region ÷ total area.)

18 A bullet is fired vertically up into the air. In the first second it has an average speed of 180 m/s; that is, it travels 180 m up into the air during the first second. Each second its speed diminishes by 12 m/s. Thus during the 2nd second the bullet has an average speed only 168 m/s and accordingly travels 168 m further up into the air.

a Find an equation for the average speed of the bullet for the *n*th second that it is in the air.

b Find the time when the average speed of the bullet is equal to zero.

c Find the maximum height of the bullet above where it was fired.

19 Coffee cools according to Newton's Law of Cooling in which the temperature of the coffee *above* room temperature drops by a constant fraction each unit of time. The table below shows the temperature of a cup of coffee in a room at 20 °C each minute after it was made.

Remember to subtract the room temperature from the temperature of the coffee before you do your calculations.

Time (min)	Temperature (°C)
1	80.0
2	74.0
3	68.6

The person who made the coffee will drink it only if it has a temperature in excess of 50 °C. What is the minimum time after the cup of coffee has been made before it becomes undrinkable?

20 Two arithmetic sequences, t_n and u_n, are multiplied together. That is, each term is multiplied by the other to form a new term.

$$t_n = 2n - 3, n \in \{1, 2, 3, \ldots\} \text{ and}$$
$$u_n = 3n, n \in \{1, 2, 3, \ldots\}$$

Show that the new sequence of numbers $t_1 \times u_1, t_2 \times u_2, t_3 \times u_3, \ldots$ is an arithmetic series and hence find the arithmetic sequence for that new series.
(*Hint:* For a sequence, a_n, with a series A_n, $a_n = A_n - A_{n-1}$)

SUMMARY

Describing sequences

- A sequence is a string of numbers or expressions. It may contain a finite or infinite number of terms and may or may not follow a recognisable pattern.
- A sequence can be described in a number of ways.
  1. As a list t_n: $\{1, 7, 2, 6, \ldots\}$ (note that $t_3 = 2$)
  2. As a function: $t_n = 2n - n^2$, $n \in \{1, 2, 3, \ldots\}$ (note that $t_5 = 2 \times 5 - 5^2 = {}^-15$)
  3. As a recursive or iterative formula: $t_{n+1} = 2t_n - 3$, $t_1 = 6$ (note that $t_2 = 2 \times 6 - 3 = 9$)
- The *logistic equation* is a model of population growth of the general form:
$$t_{n+1} = at_n(1 - t_n),$$
where $0 < t_0 < 1$ and a is a constant.
- A *convergent* sequence is a sequence whose terms settle at a certain fixed number, x, called the *limit* of the sequence. This can be written as $t_n \to x$. A sequence whose terms grow further and further apart is called *divergent*. That is, a sequence is divergent if $t_n \to \infty$, or $t_n \to {}^-\infty$ as $n \to \infty$. A sequence whose terms fluctuate between two (or more) values is called *oscillating*.

Arithmetic sequences

- An arithmetic sequence is one whose successive terms have a common difference. This common difference is given the symbol d. Thus $t_{n+1} - t_n = d$ for all values of n. The first term in the sequence is given the symbol a.
- If x, y, z are successive terms in an arithmetic sequence then the middle term (y) is called an *arithmetic* mean and is equal to the average of the two outer terms (x and z):
$$y = \frac{x+z}{2}$$
- An arithmetic sequence can be written as a, $a + d$, $a + 2d$, ... and so the nth term, t_n, is:
$t_n = a + (n - 1)d$ using the function notation, or
$t_{n+1} = t_n + d$, $t_1 = a$ using the iterative notation.

Arithmetic series

- The sum of the first n terms of the arithmetic sequence is given by
$$S_n = \frac{n}{2}(2a + (n-1)d)$$
- In general, $t_{n+1} = S_{n+1} - S_n$.

Geometric sequences

- A geometric sequence is one in which each successive term is obtained by multiplying the preceding term by a constant number. This number is called the *common ratio* and is given the symbol r. Thus $\frac{t_{n+1}}{t_n} = r$ for all values of n. The first term in the sequence is given the symbol a.
- If x, y, z are successive terms in an arithmetic sequence then y is called a geometric mean and is given by $y^2 = xz$.
- A geometric sequence can be written as a, ar, ar^2, ... and so the nth term, t_n, is
$t_n = ar^{n-1}$ using the function notation, or
$t_{n+1} = rt_n$, $t_1 = a$ using the iterative notation.

Geometric series

- The sum of the first n terms in a geometric sequence is given by

$$S_n = \frac{a(1-r^n)}{1-r} \text{ with } r \neq 1$$

$$\text{or} \quad S_n = \frac{a(r^n-1)}{r-1} \text{ with } r \neq 1$$

- When the magnitude of r is less than one, that is, $^-1 < r < 1$, the sum of a geometric sequence to infinity S_∞ is given by $S_\infty = \frac{a}{1-r}$.

Applications of sequences and series

- To solve problems, use the following guidelines.
  1. Identify the type of sequence by checking whether there is a common difference, or a common ratio.
  2. Translate given information into mathematical statements, using appropriate notation.
  3. Define what you have to find and write appropriate formula(s).
  4. Use algebra to find what is required.

CHAPTER REVIEW

SHORT ANSWER

1 Write the iterative definition for each of the following sequences:

a $\{7, 11, 19, 35, 67, \ldots\}$

b $\{^-2, 5, 26, 677, \ldots\}$

2 For the arithmetic sequence where $t_3 = 10$ and $t_6 = 478$, find

a the functional rule for the nth term in the sequence

b the iterative rule for the sequence.

3 A car at a racetrack starts from rest and travels 0.5 m in the 1st second and 1.0 m in the 2nd second following an arithmetic progression in the distances covered each subsequent second.

a How far will it travel during the 10th second?

b After 10 seconds of motion, how far will it have travelled in total?

c To the nearest whole second, how long will it take to travel 1000 m (1 km)?

4 At Bugas Heights a radiation leak in a waste disposal tank potentially exposes staff to a 1000 milli-rem h dose on the first day of the accident, a 800 milli-rem h dose on the second day after the accident and a 640 milli-rem h dose on the third day following the accident.

a Assuming a geometric sequence, find the amount of potential exposure dose by the 10th day.

b Find the total potential exposure dose in the first 5 days.

5 The infinite sum of a geometric sequence is 99 and the first term is 10. Find the common ratio for the sequence.

6 Find the sum of the following expressions:

a $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} \ldots$ b $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} \ldots$

7 Find the fraction equivalent of the following recurring decimals:

a 0.222 222

b 2.454 545 454

MULTIPLE CHOICE

1 Consider the sequence $t_{n+1} = 2t_n + 4$; $t_3 = 12$. The second term in the sequence is:

A 10 B 6 C 28
D 4 E 8

2 A series is listed as 3, 10, 21, 36, . . . The next term in the series is:

A 51 B 52 C 53
D 54 E 55

3 The 23rd term in the sequence of numbers $\{7, 3, ^-1, \ldots\}$ is:

A $^-88$ B $^-81$ C $^-74$
D $^-83$ E 90

4 Consider the arithmetic sequence 52, a, 41, b. The numerical value of the expression $a - 3b$ is:

A $^-60$ B $^-64\frac{1}{2}$ C $^-67\frac{1}{2}$
D $^-71$ E $72\frac{1}{2}$

5 Consider the arithmetic sequence $x - 2y$, $3x - 4y$, $4x - 7y$, . . . An expression for y in terms of x is:

A $y = x$ B $y = ^-x$
C $y = ^-2x$ D $y = 2x$
E $y = ^-3x$

6 A car is accelerating such that in the 1st second it travels 2.0 metres, in the 2nd second it travels 3.5 metres, in the 3rd second it travels 5.0 metres, and so on for a total of 15 seconds. The total distance travelled by the car is:

A 630 m B 93.75 m
C 187.5 m D 375 m
E 315 m

7 The sum of the first four terms in an arithmetic sequence is 70. The sum of the first six terms is 63. The sixth term of the sequence is equal to:

A $^-14$ B $^-7$
C 0 D 7
E 14

8 For a geometric sequence, the 4th term is 5 and the 7th term is $^-625$. The second term in the sequence is:

A $^-2.5$ B $^-1.25$
C 0.25 D $^-0.25$
E 0.20

9 The sum of an infinite geometric sequence is 5.6 with the common ratio equal to 0.20. The sum of the first four terms of the geometric sequence is closest to:

A 5.0 B 5.2
C 5.4 D 5.6
E 5.8

10 The sum of the first 10 terms of a geometric sequence is 400. The next term in the sequence is 3 times the previous term. The first term in the sequence is:

A $\frac{17}{731}$

B $\frac{400}{1473}$

C $\frac{100}{7381}$

D $\frac{200}{781}$

E $\frac{10}{387}$

11 The sequence $t_n = 81 \times \left(\frac{1}{3}\right)^{n-1}$ and the series $S_n = 1 - 0.1n$ are combined to form the ratio $\frac{t_n}{s_n}$. When $n = 9$ the value of the ratio is:

A $\frac{1}{27}$

B $\frac{1}{270}$

C $\frac{1}{243}$

D $\frac{1}{81}$

E $\frac{10}{81}$

EXTENDED RESPONSE

1 Consider a square of side length 2 units.

- **a** What is the perimeter of the square?
- **b** Each of the four midpoints form the vertices of a new square inscribed within the original square. Find the perimeter of this new square.
- **c** Repeat the process to find the perimeter of a third square inscribed within the second.
- **d** Give an expression for the perimeter of the *n*th square.

2 Consider the following iterative definitions:

- **a** $t_{n+1} = t_n - \frac{3}{4}$, $t_1 = \frac{1}{8}$
- **b** $t_{n+1} = at_n$, $t_1 = b^2$
- **c** $t_{n+1} = 3t_n^2 - 1.5$, $t_1 = 0.5$

If each of these definitions is used to generate a sequence of numbers:

- **i** decide whether the sequence is arithmetic, geometric or neither, and
- **ii** find its fourth term.

3 In January 2004, Rachel and Nathan inherited a small trout farm from their Uncle Michael. They were told that in any given year the trout population, p_{n+1}, could be easily calculated using the formula $p_{n+1} = 0.5p_n(1 - p_n)$, where p_n is the number of trout (in thousands) in the preceding year. They were also told that on the day of their inheritance the farm housed 800 fish.

- **a** Use the above formula to predict (to the nearest whole number) the size of the fish population on Rachel and Nathan's farm for the next three years; that is, for
  - **i** January 2005
  - **ii** January 2006
  - **iii** January 2007.
- **b** What will happen to the size of the fish population if it continues to change according to the above formula? How long will it take?

After extensive research, Rachel and Nathan decided to modernise their newly acquired farm. A new feeding system and other improvements were installed, and were completed within the first two years (that is, by January 2006). As a result, in any given year the trout population, p_{n+1}, could now be calculated using the formula $p_{n+1} = 1.6p_n(1 - p_n)$, where p_n is the number of trout (in thousands) in the preceding year.

- **c** Use this new formula, and the figure obtained in part **a ii** (that is, the trout population in January 2006) as your starting point to predict the size of the trout population for January 2007 and January 2008.
- **d** Do Rachel and Nathan still run the risk of losing all of their trout stock? Explain your answer.
- **e** Will the size of the trout population ever reach and increase beyond the initial number of 800 fish? Give reasons for your answer.

4 On an island in the Pacific Ocean the population of a species of insect (species A) is increasing geometrically with a population of 10 000 in 1990 and an annual growth rate of 12.0%. Another species of insect (species B) is also increasing its population, but arithmetically with numbers 15 000 in 1990 and an annual increment of 1000 per annum.

a Using a spreadsheet or other method, determine the difference in the numbers of the two species during the last decade of the twentieth century (that is, up to 1999).

b In what year will the first species be greater in number than the second species, assuming that growth rates remain fixed?

A scientist has a mathematical model where the species can cohabit provided that they have equal numbers in the year 2000.

c If the growth rate in species A is to remain unchanged, what would the annual increment in species B need to be to achieve this?

d If the annual increment in species B is to remain unchanged, what would the growth rate in species A need to be to achieve this?

5 **a** A series is given by the equation

$$S_n = 2n^2 + 3n.$$

Show that the sequence is arithmetic and give the expression for the nth term in the sequence, t_n.

b A series is given by the equation $S_n = an^2 + bn$. Show that the sequence is arithmetic and give the expression for the nth term in the sequence, t_n, in terms of a and b.

6 **a** Australian Heating is a company that produces heating systems. The number of heating systems produced annually is modelled by an increasing geometric sequence. The number of heating systems produced in each of the first 3 years is shown in Table 1.

Table 1: Annual production of heating systems.

Year	1	2	3
Number of heating systems produced	2000	2200	2420

i Show that the common ratio r of this sequence is 1.1.

ii What is the annual percentage increase in the number of heating systems produced each year?

iii How many heating systems will be produced in year 5?

iv The number of heating systems produced annually continues to follow this pattern. In total, how many heating systems will they produce in the first 10 years of operation?

v The geometric sequence in Table 1 can also be produced using an iterative definition of the form $P_{n+1} = bP_n + c$, where $P_1 = 2000$ and P_n is the number of heating systems produced in the nth year. Determine the values of b and c.

b The purchase and installation of a basic heating system with five outlets costs \$3500. Each additional outlet costs an extra \$80.

i Determine the cost of installing a heating system with eight outlets.

ii A customer has \$4400 to spend on a heating system and outlets. Determine the greatest number of outlets that can be bought with this heating system.

iii Australian Heating recommends that a house with 20 squares of living area should have 12 heating outlets. Using this recommended ratio, determine the cost of installing a heating system for a house having 35 squares of living area.

c The number, S_n, of heating systems sold in the nth year is defined by the iterative definition:

$$S_n = 1.2S_{n-1} - 200, \text{ where } n \leq 5 \text{ and } S_3 = 2224$$

i Use this definition to determine how many heating systems were sold in the first year.

ii What percentage (correct to 1 decimal place) of heating systems produced during the first three years was sold within the three years?

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Chapter 6

EXAM TIP Remember an iterative formula of the form: $T_n = aT_{n-1} + b$ will have a value of $b = 0$ if the sequence is geometric and $a = 1$ if the sequence is arithmetic.
[Assessment report 2004]

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6A Describing sequences

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- WE2 int-1051: Watch how to determine the behaviour of the sequence using a CAS calculator. *(page 183)*

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- History of mathematics: Learn about the mathematician Leonardo Fibonacci. *(page 186)*

6B Arithmetic sequences

Tutorial
- WE5 int-1052: Watch how to find the missing terms in an arithmetic sequence. *(page 189)*

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- SkillSHEET 6.1: Practise using elimination to solve simultaneous equations. *(page 191)*

6C Arithmetic series

Tutorial
- WE7 int-1053: Watch how to find the sum of the first 20 terms in an arithmetic sequence. *(page 193)*

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- WorkSHEET 6.1: Complete and determine arithmetic series. *(page 195)*

6D Geometric sequences

Tutorial
- WE10 int-1054: Watch how to find a specific term in a geometric sequence. *(page 197)*

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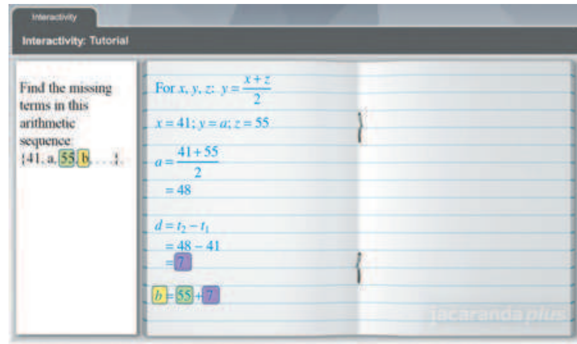
- Spreadsheet 036: Investigate Fibonacci sequences. *(page 199)*

6E Geometric series

Tutorial
- WE12 int-1055: Watch how to find the sum to infinity of a geometric sequence and a term given the sum to infinity and the first term. *(page 204)*



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- Spreadsheet 036: Investigate Fibonacci series. *(page 205)*
- WorkSHEET 6.2: Solve more complex problems with arithmetic series, complete and determine geometric series and apply geometric series theory to a worded problem. *(page 206)*

6F Applications of sequences and series

Interactivity
- Applications of sequences and series int-0973: Consolidate your understanding of sequences and series. *(page 206)*

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Chapter review

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