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Vijay Kumar Bhat

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MINIMAL PRIME IDEALS OF SKEW POLYNOMIAL RINGS AND NEAR PSEUDO-VALUATION RINGS

Vijay Kumar Bhat, Katra

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Abstract. Let R be a ring. We recall that R is called a near pseudo-valuation ring if every minimal prime ideal of R is strongly prime.

Let now σ be an automorphism of R and δ a σ-derivation of R. Then R is said to be an almost δ-divided ring if every minimal prime ideal of R is δ-divided.

Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers). Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further, if for any strongly prime ideal U of R with $\sigma(U) = U$ and $\delta(U) \subseteq \delta$, $U[x;\sigma,\delta]$ is a strongly prime ideal of $R[x;\sigma,\delta]$, then we prove the following:

(1) R is a near pseudo valuation ring if and only if the Ore extension $R[x;\sigma,\delta]$ is a near pseudo valuation ring.

(2) R is an almost δ-divided ring if and only if $R[x;\sigma,\delta]$ is an almost δ-divided ring.

Keywords: Ore extension; automorphism; derivation; minimal prime; pseudo-valuation ring; near pseudo-valuation ring

MSC 2010: 16N40, 16P40, 16S36

Introduction

In this paper we generalize Theorems 4.3 and 4.4 of [13], and thus answer (partially) the following question:

Question A (Question 1 of [13]). Let R be a near pseudo-valuation ring (NPVR), σ an automorphism of R and δ a σ-derivation of R. Is the Ore extension $O(R) = R[x;\sigma,\delta]$ a near pseudo-valuation ring (NPVR) (even if R is commutative Noetherian)?

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All the notation is the same as in Bhat and Kumari [13], but to make the paper self contained, we give the following introduction.

All rings are associative with identity. Throughout the paper R denotes a ring with identity $1 \neq 0$. The set of all nilpotent elements of R and the prime radical of R are denoted by $N(R)$ and $P(R)$ respectively. The set of prime ideals of R is denoted by $\operatorname{Spec}(R)$ and the set of minimal prime ideals of R is denoted by $\operatorname{Min}\operatorname{Spec}(R)$. The center of R is denoted by $Z(R)$. The field of rational numbers and the ring of integers are denoted by $\mathbb{Q}$ and $\mathbb{Z}$ respectively unless otherwise stated. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

Skew polynomial rings: This article concerns the study of skew polynomial rings over pseudo valuation rings. Therefore, we discuss these notions one by one.

Let R be a ring, σ an automorphism of R and δ a σ-derivation of R ($\delta\colon R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$ for all $a, b \in R$).

For example, let σ be an automorphism of a ring R and $\delta\colon R \to R$ any map.

Let $\varphi\colon R \to M_2(R)$ be defined by

$$\varphi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix} \quad \text{for all } r \in R.$$

Then δ is a σ-derivation of R if and only if φ is a homomorphism.

We denote the Ore extension $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable, i.e., $\sigma(I) = I$ and I is δ-invariant, i.e., $\delta(I) \subseteq I$, then we denote $I[x;\sigma,\delta]$ by $O(I)$. We would like to mention that $R[x;\sigma,\delta]$ is the usual set of polynomials with coefficients in R, i.e., $\left\{\sum\limits_{i=0}^{n} x^i a_i,\ a_i \in R\right\}$ with the usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take coefficients of polynomials on the left as in McConnell and Robson [19].

In case δ is the zero map, we denote the skew polynomial ring $R[x;\sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) = I$, we denote $I[x;\sigma]$ by $S(I)$.

In case σ is the identity map, we denote the differential operator ring $R[x;\delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x;\delta]$ by $D(J)$.

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example, see [12], [11], [14], [10], [15], [18], [19].

Pseudo-valuation rings (PVRs):

We recall that as in Hedstrom and Houston [16], an integral domain R with quotient field F is called a pseudo-valuation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$). Later on, Badawi and Houston in [8] showed that the definition of a strongly prime ideal is equivalent to a prime ideal being powerful.

For example, let $F = \mathbb{Q}(\sqrt{2})$. Set $V = F + xF[[x]] = F[[x]]$. Then V is a pseudo-valuation domain. We also note that $S = \mathbb{Q} + \mathbb{Q}x + x^2V$ is not a pseudo-valuation domain (Badawi [6]). For more details on pseudo-valuation rings and almost-pseudo valuation rings, the reader is referred to Badawi [6].

In Badawi, Anderson and Dobbs [7], the study of pseudo-valuation domains was generalized to arbitrary rings in the following way:

A prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion, i.e., $aP \subseteq bR$ or $bR \subseteq aP$) for all $a, b \in R$. A ring R is said to be a pseudo-valuation ring (PVR) if each prime ideal P of R is strongly prime. We note that a PVR is quasilocal by Lemma 1 (b) of Badawi, Anderson and Dobbs [7].

An integral domain is a PVR if and only if it is a PVD by Proposition 3.1 of Anderson [1], Proposition 4.2 of Anderson [2] and Proposition 3 of Badawi [4]. We denote the set of strongly prime ideals of R by $\mathrm{SSpec}(R)$.

In Badawi [5], another generalization of PVDs is given in the following way:

For a ring R with a total quotient ring Q such that $N(R)$ is a divided prime ideal of R, let $\varphi\colon Q \to R_{N(R)}$ be such that $\varphi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then φ is a ring homomorphism from Q into $R_{N(R)}$, and φ restricted to R is also a ring homomorphism from R into $R_{N(R)}$ given by $\varphi(r) = r/1$ for every $r \in R$. Denote $R_{N(R)}$ by T. A prime ideal P of $\varphi(R)$ is called a T-strongly prime ideal if $xy \in P$, $x \in T$, $y \in T$ implies that either $x \in P$ or $y \in P$. A ring $\varphi(R)$ is said to be a T-pseudo-valuation ring (T-PVR) if each prime ideal of $\varphi(R)$ is T-strongly prime. A prime ideal S of R is called a φ-strongly prime ideal if $\varphi(S)$ is a T-strongly prime ideal of $\varphi(R)$. If each prime ideal of R is φ-strongly prime, then R is called a φ-pseudo-valuation ring (φ-PVR).

Near pseudo-valuation rings (NPVRs):

Definition 0.1 (Definition 1.1 of Bhat [11])**.** A ring R is said to be a near pseudo-valuation ring (NPVR) if each minimal prime ideal P of R is strongly prime.

For example, a reduced ring is NPVR.

Here the term near may not be interpreted as near ring (Bell and Mason [9]). We note that a near pseudo-valuation ring (NPVR) is a pseudo-valuation ring (PVR), but the converse is not true. For example, a reduced ring is a NPVR, but need not be a PVR.

We recall that a prime ideal P of R is said to be divided if it is comparable (under inclusion) to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided (Badawi [3]). It is known (Lemma 1 of Badawi, Anderson and Dobbs [7]) that a pseudo-valuation ring is a divided ring.

Recall that in Bhat [11] an almost divided ring has been defined in the following way:

Let R be a ring, σ an automorphism of R and δ a σ derivation of R. An ideal I of R is called σ stable if $\sigma(I) = I$ and is called δ-invariant if $\delta(I) \subseteq I$.

Definition 0.2 (Definition 1.2 of Bhat [11]). Let R be a ring. Then R is said to be an almost divided ring if every minimal prime ideal of R is divided.

We also recall that a prime ideal P of R is σ-divided if it is comparable (under inclusion) to every σ-stable ideal I of R. A ring R is called a σ-divided ring if every prime ideal of R is σ-divided (see Bhat [12]).

Recall that an almost σ-divided ring and an almost δ-divided ring has been defined in Bhat [11] in the following way:

Definition 0.3 (Definition 1.3 of Bhat [11]). Let R be a ring. Then R is said to be an almost σ-divided ring if every minimal prime ideal of R is σ-divided.

Recall that a prime ideal P of R is δ-divided if it is comparable (under inclusion) to every σ-stable and δ-invariant ideal I of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided.

Definition 0.4 (Definition 1.4 of Bhat [11]). Let R be a ring. Then R is said to be an almost δ-divided ring if every minimal prime ideal of R is δ-divided.

It is clear that every divided ring is an almost divided ring.

$\sigma(*)$ **rings:** Recall that in Krempa [17], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies that $a = 0$ for $a \in R$.

We also recall that in [18], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring.

Example 0.5. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma\colon R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.

Main result. Let R be a Noetherian ring which is an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then

(1) $P \in \operatorname{Min}\operatorname{Spec}(O(R))$ implies that $P \cap R \in \operatorname{Min}\operatorname{Spec}(R)$, and conversely $P_1 \in \operatorname{Min}\operatorname{Spec}(R)$ implies that $O(P_1) \in \operatorname{Min}\operatorname{Spec}(O(R))$.

Further, if for any $U \in \operatorname{SSpec}(R)$ with $\sigma(U) = U$ and $\delta(U) \subseteq \delta$, $O(U) = U[x;\sigma,\delta] \in \operatorname{SSpec}(R)$, then

(2) R is a near pseudo-valuation ring if and only if $O(R) = R[x;\sigma,\delta]$ is a near pseudo-valuation ring;
(3) R is an almost δ-divided ring if and only if $O(R) = R[x;\sigma,\delta]$ is an almost δ-divided ring.

These results are proved in Theorems 1.3, 1.8 and 1.9 respectively.

1. Minimal prime ideals and near pseudo-valuation rings

Theorem 1.1. *Let R be a Noetherian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R.*

Proof. See Theorem 2.4 of [14]. □

Proposition 1.2. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then $P \in \operatorname{Min Spec}(R)$ implies $\delta(P) \subseteq P$.*

Proof. See Proposition 3.3 of [13]. □

Theorem 1.3. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then $P \in \operatorname{Min Spec}(O(R))$ implies that $P \cap R \in \operatorname{Min Spec}(R)$, and conversely $P_1 \in \operatorname{Min Spec}(R)$ implies that $O(P_1) \in \operatorname{Min Spec}(O(R))$.*

Proof. Let $P_1 \in \operatorname{Min Spec}(R)$. Then $\sigma(P_1) = P_1$ by Theorem 1.1 and $\delta(P_1) \subseteq P_1$ by Proposition 1.2. Now it can be seen that $O(P_1) \in \operatorname{Spec}(O(R))$. Suppose $O(P_1) \notin \operatorname{Min Spec}(O(R))$ and let $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$. Then $P_2 = O(P_2) \cap R \subset O(P_1) \in \operatorname{Min Spec}(O(R))$. Therefore $P_2 \cap R \subset P_1$, which is a contradiction, as $P_2 \cap R \in \operatorname{Spec}(R)$. Hence $O(P_1) \in \operatorname{Min Spec}(O(R))$.

Conversely suppose that $P \in \operatorname{Min Spec}(R)$, then it can be seen that $P \cap R \in \operatorname{Spec}(R)$, and $O(P \cap R) \in \operatorname{Spec}(O(R))$. Therefore, $O(P \cap R) = P$. We now show that $P \cap R \in \operatorname{Min Spec}(R)$. Suppose $P_1 \subset P \cap R$ is a minimal prime ideal of R. Then $O(P_1) \subset O(P \cap R)$ and as in the first paragraph $O(P_1) \in \operatorname{Spec}(O(R))$, which is a contradiction. Hence $P \cap R \in \operatorname{Min Spec}(R)$. □

Remark 1.4. Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then if $P \in \operatorname{Min Spec}(O(R))$, then $P \cap R \in \operatorname{Min Spec}(R)$ with $\sigma(P \cap R) = P \cap R$ and $\delta(P \cap R) \subseteq P \cap R$, and if $P_1 \in \operatorname{Min Spec}(R)$ such that $\sigma(P_1) = P_1$, and $\delta(P_1) \subseteq P_1$, then $O(P_1) \in \operatorname{Min Spec}(O(R))$.

P r o o f. The proof follows from Theorem 1.3 above. □

Theorem 1.5 (Hilbert Basis Theorem)**.** *Let R be a right/left Noetherian ring. Let σ and δ be as usual. Then the Ore extension $O(R) = R[x;\sigma,\delta]$ is right/left Noetherian.*

P r o o f. See Theorem 1.12 of Goodearl and Warfield [15]. □

The following example shows that the extension of a strongly prime ideal need not be a strongly prime ideal:

Example 1.6 (Example 3.1 of [10])**.** Let $R = \mathbb{Q}[t] = (t^2)$. Let $\sigma = \mathrm{id}$ and $\delta = 0$. For all $p(t) \in Q[t]$, we denote by $\overline{p(t)}$ the image of $p(t)$ under the natural projection $\mathbb{Q}[t] \to R$.

Now $P = \bar{t}R$ is a strongly prime ideal of R. Let $a = 1$ and $b = x$ and $J = PR[x] = \bar{t}R[x]$. Then neither $aJ \subseteq bR[x]$ nor $bR[x] \subseteq aJ$. Therefore, J is not a strongly prime ideal of $R[x]$.

Example 1.7 (Example 3.2 of [10])**.** $R = \mathbb{Z}_{(p)}$. This is in fact a discrete valuation domain, and therefore, its maximal ideal $P = pR$ is strongly prime. But $pR[x]$ is not strongly prime in $R[x]$ because it is not comparable with $xR[x]$ (so the condition of being strongly prime in $R[x]$ fails for $a = 1$ and $b = x$).

In view of Examples 1.6, 1.7 we are not able to answer Question A completely and moreover, in answering it partially we impose some conditions as given in the statements of Theorems 1.8, 1.9 below:

Theorem 1.8. *Let R be a Noetherian ring which is an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further, let $U \in \mathrm{SSpec}(R)$ with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ imply $O(U) \in \mathrm{SSpec}(R)$. Then R is a near pseudo-valuation if and only if $O(R)$ is a near pseudo-valuation ring.*

P r o o f. Let R be a near pseudo-valuation ring which is also an algebra over $\mathbb{Q}$. Now $O(R)$ is Noetherian by Theorem 1.5. Let $J \in \mathrm{Min\,Spec}(O(R))$. Then by Theorem 1.3, $J \cap R \in \mathrm{Min\,Spec}(R)$. Since R is a $\sigma(*)$-ring, $\sigma(J \cap R) = J \cap R$ and $\delta(J \cap R) \subseteq J \cap R$ by virtue of Theorem 1.1 and Proposition 1.2. Now R is a Noetherian near pseudo-valuation $\mathbb{Q}$-algebra, therefore $J \cap R \in \mathrm{SSpec}(R)$. Now by hypothesis $O(J \cap R) \in \mathrm{SSpec}(O(R))$. Now it is easy to see that $O(J \cap R) = J$. Therefore $J \in \mathrm{SSpec}(O(R))$. Hence $O(R)$ is a Noetherian near pseudo-valuation ring.

Conversely, let $O(R)$ be a near pseudo-valuation ring. Let $U \in \mathrm{Min\,Spec}(R)$ and $a, b \in R$. Then $O(U) \in \mathrm{Min\,Spec}(O(R))$, by virtue of Theorem 1.3. Since $O(R)$

is a near pseudo-valuation ring, so $O(U) \in \mathrm{SSpec}(O(R))$. Therefore $a(O(U))$ and $b(O(R))$ are comparable (say $a(O(U)) \subseteq b(O(R))$). So $a(O(U)) \cap R \subseteq b(O(R)) \cap R$, i.e., $aU \subseteq bR$. Hence R is a near pseudo-valuation ring. □

Theorem 1.9. *Let R be a Noetherian ring which is an algebra over $\mathbb{Q}$ and let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further, let $U \in \mathrm{SSpec}(R)$ with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ imply $O(U) \in \mathrm{SSpec}(R)$. Then R is an almost δ-divided ring if and only if $O(R)$ is a Noetherian almost δ-divided ring.*

Proof. Let R be an almost δ-divided ring which is also an algebra over $\mathbb{Q}$. Hence $O(R)$ is Noetherian by Theorem 1.5. Let $J \in \mathrm{Min\,Spec}(O(R))$. Since R is a $\sigma(*)$-ring, we have $\sigma(J \cap R) = J \cap R$ and $\delta(J \cap R) \subseteq J \cap R$ by Theorem 1.1 and Proposition 1.2. Let K be a proper ideal of $O(R)$ such that $\sigma(K) = K$ and $\delta(K) \subseteq K$. Now by Theorem 1.3, $J \cap R \in \mathrm{Min\,Spec}(R)$. Also $K \cap R$ is an ideal of R with $\sigma(K \cap R) = K \cap R$ and $\delta(K \cap R) \subseteq K \cap R$. Now R is almost δ-divided, therefore $J \cap R$ and $K \cap R$ are comparable under inclusion. Say $J \cap R \subseteq K \cap R$. Therefore, $O(J \cap R) \subseteq O(K \cap R)$. Thus $J \subseteq K$. Hence $O(R)$ is a Noetherian almost δ-divided ring.

Conversely, suppose that $O(R)$ is an almost δ-divided ring. Let $U \in \mathrm{Min\,Spec}(R)$. Since R is a $\sigma(*)$-ring, we have $\sigma(U) = U$ and $\delta(U) \subseteq U$, using Theorem 1.1 and Proposition 1.2. Let V be an ideal of R such that $\sigma(V) = V$ and $\delta(V) \subseteq V$. Theorem 1.3 implies that $O(U) \in \mathrm{Min\,Spec}(O(R))$. Now $O(R)$ is an almost δ-divided ring implies that $O(U)$ and $O(V)$ are comparable under inclusion, i.e., $O(U) \subseteq O(V)$ (say). This implies that $O(U) \cap R \subseteq O(V) \cap R$, i.e., $U \subseteq V$. Hence R is an almost δ-divided ring. □

Question 1.10. Let R be an NPVR. Let σ be an automorphism of R and δ a σ-derivation of R. Is $O(R) = R[x; \sigma, \delta]$ an NPVR (even if R is commutative Noetherian)?

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Author's address: V. K. Bhat, School of Mathematics, SMVD University, P/o Kakryal, Katra, J and K, India 182301, e-mail: `vijaykumarbhat2000@yahoo.com`.