BEAM BUCKLING ON RANDOM ELASTIC FOUNDATIONS

by

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### ABSTRACT

This thesis explores the impact of the seafloor on the buckling load of an undersea pipeline via beam on elastic foundation buckling theory. Undersea pipelines are used for the conveyance of hydrocarbons from wells located on the seafloor to facilities located on the ocean surface. As worldwide demand for hydrocarbon fuels increases and onshore reserves are depleted hydrocarbon production is forced offshore and increasingly into deep water.

Increases in pipeline temperature and pressure result in axial loads sufficient to cause buckling. Pipelines do not require trenching in deep water as fisherman's trawling equipment is unlikely to come into contact with them. This lack of confinement results in lateral displacements from the as-laid position multiple pipe diameters in length. Pipeline design must predict and accommodate these lateral movements to avoid ruptures, which makes lateral buckling a subject of both industry and academic interest.

A finite element model of a beam on an elastic foundation of randomly varying stiffness (BOREF) has been developed to explore the impact of the seafloor foundation on the critical buckling load of a pipeline. The Monte-Carlo method is used in conjunction with a random field generator to calculate a probability of failure due to buckling for a given axial load. A realistic example is presented in the thesis.

This research demonstrates the importance of describing the mechanical properties of natural materials stochastically when modeling. In geotechnical analysis deterministic descriptions of natural materials can result in non-conservative predictions even in instances where the material has been characterized extensively.

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## **CHAPTER 1 INTRODUCTION**

Deepwater oil and gas production has experienced phenomenal growth since the mid 1990's. Eighty percent of oil production and forty-five percent of gas production in the Gulf of Mexico comes from wells in water depths greater than 1000 meters (3281 ft). These deepwater platforms account for less than 1% of the platforms in the Gulf. To support these platforms there are more than 33,000 miles of pipeline on the seafloor (McCarron 2011). These pipelines are subject to buckling caused by thermal expansion during operation which must be accommodated for in design.

### 1.1 Undersea Pipeline Buckling

The study of undersea pipeline buckling began with upheaval buckling. In shallow waters where hydrocarbon exploration began it was necessary to trench pipelines to protect them from fishing equipment and boat anchors. Trenching has the additional benefit of providing insulation for the pipeline (Guijt 1990, White and Barefoot 2000, Craig and Nash 1990). During operation undersea pipelines can increase in temperature and pressure by as much as 100°C and 10N/mm<sup>2</sup> respectively above ambient conditions (Taylor 1993). Axial loading caused by thermal expansion and internal pressure of the pipeline accumulate due to the pipe being constrained by soil friction. The pipeline's self-weight, in addition to any overburden, offers less resistance to buckling than the walls of the trench. This results in the vertical movement of sections of the pipeline and is commonly referred to as upheaval buckling.

Conveyance pipelines do not require trenching as hydrocarbon production ventures into deeper water because they are no longer susceptible to unpredictable external loading (Merifield and White 2008). In untrenched pipelines the path of least resistance to buckling transitions from vertical, in trenched pipelines, to horizontal. This thesis focuses on calculating the probability an untrenched pipeline will buckle laterally under a given axial load.

#### 1.2 Motivation for Research and Problem Statement

Pipeline failure due to the lateral buckling of untrenched pipelines has occurred in operation. The lateral movements associated with buckling are sufficiently large to cause excessive yielding in the pipe material and in some instances rupture (Helvoirt and Sluyterman

1990, Schaminee and Zorn 1990). Three full-bore ruptures in the North Sea, West Africa and Brazil prompted an acknowledgement in 2002 that the industry's understanding of lateral buckling was not mature (Bruton et al. 2005). In the first two instances lateral buckling was not addressed correctly in design. In the last instance buckling was caused by the unexpected exposure and global instability of a pipeline buried in very soft clay. It was proposed that these shortcomings should be addressed by a joint industry project (JIP) called 'SAFEBUCK'.

The desire for a probabilistic approach to lateral buckling was identified in (Brown et al. 2006). The paper acknowledges that defining the statistical variations in key design variables is difficult but goes on to say that incorporating these uncertainties in the input allows the response to be assessed in a rational way. The field of probabilistic geotechnics is not young and the areas of its application continue to increase rapidly (e.g. Griffiths et al. 2011, Phoon and Kulhawy 1999, Ahmed and Soubra 2012). The ability to describe a natural material stochastically instead of deterministically in engineering analysis permits a rigorous quantitative assessment. It is this quantitative assessment that industry and consumers are increasingly demanding.

The goal of this research is to create a computational tool that can calculate the probability a given axial load will cause an untrenched pipeline to buckle laterally. The development of the computational tool and a demonstration of the tool are presented in this thesis.

### **CHAPTER 2 LITERATURE REVIEW**

An extensive literature review was conducted in order to gain an understanding of the current state of undersea pipeline modeling, the application of the Winkler Model for beams on elastic foundations and the generation of random fields used to assign finite element properties.

Undersea pipeline buckling continues to be a subject of interest in both industry and academia. This area of research has predominately been addressed by the JIP SAFEBUCK. SAFEBUCK involves most major operators, pipeline installers, pipe manufacturers, internationally recognized regulatory authorities and a number of research universities. Since the SAFEBUCK JIP's inception it has produced numerous publications on the topic of undersea pipeline buckling. This literature review contains the most relevant of these publications.

The Winkler foundation was chosen to represent the sea floor as a simplified elastic support. The use of a more complex elastic foundation model is not warranted due in part to the physical uncertainties inherent in soil mechanics. Hetényi asserts that in spite of the simplicity of the Winkler theory, it may often more accurately represent the conditions existing in soil foundations than more complex models in which the foundation is regarded as a continuous isotropic elastic body (Hetényi 1946).

Finally, two distinct methods used to generate the random field needed to define the stiffness properties of the elastic foundation were evaluated. The two methods, the Local Averaging Subdivision (LAS) method (Fenton and Vanmarcke 1990) and a method based on the Karhunen–Loève (K-L) theorem (Spanos and Ghanem 1989), were used to produce one-dimensional (1D) random fields (RFs). These methods were evaluated by comparing 5000 finite samples drawn from the infinite 1D RFs created by these two methods. Theoretically, if it were possible to determine the mean and standard deviation of an infinite RF, the RF's properties should match the user specified input perfectly. Because entire RFs cannot be sampled, this thesis relies on 5000 samples to estimate the properties of the infinite RFs generated by both methods. The analysis of the 5000 samples drawn from each field suggests that the LAS method is able to produce a RF that more closely matches the user specified input properties than that K-L method. The method based on the K-L theorem is also shown to

suffered from convergence issues in addition to ambiguity surrounding the number of terms to include in the expansion.

## 2.1 Lateral Buckling of Untrenched Undersea Pipeline

Untrenched pipelines have less resistance than trenched pipelines to buckling induced by thermal expansion and internal pressure. Lateral movements (Figure 2.1) of ten to twenty pipe diameters, absolute movements of two to ten meters, have been observed in operating pipelines (Bruton et al. 2008). These movements are sufficiently large to cause excessive yielding in the pipe material and in some instances rupture (Helvoirt and Sluyterman 1990, Schaminee and Zorn 1990, Bruton et al. 2005). Lateral buckles must therefore be taken into account and either prevented or accommodated for in design.



Figure 2.1 - Side-scan sonar image of a Lateral Buckle (Bruton et al. 2005)

## 2.1.1 Joint Industry Project (JIP) on Design of Safe Buckling

The occurrence of catastrophic full-bore ruptures caused by unexpected lateral buckles prompted the acknowledgement by industry that lateral buckling in pipelines was not well understood. The SAFEBUCK JIP was initiated in 2002 to address the uncertainties and deliver a demonstrably safe and effective lateral buckling design approach (Bruton et al. 2005).

The JIP began working towards the goal of creating safe and effective lateral buckling design guidelines by assembling a knowledge base comprised of participants' and contractors'

experiences. This allowed the JIP to identify the gaps in understanding the phenomenon and helped devise appropriate testing programs.

The most elegant and cost effective solution to the problem of lateral buckling in pipelines involves purposely initiating buckles (Figure 2.2) at controlled intervals (Cheuk et al. 2007). Initiating buckling at controlled intervals distributes the stress induced in the pipeline evenly rather than concentrating them in unpredictable locations.



Figure 2.2 - Sleeper Buckle initiator (Brown et al. 2006)

The selection of an optimal spacing between buckle initiators is not straightforward however. Spacing the buckle initiators too far apart negates their use. Spacing buckle initiators too close together increases the probability that the buckles will not form in the desired locations (Bruton et al. 2008, Brown et al. 2006, Wang et al. 2010). Buckle initiation, according to (Bruton et al. 2008), is governed by three parameters: (i) the effective compressive force in the pipeline, (ii) the out-of-straightness features and (iii) the lateral breakout resistance. Lateral breakout resistance refers to the resistance provided by the soil material that must be physically disrupted in order for the pipeline to buckle laterally. This parameter is thought to have the greatest impact on buckle formation and hold the largest degree of uncertainty.

### 2.1.2 Embedment and Lateral Breakout Resistance

Lateral breakout resistance is largely governed by the depth a pipeline is embedded in the ocean floor. Embedment depth depends on the soil properties of the ocean floor, the pipeline's operating weight and the additional loads imparted on the pipe during placement

(Cheuk et al. 2007, White and Randolph 2008). Quantifying the lateral breakout resistance provided by the soil foundations underlying pipelines is the subject of numerous research papers. Researchers have approached this problem using analytical models (White and Randolph 2008, Li and Batra 2007), computational models (Merifield and White 2008, Wang et al. 2010), small scale centrifuge testing (Bruton et al. 2008) and full scale laboratory testing (Bruton et al. 2005, Cheuk et al. 2007).

Each approach has its own set of unique challenges. In analytical models the failure mechanism must be assumed. In (White and Randolph 2008) a rotational mechanism for the lateral motion of a cylinder through soil, called the Martin mechanism, is used to calculate the lateral breakout resistance. Assumptions about the failure surfaces in the soil were made in order to develop the Martin mechanism, pictured below in (Figure 2.3).





Computational models, generally relying on finite element analysis (FEA), must first predict the pipe embedment depth before they can accurately predict lateral breakout resistance. Both pipeline embedment and the subsequent lateral movement calculations rely heavily on assumptions. These assumptions range from the behavior of the in-situ pipeline foundation material to the interactions between the pipeline and the foundation (Wang et al. 2010).

Computational models must also be greatly simplified to reduce run time. Researchers must choose between modeling a short section of pipeline with a dense mesh or a long section with a course mesh (Bruton et al. 2008). When modeling a short section of pipeline the boundary conditions at the ends of the pipeline segment must be assumed and hard coded by the user. The boundary conditions naturally develop when modeling long sections of pipeline at the cost of accuracy, runtime or both.

Small scale centrifuge testing becomes an attractive option as it reduces the overall number of assumptions. Foundation material behavior and foundation-pipeline interactions are traded for assumptions associated with scale. Full scale laboratory testing is limited by the length of pipeline able to be tested and the cost of the required equipment.

To calculate the lateral breakout resistance for an example pipeline this thesis relies on an empirical model, based on laboratory data and numerical results. The lateral breakout resistance is used to define the foundation stiffness in a finite element (FE) model of a beam on an elastic foundation (BOEF). The lateral breakout resistance/foundation stiffness is described statistically in order to model the beam on a random elastic foundation (BOREF). The BOREF model is used in conjunction with the Monte-Carlo method to calculate the critical buckling load for an example pipeline resting on thousands of randomly varying foundations. The buckling loads from each realization are used to create a probability density function of the critical buckling load. A probability that a given axial load will buckle the pipeline will then be determinable.

### 2.2 Beam Buckling on Elastic Foundation

Elastically supported beams appear in a large variety of technical problems. Hetényi (1946) points out that these problems extend beyond those in which the beam and the foundation are easily identifiable to problems where the elastic foundation for the beam is supplied by the resilience of the adjoining portions of a continuous elastic structure. Winkler (1867) developed the mathematical description for a BOEF model, now known as the Winkler

model, where the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point. This model's assumptions are rigorously satisfied for networks of beams, which are characteristic in the construction of floor systems, and for thin shells of revolution like pressure vessels. When applied to soil foundations however the model serves as a reasonable approximation. Models more complicated than the Winkler model are difficult to justify due to the difficulty of capturing the physical properties of soil.

#### 2.2.1 Winkler Model

The derivation of the differential equation for the deflection curve of a Winkler BOEF begins with the assumption that the supporting medium obeys Hooke's law. The deflection of the beam caused by loading deflects the underlying foundation which supplies a reaction force proportional to the deflection. The reaction force provided by the foundation is calculated using the familiar expression p = ky, were k is the foundation stiffness, or the modulus of the foundation (Figure 2.4).



Figure 2.4 - Load deflected beam supported by elastic medium (Hetényi 1946) Shown in (Figure 2.5) is a section of width dx of a BOEF subjected to a uniformly distributed load q. In equilibrium, the summation of forces on the section results in:

$$Q - (Q + dQ) + ky \, dx - q \, dx = 0 \tag{2-1}$$

Where: Q is a shear force; k is the foundation stiffness; y is a vertical displacement; dx is the section width; q is a distributed load; M is a moment.

Equation (2-1) simplifies to equation (2-2).

$$\frac{dQ}{dx} = ky - q \tag{2-2}$$



Figure 2.5 - Free body diagram of BOEF section (Hetényi 1946)

The relationship Q = dM/dx allows:

$$\frac{dQ}{dx} = \frac{d^2M}{dx^2} = ky - q \tag{2-3}$$

Differentiating twice  $EI\left(\frac{d^2y}{dx^2}\right) = -M$ , the differential equation of a beam in bending, gives:

$$EI\frac{d^4y}{dx^4} = -\frac{d^2M}{dx^2}$$
(2-4)

Where: *E* is the elastic modulus; *I* is the area moment of inertia; together *EI* is the bending stiffness of the beam.

Substituting  $\frac{d^2M}{dx^2} = ky - q$  from (2-3) into equation (2-4) results in (2-5), the differential equation for the deflection curve of a beam supported on an elastic foundation.

$$EI\frac{d^4y}{dx^4} = -ky + q$$
 (2-5)

Hetényi's book, Beams on Elastic Foundation, focuses on the analysis of elastically supported beams described by this differential equation.

#### 2.2.2 Hetényi: Analytical Solution for Critical Buckling Load

The analytical solution to the critical buckling load for beams with hinged ends is found in chapter VII, Elastic Stability of Straight Bars, of Hetényi's book. Hetényi demonstrates that the critical buckling load,  $N_{cr}$ , must occur at a value greater than  $2\sqrt{kEI}$ , the critical buckling load for an infinite beam. For a beam of unit width the modulus of the foundation has units of  $k = \frac{F}{L^2}$  and beam stiffness  $EI = F \cdot L^2$ , where F is force and L is length.

Hetényi's critical buckling load equation is derived from equation (2-6), the deflection of a hinged-end beam resting on an elastic foundation with a concentrated moment at one end (Figure 2.6).

$$y = \frac{M_0}{EI} \frac{1}{2\alpha\beta} \frac{\cosh \alpha l \sinh \alpha x' \sin \beta x - \cos \beta l \sinh \alpha x \sin \beta x'}{\cosh^2 \alpha l - \cos^2 \beta l}$$
(2-6)



Figure 2.6 - Hinged-end BOEF with concentrated moment

Interchanging  $\alpha$  and  $\beta$  in equation (2-6) transforms the equation from tension to compression. Equation (2-6) becomes equation (2-7).

$$y = \frac{M_0}{El} \frac{1}{2\alpha\beta} \frac{\cosh\beta l \sinh\beta x' \sin\alpha x - \cos\alpha l \sinh\beta x \sin\alpha x'}{(\cosh\beta l + \cos\alpha l)(\cosh\beta l - \cos\alpha l)}$$
(2-7)

With no moment applied, finite deflection of the bar can only occur if the denominator in equation (2-7) equals zero. Two conditions satisfy this:  $\alpha\beta = 0$ , which results in a trivial solution, and  $\cosh\beta l \pm \cos\alpha l = 0$ . No real values of  $\alpha$  or  $\beta$  can satisfy the second condition, therefore  $\beta$ 

must be transformed into  $i\overline{\beta}$ , where  $\overline{\beta} = \sqrt{\frac{N}{4EI} - \sqrt{\frac{k}{4EI}}}$ . This results in  $\cosh\beta l \pm \cos\alpha l = 0$ 

becoming  $\cosh i\overline{\beta}l \pm \cos \alpha l = 0$ . This further simplifies to  $\cos \overline{\beta}l \pm \cos \alpha l = 0$  which is satisfied by

$$(\alpha - \overline{\beta})l = n\pi \tag{2-8}$$

Where: *n* is an integer

A substitution of  $\alpha$  and  $\overline{\beta}$  into equation (2-8) results in Hetényi's equation (2-9) for the critical buckling load of a beam on an elastic foundation (Figure 2.7).

$$N_{cr} = n^2 \left(\frac{\pi^2 EI}{l^2}\right) + \frac{1}{n^2} \left(\frac{kl^2}{\pi^2}\right)$$
(2-9)

Where:  $N_{cr}$  is the critical buckling load; *EI* is the beam's bending stiffness; *n* is an integer; *l* is the beam length; *k* is the foundation stiffness.



Figure 2.7 - Hinged-end BOEF in compression

The integer n in equation (2-9) corresponds to the number of waves in the beam's deflected form (Figure 2.8). Henceforth n will be referred to as the mode shape number. Odd and even values of n are associated with symmetric and antisymmetric buckles respectively.

Given values of *k*, *l* and *EI*, *n* must be selected to minimize the  $N_{cr}$  expression. In order to determine the appropriate value for *n*, Hetényi solves  $\frac{dN_{cr}}{dn} = 0$  for *n*, resulting in equation (2-10).

$$n = \frac{l}{\pi} \sqrt[4]{\frac{k}{EI}}$$
(2-10)



Figure 2.8 - Frist three beam buckling modes

Equation (2-9) requires n to be an integer. Hetényi therefore states it must be taken as the nearest integer to the value determined by equation (2-10). A FE program developed in the course of this thesis revealed a discrepancy between the critical buckling loads computed using Hetényi's equations (2-9) and (2-10) and the FE method in the transition zones between buckling modes. The program will be discussed in further detail in the methodologies section.

The discrepancy is illustrated in (Figure 2.9) where the transition from mode shape one, n=1, to mode shape two, n=2 occurs. Plots showing the discontinuity in the Hetényi solution can be produced using any consistent system of units.

The discontinuity in the critical buckling load observed when using Hetényi's analytical solution is resolved when the problem is analyzed using the finite element (FE) method. It can be seen (Figure 2.9) that the two solutions agree almost perfectly, with the exception of the region where the transition to a higher buckling mode occurs. This is observed in all mode transition regions. However, it is more pronounced in the lower buckling modes. Included in (Figure 2.9) is the number of iterations the FE solver required to converge as it solved for the critical buckling load. The number of iterations required at the transition of mode shapes offers some insight into the difficulty of pinpointing the transition. Hetényi's equation (2-10), derived to calculate n, does not determine the value of n required to minimizes equation (2-9) in transition regions.



Figure 2.9 - Hetényi analytical and FE solution comparison

Equation (2-10) produces a real decimal number which Hetényi states must be rounded to the nearest whole number. Manipulation of Hetényi's equations (2-9) and (2-10) permit the determination of the exact value of *n* at the transition point between modes.  $N_{cr_n} = N_{cr_{n+1}}$  at the transition between buckling modes which permits the equality seen in (2-11).

$$n^{2} \frac{\pi^{2} EI}{l^{2}} + \frac{1}{n^{2}} \frac{kl^{2}}{\pi^{2}} = (n+1)^{2} \frac{\pi^{2} EI}{l^{2}} + \frac{1}{(n+1)^{2}} \frac{kl^{2}}{\pi^{2}}$$
(2-11)

Equation (2-11) becomes equation (2-12) for n = 1.

$$\frac{\pi^2 EI}{l^2} + \frac{kl^2}{\pi^2} = \frac{4\pi^2 EI}{l^2} + \frac{kl^2}{4\pi^2}$$
(2-12)

Equation (2-12) can be rearranged into (2-13). The left-hand side of (2-13) is equal to the righthand side of Hetényi's equation (2-10).

$$\frac{l}{\pi} \sqrt[4]{\frac{k}{EI}} = \sqrt{2} \tag{2-13}$$

This procedure is followed for n = 1, 2, 3, 4... to produce the results in (Table 2.1).

Mode shape number	Value of <i>n</i> at transition		
1	$\sqrt{2}$	1.414	
2	$\sqrt{6}$	2.449	
3	$\sqrt{12}$	3.464	
4	$\sqrt{20}$	4.472	

Table 2.1 - Mode shape number with transition value of *n* 

A pattern for the exact value of *n* at the transition between buckling modes emerges as  $\sqrt{n(n + 1)}$ . This allows the determination of the buckling mode number without the rounding requirement. This result indicates that rounding to the nearest whole number will never predict the correct transition to the higher wave number. Shown in (Table 2.2) are two inequalities, equation (2-14) for the first mode shape and equation (2-15) for all subsequent mode shapes, that if satisfied will determine the correct buckling mode without using equation (2-10) in combination with the rounding requirement. It should be noted that the observed discrepancy is the result of the method Hetényi developed for determining *n*, not of the critical buckling load equation. Hetényi did note that *n* should be chosen to minimize  $N_{cr}$  and was certainly aware of the mode transition points, he simply included a method in his book for determining *n* that does not always accurately determine the value of *n* required to minimize  $N_{cr}$ .

Mode shape number	Determination of correct buckling mode	
1	$\frac{l}{\pi} \sqrt[4]{\frac{k}{EI}} < \sqrt{n(n+1)}$	(2-14)
2, 3, 4, 5	$\sqrt{n(n+1)} < \frac{l}{\pi} \sqrt[4]{\frac{k}{EI}} < \sqrt{(n+1)((n+1)+1)}$	(2-15)

Table 2.2 - Value of *n* at transition

#### 2.2.3 Finite Element Formulation

The FE method is a highly versatile numerical method used to find approximate solutions to partial differential equations. The FE method approximates the exact solution to a differential equation using a trial solution of the form  $\tilde{y} = \sum_{1}^{n} c_{i}\psi_{i}(x)$ , where  $c_{1}$ ,  $c_{2}$ ... $c_{i}$  are undetermined parameters and  $\psi_{1}(x)$ ,  $\psi_{2}(x)$ ...  $\psi_{i}(x)$  are trial functions. The "c" coefficients are

initially unknown and will be optimized via the weighted residual method in order to approximate  $\tilde{y}$  as accurately as possible while the  $\psi(x)$  functions are simple functions of x selected before the process beings.

The FE method solves for the buckling load of a BOEF by assembling global "stiffness" and "geometric" matrices which are comprised of individual element stiffness and geometric matrices. The contribution from each element is accounted for in the assembly process. The resulting global matrix equation in the case where there is no initial transverse load is:

$$[K_m]{U}-P[K_g]{U} = \{0\}$$
(2-14)

Where:  $[K_m]$  is the global stiffness matrix; {U} is a vector containing unknown nodal variables; P is the axial force;  $[K_a]$  is the global geometric matrix.

The global matrix equation can be rewritten as  $[K_m]{U} = P[K_g]{U}$  which can then be converted to an eigenvalue problem in standard from  $[K_g]^{-1}[K_m]{U} = P{U}$ . The eigenvalues P and the corresponding eigenvectors {U} can be solved for using an appropriate numerical method. Physically, P represents the buckling load and {U} contains the corresponding "mode shape" of the beam when it buckles.

To construct the global stiffness and geometric matrices the elemental stiffness and geometric matrices must first be calculated. In order to calculate these matrices the governing differential equation for beam buckling on an elastic foundation (2-15) is converted into the matrix equation (2-16).

$$EI\frac{d^4w}{dx^4} + P\frac{d^2w}{dx^2} + kw = q$$
(2-15)

Where: *EI* is the bending stiffness of the beam; w is a transverse displacement; x is the distance along the beam; P is the axial load; k is the foundation stiffness; and q is a uniformly distributed transverse load.

$$[k'_m]{u}-P[k_g]{u}={f}$$
 (2-16)

Where:  $[k'_m] = [[k_m]+[m_m]]$ ;  $[k_m]$  is the elemental stiffness matrix;  $[m_m]$  is the elemental mass matrix;  $\{u\}$  is a vector of unknowns containing the element nodal displacements; P is the axial force;  $[k_q]$  is the "geometric" matrix;  $\{f\}$  is a vector containing the element nodal forces.

Note that elemental matrices are specified using lower case letters. It can be seen in the equation for the elemental stiffness matrix (Figure 2.10) that each element can have a different k, *EI* and length, *L*. It is at this level where the FE method incorporates the random foundation. A random pipe bending stiffness could also be accommodated.

	12 <i>EI</i> 13	kL 6EI	$11kL^{2}$	12 <i>EI</i>	9 <i>kL</i>	6EI	$13kL^{2}$
	$L^{3}$ + 3	5 $L^2$	210	$L^3$	70	$L^2$	420
		4 <i>E1</i>	$kL^3$	6EI	$13kL^2$	2EI	$kL^3$
$\left[\mathbf{k}_{m}^{\prime} ight] =$		L	$+\frac{105}{105}$	$L^2$	420	$\overline{L}$	140
				12 <i>EI</i>	13 <i>kL</i>	6EI	$11kL^2$
				$L^3$	35	$L^2$	210
	:	symmetric				4EI	$kL^3$
	_					L	105

Figure 2.10 - Modified stiffness matrix (Griffiths 2013)

The elemental stiffness matrix [k'<sub>m</sub>] is determined using the previously derived BOEF differential equation (2-17) which is converted into the matrix equation (2-18).

$$EI\frac{d^4w}{dx^4} + kw = q \tag{2-17}$$

Where: *EI* is the bending stiffness of the beam; k is the foundation stiffness; w is a transverse displacement; x is the distance along the beam; and q is a uniformly distributed transverse load.

$$[k_m]{u}+[m_m]{u}={f}$$
(2-18)

Where:  $[k_m]$  is the elemental stiffness matrix;  $[m_m]$  is the element mass matrix;  $\{u\}$  is a vector of unknowns containing the element nodal displacements;  $\{f\}$  is a vector containing the element nodal forces.

Equation (2-18) can be rearranged into equation (2-19):

$$[[k_m]+[m_m]]{u}={f}$$
(2-19)

This allows the combination of  $[[k_m]+[m_m]]$  into  $[k_m]$ . For more detailed information on the finite element method the reader is directed to (Smith and Griffiths 2004).

#### 2.3 Random Field Generation Techniques

What distinguishes the work in this thesis from work previously done on BOEF is the incorporation of a foundation of random stiffness. This incorporation allows a statistical, versus deterministic, description of the foundation stiffness. Natural materials, like the ocean floor underlying the beam, are more aptly described statistically than with a single deterministic value.

The analysis is no longer analytically tractable when the foundation is converted from a single stiffness to a stiffness which varies randomly within a specified range. The FE method was demonstrated in the previous section to have no trouble coping with different foundational stiffness underlying each element. This model will be referred to as a beam on a random elastic foundation (BOREF).

The difference between a BOEF and a BOREF is depicted in (Figure 2.11). It can be seen that the supporting medium underlying each element in the BOREF has a different stiffness. A 1D RF is used to assign stiffness values to the foundation medium underlying each element.

For a RF to be useful it must be constrained by input parameters that force the RF to represent the phenomenon one is interested in. In the case of natural materials three such constraints arise as an artifact of the method, discrete sampling, used to determine the material properties. For example, if one were to collect soil samples along the planned route of a pipeline a collective mean and standard deviation of those samples could easily be calculated. In addition, a special correlations length, which describes the distance over which properties tend to be spatially correlated, can also be determined (Griffiths et al. 2011). (Figure 2.12) highlights the difference between a large correlation length on the top and a small correlation length on the bottom.



Figure 2.11 - BOEF versus BOREF



Figure 2.12 - Large and small spatial correlation (Griffiths et al. 2011)

A useful RF generator must be capable of generating a random field that closely matches the input parameters of: mean, standard deviation and spatial correlation length. Samples can be drawn from the generated random field and used in analysis. Two RF generators were reviewed for their ability to generate useful RF's, the Karhunen–Loève Expansion (K-L) method and the Local Average Subdivision (LAS) Method.

#### 2.3.1 The Karhunen–Loève Expansion

A random process can be represented as a series expansion involving a complete set of deterministic functions with corresponding random coefficients. The K-L series expansion relies specifically on the eigen-decomposition of a covariance function (Huang et al. 2001). Covariance functions are used in both the K-L and LAS methods to determine the correlation between two points in a random field. The functions dictate the degree of similarity between two points in a RF based on their absolute distance from each other.

By defining the similarity between two points the covariance function permits the incorporation of the spatial correlation length parameter previously mentioned. There are many covariance functions available to define the similarity between two points in the RF. A linear covariance function would be well suited for a process in which the correlation between two points was thought to decreases linearly.

In soil mechanics the correlation between properties is thought to decrease exponentially with distance (Griffiths and Fenton 2004). Exponential functions are also used due to their relative ease of implementation (Griffiths et al. 2011, Ahmed and Soubra 2012).

The covariance function underlying the K-L RF generator used in this thesis is  $C(i,j) = \sigma_s^2 e^{-c|i-j|}$ , where:  $\sigma_s$  denotes the standard deviation of the random process and *c* is a parameter that dictates how fast the correlation attenuates. This function is related to a first-order Markovian process and is used extensively in geophysics and earthquake engineering (Spanos and Ghanem 1989). The K-L series is approximated by equation (2-20) using a finite number of M terms.

$$\widehat{\varpi}(x,\theta) = \overline{\varpi}(x) + \sum_{i=1}^{M} \sqrt{\lambda_i} \xi_i(\theta) f_i(x)$$
(2-20)

Where:  $\widehat{\varpi}(x, \theta)$  is a random process with a finite variance and mean  $\overline{\varpi}(x)$ ;  $\lambda_i$  and  $f_i(x)$  are the eigenvalues and eigenfunctions of the covariance function;  $\xi_i$  is an uncorrelated random variable.

The K-L expansion can more easily be understood visually. Shown in (Figure 2.13) are the first four terms, M = 1 to 4 of a K-L expansion. Note that the term number, M, dictates the number of peaks and troughs in the eigenfunctions. The superposition of the first 4 terms is shown in (Figure 2.14). The resulting superposition is a 1-D random field produced using the first 4 terms of a K-L expansion.







Figure 2.14 - Superposition of first 4 terms

The expansion can be thought of as the summation of eigenfunctions, derived from the chosen covariance function, scaled by an eigenvalue and an uncorrelated random variable.

#### 2.3.2 Local Average Subdivision Method

The LAS method, like the K-L method, can use a variety of functions to define the relationship between two points separated by an absolute distance in a random field. The LAS method uses a correlation coefficient,  $\rho_{ij}$ , to define the covariance between two points  $X_i$  and  $X_j$ .  $Cov[X_bX_j] = \rho_{ij}\sigma_{Xi}\sigma_{Xj}$ , where:  $\rho_{ij}$  is the correlation coefficient between  $X_i$  and  $X_j$ , calculated using a correlation function. The LAS RF generator used in this thesis uses a correlation function similar to the covariance function used by the K-L method. Both functions are exponential and first-order Markovian. The LAS correlation function is  $\rho_{ij} = e^{\frac{-2\tau_{ij}}{\theta_E}}$  where  $\tau_{ij}$  is the absolute distance between two points *i* and *j* and  $\theta_E$  is the spatial correlation length (Griffiths and Fenton 2009).

The construction of LAS RF proceeds in a top-down recursive fashion. (Figure 2.15) illustrates how this is accomplished. In stage 0, a global average is generated for the process. In stage 1, the domain is subdivided into two regions whose local averages must in turn average to the global, or parent, value. Subsequent stages are obtained by subdividing each parent cell and generating values for the resulting two regions. This process ensures that the global average remains constant throughout the subdivision process (Fenton and Vanmarcke 1990).

Stage 0		$Z_1^0$						
Stage 1	$Z_1^1$			Z <sup>1</sup>				
Stage 2	Z	'2 1	Z	2	Z	23	Z	2 4
Stage 3	$Z_{1}^{3}$	$Z_{2}^{3}$	$Z_{3}^{3}$	$Z_4^3$	$Z_{5}^{3}$	$Z_{6}^{3}$	$Z_{7}^{3}$	$Z_{8}^{3}$
Stage 4								

Figure 2.15 - Construction of local average random process (Fenton and Vanmarcke 1990) Pseudocode describing how the LAS algorithm works was used by Fenton and Vanmarcke (1990) to succinctly and comprehensively explain the process. The pseudocode reads as follows:

- 1) Generate a normally distributed global average (labeled  $Z_1^0$  in Figure 12) with mean zero and variance obtained from local averaging theory.
- 2) Subdivide the field into two equal parts.
- 3) Generate two normally distributed vales,  $Z_1^1$  and  $Z_2^1$ , whose means and variances are selected so as to satisfy three criteria: (a) That they show the correct variance according to local averaging theory; (b) that they are properly correlated with one another; (c) that they average to the parent value,  $\frac{1}{2}(Z_1^1 + Z_2^1) = Z_1^0$ . This is, the distribution of  $Z_1^1$  and  $Z_2^1$  are conditioned on the value of  $Z_1^0$ .
- 4) Subdivide each sell in stage 1 into two equal parts.
- 5) Generate two normally distributed vales,  $Z_1^2$  and  $Z_2^2$ , whose means and variance are selected so as to satisfy four criteria: (a) That they show the correct variance according to local averaging theory; (b) that they are properly correlated with one another; (c) that they average to the parent value,  $\frac{1}{2}(Z_1^2 + Z_2^2) = Z_1^1$ ; and (d) that they are properly correlate with  $Z_3^2$  and  $Z_4^2$ . The third criterion implies conditioning of the distributions of  $Z_1^2$  and  $Z_2^2$  on the value of  $Z_1^1$ . The forth criterion will only be satisfied approximately by condition their distributions also on  $Z_2^1$ .

More detailed descriptions of the K-L and LAS methods can be found in the literature referenced in the two preceding sections.

#### 2.3.3 Random Field Comparison

A comparison of the random fields generated using the K-L and LAS methods is presented in this section. However, because the methods use slightly different functions to define the covariance between two points in their respective fields it is not possible to directly compare them. Despite the covariance and correlation functions playing an integral role in the construction of the RF in both methods the impact that these functions have on the method's ability to reproduce the user specified mean and standard deviation should be negligible. Ideally, a RF generator would maintain the input mean and standard deviation, independently of the covariance of correlation function selected. If a phenomenon was better described using a linearly decaying relationship between points in a RF, verses exponentially in soil mechanics, the selection of a linearly decaying covariance function should have no impact on the mean and standard deviation of the fields as a whole. For this reason, the comparison of the LAS and K-L method will focus on the methods' abilities to reproduce the user specified mean and standard deviation in the fields they produce. It is the author's opinion that the ability to match the two properties of input mean and standard deviation are therefore a more objective way of determining the quality of a RF generator.

The analysis of the K-L method began with an attempt to determine how many terms were required to reach steady state in the expansion. The literature suggested a broad range for the number of terms to include. Li (1993) suggests that usually only a few of the terms associated with the largest eigenvalues are important. Recall that the eigenfunctions and eigenvalues are derived from the selected covariance function. The eigenvalues are then used in conjunction with random variables to scale the eigenfunctions in the K-L expansion. Therefore, as the eigenvalues get smaller their impact decreases until they can be considered negligible.

Ahmed and Soubra (2011) suggest that the number of terms used depends on the desired accuracy of the problem being treated and must be sufficient to rigorously represent the target random field. They determined that 100 terms would be sufficient based on the eigenvalue coefficient vanishing for most problems. Ahmed and Soubra (2012) determined that the circumstance warranted the use of 500 terms. The effects of the number or terms in the K-L expansion are explored in great detail by Zhang and Lu (2004). The authors show that if the correlation length is relatively small the number of terms required for a reasonable approximation is large. The authors explored multiple combinations of correlation length and variance ( $\sigma^2$ ). They showed that for a correlation length and variance of 1 there was no difference between including 100, 200 or 500 terms.

The following (Figure 2.16 – Figure 2.19) show 1D K-L RF's creating using 100, 200, 500 and 1500 terms in the expansion respectively. A change in appearance of the RF's as the number of terms increases is detectable in the figures. RF input properties: mean=0, standard deviation = 1, correlation length=1, domain length=10, sampling points over domain=1000, random scaling variable drawn from normally distributed pseudorandom number function.



Figure 2.16 - K-L RF sample with 100 terms in expansion



Figure 2.17 - K-L RF sample with 200 terms in expansion



Figure 2.18 - K-L RF sample with 500 terms in expansion



Figure 2.19 - K-L RF sample with 1500 terms in expansion

Shown in (2.20) is a 1D LAS RF's. The LAS process does not rely on an infinite series thereby eliminating the need to address steady state concerns. RF input properties: mean=0, standard deviation=1, correlation length=1, domain length=10, sampling points over domain=1000, normally distributed process.



Figure 2.20 - 1D LAS RF sample

The analysis of the 5000 samples drawn from each methods' RF revealed that the LAS method produces a RF more closely aligned with the input properties that the K-L method. This is apparent when viewing (Figure 2.21) and (Figure 2.22). The mean and standard deviation of each of the 5000 samples drawn from their respective RF was calculated and used to generate the probability density functions shown in (Figure 2.21) and (Figure 2.22). For this comparison 500 terms were used in the K-L expansion. The figures indicate that the LAS method has a higher probability of producing a RF that matches the input mean and standard deviation. For this reason, the LAS method was selected as the method of choice for generating the 1D RF field used to populate the elastic foundation proprieties for the BOREF problem.



Figure 2.22 - PDF of standard deviations from 5000 samples drawn from RFs

## **CHAPTER 3 METHODOLOGIES**

The development of the FE program used to solve the BOREF problem is discussed in this section. In addition, the program that brought to light the discrepancy in the transition zones between the FE solution and Hetényi's analytical solution for a BOEF will be discussed.

Also addressed are the assumptions inherent in the Winkler model of a BOEF and the assumptions surrounding the physical interactions between the pipeline and the foundation. The range of the model's validity is also discussed and its limitations highlighted. Finally, the literature used to determine realistic input parameters for the BOREF program is reviewed in this section.

#### 3.1 Development of Computational Tools

As indicated earlier, the FE method was selected to solve the BOREF problem. The FE method was the apparent choice due to its ability to accurately approximate the solution to differential equations and the ease in which the random foundation could be incorporated into the BOEF model. A BOEF program was developed to explore BOEF behavior in the transition zones between buckling modes. The program was also used to develop and verify the output of the BOREF program. The BOEF and BOREF programs were developed in Fortran by the author and rely on subroutines written by either D.V. Griffiths, G.A. Fenton or both. The remaining programs were written by the author or are adaptations of work done by K. K. Phoon (2005) using the MATLAB language.

#### 3.1.1 Beam on Elastic Foundation: Varying Beam Length

Program **increment\_length.f95** solves for the buckling load of a BOEF, records the value, increases the length of the beam by a user specified number of elements and repeats the process. The user specifies the global element length as a real number. The starting length of the beam can be specified via the number of elements in the first iteration. The length of the beam is reported with the buckling load and is simply the number of element in that iteration multiplied by the element length. This program was used to produce (Figure 2.9). The BOEF program is an adaptation of program p46 from Smith and Griffiths (2004), where the source code can be found. Program p46 performs the stability analysis of elastic beams using 2-node beam elements with or without an elastic foundation. A user specified homogeneous elastic

foundation was hard coded into **increment\_length.f95**. The subroutines responsible for creating the global matrix equation and solving the eigenvalue problem, described in section 2.2.3 were put inside a loop that incrementally increased the length of the beam after solving each iteration. The buckling load value is recorded prior to the deallocation, or clearing, of all variable arrays. The beam is incremented and the arrays repopulated for the subroutine to solve again. The output of the program was significantly altered to produce files that could be imported readily into Microsoft Excel of MATLAB.

*Program Input*: beam EI, foundation stiffness, element length, number of elements to increment beam length by for each iteration, boundary conditions on first and last node.

#### Program Output:

buck\_load.res; contains beam length, buckling load and element length run\_setup.res; contains a readout of input parameters rota.res and trans.res; contain the rotation and translation of every node in each iterations iters.res; contains the number of iterations the solver required to converge

#### 3.1.2 Beam on Random Elastic Foundation

The BOREF program was also developed by incorporating a standalone 1D LAS RF generator into program p46. The Monte-Carlo method was also implemented within the program to take full advantage of the BOREF program's ability to sample the RF to provide a unique foundation for each realization. The Monte-Carlo method relies on repeatedly running the same problem with only the input parameters of the problem changing. In this case the foundation underling each element was the varying input parameter. The Monte-Carlo method works well with phenomena that have a large degree of uncertainty in input parameters, making it ideal for soils modeling. Henceforth, in reference to the BOREF program, a "run" will be comprised of the user defined number of Monte-Carlo "realizations". Each realization is associated with a unique sample of the random field.

Instead of the homogenous foundation used by **increment\_length.f95** the BOREF program uses the LAS method to create a 1D array of stiffness values which is used as the foundation. The random foundation stiffness is incorporated into each elemental stiffness matrix, [k'<sub>m</sub>], as described in section 2.2.3. The subroutines responsible for assembling the

global matrix equation and solving the eigenvalue problem are once again put in a loop. Instead of the beam getting incremented in length, the foundation is replaced with a different random sample for each realization. As before, the arrays are cleared after the buckling load value has been recorded and the programs begins to solve another realization.

The program algorithm is described in pseudo code as follows:

- Read program input from text file: beam properties; foundation stiffness mean, standard deviation and spatial correlation length; number of Monte-Carlo realizations in run; element length; boundary conditions.
- 2) Generate random field of appropriate length.
- 3) Assign foundation stiffness properties using random field.
- 4) Solve for buckling load and record value.
- 5) Clear variables.
- 6) Generate random field.
- 7) Repeat steps 3 through 6 until desired number of realization reached.

The program's output was once again optimized for importation into Microsoft Excel or MATLAB for analysis. For example, the buckling load for each realization is output into a separate file as a single column vector easily interpreted by MATLAB or Excel.

The program generates three output files:

- 1) *"input\_file\_name".res* echoes the input data back to the user.
- "input\_file\_name"\_buck\_loads.res lists the buckling load calculated for each realization in a column vector.
- *iterations.res* lists the number of iterations required for the solution to converge for each realization with the random seed used by the LAS generator.

Real time program progress output was also included by the author. While running the program supplies the user with updates using the command window. The program reports its progress as a percentage (number of realization completed over desired number of realization) as well as the time in seconds for each completed realization. The time per realization output allows users to estimate the total time to complete a run. The program will continue to run, by

moving onto the next realization, if it fails to converge before reaching the user specified iteration limit on the current realization. The program will however notify the user with a warning message in the command window as well as flag the non-convergent realization in the *iteration.res* file. This allows the user to easily identify and remove the non-convergent realization realization from the dataset thereby salvaging the time already invested in the run.

The effects of the random foundation on the solver are reflected in the number of iterations per realization in the *interations.res* file and the time per realization displayed in the command window. Very rarely, and only by coincidence, does a realization converge in the same number of iterations or in the same amount of time.

The analysis of the buckling load data is completed in MATLAB via the *Buck\_Load\_Analysis* program. The MATLAB program reads the buckling load file directly once the user has specified its location. MATLAB's built in function histfit(data, nbins, dist) plots a histogram of the values in the vector data using nbins bins and then superimposes a fitted distribution dist. A non-parametric kernel-smoothing distribution was selected to fit the data due to its adaptability. Non-parametric in this instance means that the fit does not assume that the structure of the distribution is fixed. The density is evaluated at one hundred equally spaced points that cover the range of the data and fitted using those points.

After the location of the buckling load file has been specified the program runs and produces the following output:

- A histogram of the values in the *"input\_file\_name"\_buck\_loads.res* file binned into fifty bins with a superimposed fitted distribution.
- 2) A plot of the probability density function fit to the data.
- 3) A plot with axial load on the x-axis and probability the pipeline will buckle on the y-axis.

The third figure is produced by approximating the area under the probability density function plotted in the second figure. This is accomplished with the trapezoid rule using three thousand equal slices taken between the minimum and maximum buckling load values in the *"input\_file\_name"\_buck\_loads.res* file. The area is converted to a percentage and then plotted against the three thousand buckling load values between the minimum and maximum.

### 3.2 Assumptions, limitations and range of validity

There are three primary assumptions that must be made to justify the use of the Winkler BOEF model used in this work. The first assumption is that the pipeline buckles laterally; meaning the initial movement must be parallel to the ocean floor. The second is that the two mechanisms supplying the lateral breakout resistance, brittle breakout and soil suction, which causes tensile bonding (Bruton et al. 2008, Cheuk et al. 2007), can be combined and used to determine the Winkler model's foundation stiffness. Finally, the third assumption is that buckling is imminent at, or prior to, the pipeline mobilizes the peak value of horizontal breakout resistance. Meaning, the out-of-straightness of the pipe required to mobilize the peak horizontal breakout resistance results in an instability that the foundation cannot support.

Shown in (Figure 3.1) is the horizontal breakout resistance plotted against the pipe displacement. It will be shown that the peak resistance coincides with a relatively small lateral displacement of .01D to .1D. Note the sharp drop in horizontal resistance as the trailing edge of the pipe breaks free form the soil and suction is lost. The dashed line superimposed on (Figure 3.1) illustrates how the Winkler model will approximate the lateral breakout resistance up to the peak resistance value. Beyond the peak value, the Winkler model will deviate from reality and continue to supply a linearly increasing resistance following the "long dashed dot dot" line.



Figure 3.1 - Lateral breakout resistance vs. displacement (Cheuk et al. 2007)

Lateral buckling is a reasonable assumption to make for nearly all untrenched pipelines placed on marine sediment. This includes all deep water pipelines, as noted earlier they are laid untrenched, as well some shallow pipelines. A lateral path typically offers the least resistance to buckling for an untrenched pipeline and therefore is frequently observed in operating pipelines (Bruton et al. 2005). Grouping the foundation's suction and brittle breakout resistance is also reasonable as both exert their greatest influence in the instant prior to initial buckling (Bruton et al. 2008).

The lateral buckling assumption makes the Winkler BOEF model ideally suited for describing this problem. The contributions from the two components comprising the foundation stiffness can be assumed to remain in proportion along the length of the pipeline. If the motion is not purely horizontal the proportions of the breakout resistance caused by suction and brittle breakout would be in constant flux along the length of the pipe. In the extreme case of vertical pipeline motion, there would be zero contribution from brittle breakout. Therefore, purely horizontal movement allows the variability in the soil properties and embedment depth to be the sources of uncertainty instead of which component, brittle breakout of suction, is contributing more to the breakout resistance. It will be shown that there are many empirical formulas available to calculate the lateral breakout resistance.

This model is limited to the estimation of the probability a pipeline will buckle under a given axial load. It cannot produce estimates of lateral pipeline displacement. Because this research is only concerned with the axial load required to cause buckling the magnitude of lateral movements is out of the scope. This might seem to be a severe limitation, but recall that the most elegant and cost effective solution to the problem of lateral buckling in pipelines involves purposely initiating buckles at controlled intervals (Cheuk et al. 2007). This computational tool associates a probability of buckling with a given axial load which can help the user either define pipeline temperature and pressure operating specifications of facilitate the selection of an appropriate spacing between buckle initiators. If the spacing between buckle initiators has been predetermined, for whatever reason, the program can indicate the likelihood of buckling under a given axial load, which can be estimated with relative ease based on operating temperature and pressure. The program can also be used iteratively, by altering the

pipeline section length between buckle initiators, to achieve an acceptable probability of buckling for an anticipated axial load resulting from increases in temperature and pressure.

The validity of this model is closely tied to the statistical description of the Winkler foundation stiffness. The mean, standard deviation and spatial correlation length are generally more representative of the foundation over shorter lengths. The foundation along the planned route can more accurately be characterized by dividing it into smaller, rather than larger, sections. The utility of the program's output is diminished as: the standard deviation of the foundation material increases, the length of the pipeline increases, the mean and correlation length describe larger and larger sections. Breaking the planned route into sections in which the foundation properties maintain similarity will maintain the validity of the program's output.

#### 3.3 Determination of Program Input Parameters

Realistic input parameters were desired to demonstrate the BOREF program. A means of calculating the lateral breakout resistance of the seabed foundation was sought in the literature. This resistance then needed to be converted into the Winkler foundation modulus. Recall that the Winkler foundation modulus has units of kN/m<sup>2</sup>. A means of converting the lateral breakout resistance into the Winkler foundation modulus was crucial to accurately describing the example problem. The model pipeline properties of diameter and bending stiffness were required, in addition to the boundary conditions at the ends of the pipeline section. The following section identifies the sources and logic behind the selection of the input parameters for the example problem presented in this thesis.

#### 3.3.1 Characterization of Foundation Modulus

A large body of work related to pipeline breakout resistance exists as it has historically been the focus of lateral pipeline/soil interaction tests (Bruton et al. 2006). This focus is the result of lateral breakout resistance having an extremely significant influence on the critical buckling load of a pipeline while simultaneously being the parameter with the largest degree of uncertainty (Bruton et al. 2008).

The largest contributor to uncertainty in the breakout resistance is the embedment depth of the pipeline. Embedment depth is in turn inextricably linked to the seafloor soil properties, the

pipeline's weight and the installation process. This means that in order to accurately determine the breakout resistance the pipeline embedment must first be determined.

Researchers have developed analytical and computer models as well as done centrifuge and full scale testing in an effort to try and determine pipeline embedment depth e.g. (Merifield et al. 2008, Merifield and White 2008, White and Randolph 2008, Wang et al. 2010). An amazing body of work has been produced on the subject of cylinders penetrating clays. The results however are difficult to apply to thousand meter long sections of pipeline. Embedment depth has been shown to largely depend on factors that cannot be input into static models. Mainly, pipeline motions at the seabed, which are a result of sea state, vessel response, lay ramp configuration, pipeline rigidity, installation rate, water depth and seabed stiffness. During offshore pipeline installation, seen in (Figure 3.2), pipeline motions increase the pipeline embedment depth above that predicted by static penetration models (Westgate et al. 2010).



Figure 3.2 - Pipeline installation (Westgate et al. 2010)

The complexity of predicting the embedment depth has prompted the inclusion of embedment depth along with soil shear strength as the key parameters defining the variability of the breakout resistance, and therefore the Winkler foundation modulus stiffness. The impact of dynamic lay effects on pipeline embedment were studied in (Westgate et al. 2010) during installation of a 0.39 m (12 inch) diameter steel pipeline in a water depth that varied from 134 m to 153 m along the 25 km lay route. It should be noted that although this pipeline is not in deep water, it is untrenched and is therefore subject to lateral buckling. The researchers found that

the mean embedment was equal to 0.37D, and ranged from 0.26D to 0.48D within one standard deviation and from 0.15D to 0.59D within two standard deviations. The seabed along the lay route had the characteristic of a medium plasticity marine clay. This paper provided a data set perfect for the demonstration of the BOREF program. It should be noted that there are many methods available to determine embedment depth prior to pipeline installation. Any one of these methods, in addition to petroleum engineers' acquired experience with pipeline embedment, would provide the program with useful input data.

The properties of the pipeline and seabed presented in (Westgate et al. 2010) are shown in (Table 3.1). The pipeline was installed in the UK North Sea sector, approximately 210 km northeast of Aberdeen. The field is located in the Witch Ground Graben of the northern North Sea. The geotechnical site investigation along the pipeline route included 11 shallow cone penetrometer tests (CPTs) and 10 vibrocores spaced evenly over 25 km. The surface stratigraphy comprises a thin very soft silty clay layer within the upper 0.2 m (the Glenn Member), transitioning to soft silty clay with thin interbedded sand layers below 0.2 m (the Witch Member). The undrained shear strength profile of the seabed was idealized using a zero strength intercept at the mudline and a strength gradient of 10 kPa/m.

Flowline property		Value	
Outside steel diameter	0.3239 m		
Wall thickness		15.9 mm	
Coating 1		0.45 mm FBE	
Coating 2		34 mm SPU	
Overall outside diameter		0.3928 m	
Unit weight in air		1.44 kN/m	
Submerged unit weight (empty)	Submerged unit weight (empty)		
Modulus of elasticity, E <sup>a</sup>	200 GPa		
Bending rigidity, El	36.6 MN m <sup>2</sup>		
<sup>a</sup> Assumed value based on typical st	eel pipes.		
Soil property	Glenn Member	Witch Member	
Depth to bottom of layer (m)	~0.2	>1	
Effective unit weight (kN/m3)	6.5 to	o 8.5	
Shear strength gradient (kPa/m) <sup>a</sup>	1 to 2		
Sensitivity	2 to	53	

Table 3.1 – Pipeline and seabed properties (Westgate et al. 2010)

<sup>a</sup> Mudline strength intercept is considered negligible.

The fully laden pipeline mass calculations are presented in (Table 3.2). The seabed must support a linear force of 0.6 kN per meter when the pipeline is operating. This estimate assumes that the pipeline is completely full of oil with a density of 950 kg/m<sup>3</sup> and does not take into account the buoyancy of the coatings. The force the pipeline imparts on the seabed is largely dependent on the weight of the fluid being transported by the pipeline and must be adjusted accordingly.

Component *Per Meter of Pipeline	OD (m)*	Wall Thickness (m)*	Volume (m³)*	Density (kg/m³)*	Mass (kg)*
ASTM A106 – Steel	0.3239	0.0159	0.0154	7850	121
Pipe with Coatings (FBE & SPU)	0.3928				
Oil	0.2921		0.0670	950	64
Mass of Operating Pipe					184
Water Displaced	0.3928		0.1212	1037	126
Buoyant Pipe					59

#### Table 3.2 - Pipeline mass

The seabed along the planned pipeline installation route would most likely be characterized via sampling in a manner similar to that presented in the North Sea example. However, characterizations of large underwater areas exist (Figure 3.3) and could be used to make preliminary calculations with this program.

An example problem in the Gulf of Mexico was also considered for this thesis. The Sigsbee abyssal plain, shown in (Figure 3.4), is a roughly triangular basin at the base of the western limit of the Mississippi Fan. The plain is approximately 400 km (250 miles) long and 200 km (120 miles) wide. The sediments on the abyssal plain are primarily terrigenous silts and clays. Shear strength characterization of the Gulf of Mexico was completed by (Bryant and Delflache 1971). The water depth is over 3300 meters (10,800 ft), the deepest in the Gulf, and additionally the plain is extremely smooth and flat with a gradient less than 1:8000. The abyssal plain would provide a textbook location for the application of this method due to aforementioned qualities.



Figure 3.3 - Average shear strength (cohesion) in lb/sq ft, 1 ft below sediment-water interface (Bryant and Delflache 1971)



Figure 3.4 - Physiographic chart of the Gulf of Mexico (Bryant and Delflache 1971)

Once the pipeline's planned route has been characterized the breakout resistance is relatively easily calculated using one of many available empirical analytical expressions. Some examples of the available analytical expressions used to calculate the horizontal force required for lateral breakout, and the research underlying their derivation, can be found in (Merifield and White 2008, Bruton et al. 2006, Merifield et al. 2008). The lateral breakout resistance, for the example presented in this thesis, will be calculated using N<sub>cH</sub> = 2.85 $w^{0.6}$  from (McCarron 2011), where: N<sub>cH</sub> = H/DS<sub>u</sub>, H = lateral force per meter of pipeline, D = pipeline diameter, S<sub>u</sub> = shear strength of the soil at the pipe invert and w = ratio of embedment to pipeline diameter (z/D). To directly solve for the horizontal force required to cause horizontal breakout the equation has been rearranged into the form  $H = 2.85DS_u \left(\frac{z}{D}\right)^{0.6}$ .

As indicated earlier, once the horizontal breakout resistance force has been determined it must be converted into the Winkler foundation modulus. This requires a dimension known as the axial mobilization displacement (AMD). This is the distance required to mobilize the full horizontal breakout resistance. In Bruton et al. (2008) it is stated that this distance varies considerably but is typically over 10 mm and can reach 100 mm or more, depending on the test method, soil conditions and pipe geometry. These absolute distances compare well with data from other researches. The normalized breakout resistance plotted against the normalized lateral displacement from (Wang et al. 2010), (Dingle et al. 2008) and (Cheuk et al. 2007) are shown in (Figure 3.5), (Figure 3.6) and (Figure 3.7) respectively.



Figure 3.5 - AMD from (Wang et al. 2010)



Figure 3.7 - AMD from (Cheuk et al. 2007)

In all three figures the peak value of lateral breakout resistance appears to occur at or before 0.1D. Additionally, the lateral resistance appears to increase fairly linearly up to the peak value making the Winkler model a good approximation.

A more accurate approximation of the AMD value could be determined with additional knowledge specific to the pipeline in question and industry experience. The rate at which the axial load develops in an operating pipeline would help refine the AMD value. This rate depends on many factors including: ambient water temperature, the fluid temperature in the pipe, flow rate and pressure. Increases in pore water pressure, which results in increases in the breakout resistance, are directly related to the rate at which the pipe loads the soil foundation. The tensile force generated at the trailing edge of the pipe by suction is also dependent on the pipeline's velocity during breakout. The initial buckling event is reported in (Bruton et al. 2008) as usually happening very quickly.

With the AMD value chosen it is now possible to equate the lateral breakout resistance to the Winkler foundation stiffness. An explanation of why the Winkler foundation modulus has units of kN/m<sup>2</sup> is warranted. The foundation's reaction to a section of beam displaced into the foundation by a distance of *y* is shown in (Figure 3.8). As stated, the foundation produces a reaction force proportional to the displacement. The resultant force acting on the beam section is therefore  $F_R = k \cdot y \cdot dx$ . It is apparent that the foundation must have units of N/m<sup>2</sup> if the result force is to have units of force. Stated another way, the resultant force produced by the foundation on a section of beam is a function of the length of the beam section and the displacement of that section into the foundation.



Figure 3.8 - Foundation reaction force

The AMD can be thought of as the *y*, or displacement into the foundation of the Winkler model. The peak horizontal breakout resistance per unit length, *dx*, of the pipeline was determined analytically previously. Using the assumption that the foundation should mimic the support supplied by the lateral breakout resistance until the peak value is reached, the Winkler foundation modulus can be determined using k = H/AMD, where *H* is the peak horizontal breakout resistance. The assumption that the horizontal breakout resistance increases linearly until the peak value is reached is important because the Winkler model can only describe linearly increasing stiffness.

(Table 3.3) is produced using an Excel file called **k\_calc.xlsx** included with the other programs associated with this thesis.

Soil and Pipe Properties				
D	0.3928	М		
V	0.6	kN/m		
Su @ ml	0.0	kN/m²		
Su Gradient	12.5			

Table 3.3 - Foundation stiffness calculator output

Embedment Ratio (z/D)	H (kN/m)
0.26	0.6
0.37	1.1
0.48	1.7

Embedment	z (m)	Su @ z (kN/m <sup>2</sup> )
Low	0.1021	1.2766
Medium	0.1453	1.8167
High	0.1885	2.3568

AMD ratio	0.1	x/D
AMD	0.03928	М
$k_{low}$	16.2	kN/m²
$k_{med}$	28.5	kN/m <sup>2</sup>
$k_{high}$	43.2	kN/m²

Where:

D = Pipeline Diameter
k = Winkler Foundation Stiffness
z = Distance below Mud Line
AMD = axial mobilization distance

Su Gradient = Shear Strength Gradient (considered linear) V = Weight of Operating Pipeline

Su @ ml = Shear Strength of Soil at Mud Line

## 3.3.2 Pipeline Boundary Conditions and Length

The boundary conditions at the ends of the pipeline in the model can be independently configured to model a variety of situations. Vertical and rotational displacement at each end of the pipe can independently be configured as either fixed or free. The boundary conditions at the ends of the pipeline will vary depending on the situation. Short pipes shallowly embedded in weak soil will not develop a fully constrained condition at the pipe ends. A short pipe might be

constrained on one end by a well head or other underwater feature and free at the other end. Long pipes develop fully constrained conditions after the friction between the pipe and the foundation has mobilized the soil's shear strength over a great enough length (Bruton et al. 2005). The first two buckling modes of a pipe modeled with fully constrained boundary conditions, e.g. no rotation or vertical translation, are shown in figure (Figure 3.9). A detailed method for calculating the length of pipe needed to create the fully constrained condition is presented in (Taylor and Gan 1986). The fully constrained condition has been selected for the example presented in this thesis as long lay routes are common in deep water.



Figure 3.9 - Fully constrained pipeline boundary conditions

In (Brown et al. 2006) a simple example of a typical deep water pipe lay was presented. The pipe in the example was 8000 meters (5 mile) long and had a diameter of 323.95mm (12.7 in). Buckles initiators were placed every 2000 meters (1.25 miles). This pipeline has the same diameter as the pipe selected for the example presented in this thesis which prompted the selection of 2000m as the pipe length to model.

### **CHAPTER 4 RESULTS**

The buckling load analysis of the example pipeline presented in the methodologies chapter can be seen in (Figure 4.1) and (Figure 4.2). The modeled pipeline was 2000 meters (6560 ft) in length and contained 4000 finite elements of equal length. The pipeline's bending stiffness was 36.6 kNm<sup>2</sup> and the special correlation length of the foundation was chosen to be 200 meters (656 ft) or 10% of the model's length. The first and last nodes were fixed in both translation and rotation. Lastly, the foundation stiffness had a mean and standard deviation of 28.5 kN/m<sup>2</sup> and 13.5 kN/m<sup>2</sup> respectively. The plots presented in (Figure 4.2) and (Figure 4.1) were constructed with 4500 Monte-Carlo simulations.

Shown in (Figure 4.1) is a histogram of the buckling loads constructed using 50 bins. The kernel distribution can be seen in red overlaying the histogram. Recall that the kernel distribution was fit using 100 equally spaced points covering the range of the data.



Figure 4.1 - Histogram with fitted distribution

A plot showing probability the pipeline will buckle under a given axial load is shown in (Figure 4.2). This plot, in conjunction with the pipeline's operating and environmental conditions will allow an engineer to determine the likelihood a pipeline will buckle. The axial load a pipeline will carry can be calculated using expected operating pressures and temperatures. This information allows the engineer to enter the plot on the x-axis and read off the probability the pipeline will buckle over the modeled length. If there is a high likelihood the pipeline will buckle the engineer can insert a buckle initiator at the midpoint of the modeled pipeline and run the problem again at half the length. This shorter section can then be checked to see if it falls safely under what is determined to be an acceptable probability of buckling.

The program can be used on pipelines that have already been installed and are being repurposed or having maintenance performed. In these situations the buckling initiation spacing has already been determined. The program can provide guidance on the temperatures and pressures the pipeline can safely be operated at to maintain the probability of buckling at an acceptably low level.



Figure 4.2 - Probability pipe will buckle under given axial load

### 4.1 Buckling Load: Foundation with Random Stiffness versus Constant

#### Stiffness

In order to verify the results of the BOREF program the buckling load was calculated on three homogeneous foundations. The high, low and mean stiffness values from (Table 3.4) were used for the calculations and the results are presented in (Table 4.1). The BOREF program is able to determine the buckling load for a beam on a homogeneous foundation by specifying a standard deviation of zero for the foundation. With the standard deviation of the foundation set to zero the program effectively becomes a BOEF program. The output from the BOREF using a standard deviation of zero was compared with the output from a BOEF program during development and was verified to produce the same results.

Table 4.1 - Buckling	loads using	homogeneous	foundations
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k (kN/m²)	Buckling Load (kN)
16.2	1540
28.5	2043
43.2	2515

It is possible to determine if the buckling load values produced by the BOREF program are reasonable using (Figure 4.2). The buckling load for a beam on a homogeneous foundation of 28.5 kN/m<sup>2</sup>, 2043 kN, corresponds to roughly a 100% probability of buckling in (Figure 4.2). Recall that each Monte-Carlo simulation was performed on a foundation with a mean stiffness of  $28.5 \text{ kN/m}^2$ .

It has been shown previously by (Griffiths and Fenton 2001) that the response of a nonhomogeneous material is disproportionally dictated by weak spots. This phenomenon is again demonstrated here with beam buckling on elastic foundations. The response is governed by the sections of the foundation that are less stiff than the mean. This is evident by the fact that above the axial load required to buckle the beam on the homogenous foundation the probability of buckling is over 100%. (Figure 4.2) indicates that the beam is likely to buckling well before it reaches the buckling value determined using a homogeneous foundation.

## **CHAPTER 5 CONCLUSIONS**

### 5.1 Significance of Work

This work is significant because it introduces a new tool to the field of petroleum engineering. The ability to incorporate uncertainty, and therefore perform a reliability calculations has been identified as important by researchers in the field (Brown et al. 2006). This computational tool has the ability to capture the uncertainty in the largest factor driving lateral pipeline buckling, lateral breakout resistance. The uncertainty in all of the parameters needed to calculate the lateral breakout resistance can now be accounted for. Their impact on the buckling load can be explored in one of thousands, or tens of thousands, of Monte-Carlo simulations. From these simulations flow statistically meaningful guidance in determining the answer to some of the most important questions facing the industry regarding safety and cost.

The Winkler BOEF model is extremely well suited for describing the lateral buckling phenomenon of untrenched pipelines. Even if the methodology used to determine the Winkler foundation stiffness presented in this thesis is based on a poor assumption it is not difficult to imagine that a foundation stiffness exists that would allow the model to make very accurate predictions.

Stochastically describe parameters used in engineering analysis were not encountered in the literature regarding pipeline buckling. This research describes a technique that the industry might find useful in this and other modeling situations.

With different input parameters this tool could be repurposed to perform the same analysis on a variety of problems that can accurately described using the Winkler model. As previously discussed these problems include: networks of beams, which are characteristic in the construction of floor systems for ships, buildings and bridges; and shells of revolution, which includes pressure vessels and large-span reinforced concrete halls and domes (Hetényi 1946).

### 5.2 Future Work: Improving the Model

Many paths present themselves for future development of this computational tool. One path is concerned with the validation of this model and the methodology used to define the

Winkler foundation stiffness via experimentation. Another path is concerned with further development of the computational tool's foundation.

Additionally, a suite of programs that would complement the BOREF program can be developed. A program for calculating embedment that incorporated a random foundation would greatly facilitate the determination of the mean and standard deviation of a particular pipeline's embedment. This program could directly calculate the horizontal breakout resistance needed to define the Winkler foundation stiffness. A program that could calculate the axial force within a pipeline due to increases in temperature and pressure along with the length of pipeline contacting the seafloor required to develop the fully fixed boundary condition would also facilitate the selection on of the input parameters for the BOREF program.

Finally, extensive testing of the tool needs to be completed. A wide range of input parameters should be experimented with and the number of Monte-Carlo simulations required to reach convergence should be determined.

#### 5.2.1 Experimental Validation

The methodology developed to determine the Winkler foundation modulus is unique and unproven. The elastic foundation in the model described in this thesis can only supply the pipe with a linearly increasing reaction force. Experimental data suggests that the horizontal resistance can be idealized as linearly increasing until the peak value is reached. The reaction force in the model was configured to reach the peak horizontal breakout resistance by the time the pipeline reaches the AMD. The model no longer accurately represents reality after the pipeline has moved past the AMD. In reality, the horizontal resistance supplied by the foundation drops sharply once the AMD has been reached, in the model the resistance increases linearly forever. Experimental validation is needed to determine if a pipeline will indeed buckle prior to or at the AMD. In order to reach the AMD the pipeline must deflect, it is not unreasonable to think that these deflections may result in an instability that cannot be supported, no matter how stiff the foundation gets, and buckling is initiated at or before the pipeline reaches the AMD. Experimentation could demonstrate this. If the degree of out-ofstraightness required to mobilize the full horizontal breakout resistance results in buckling being imminent then the model is indeed appropriate. If not, the Winkler BOEF model might not be suitable for representing the complex behavior of the horizontal breakout resistance.

#### 5.2.2 Computational tool development

The simplicity of this model is simultaneously a strength and a weakness. The model is easy to work with and can run relatively quickly, depending on the problem. The tradeoff for these qualities comes in the simplicity of the model's foundation. A linearly increasing reaction force, as stated, is the only type of foundation able to be represented. This model performs ideally in situations where the foundation medium is kept in the elastic range, and this indeed represents a great deal of engineering analysis. This model does not perform well however in situations where the foundation does not supply an entirely linearly increasing reaction force. The horizontal breakout resistance is a prime example of this. It can be idealized as linear until the peak breakout resistance is reached but, beyond that point the behavior simply cannot continue to be accurately represented linearly. An extension of this work could involve the development of a probabilistic computational tool able to capture the nonlinear behavior of the horizontal breakout resistance.

A model capable of capturing the nonlinear behavior of the horizontal breakout force would require a means of determining the horizontal displacement of the pipeline. It would need the horizontal displacement to vary the reaction force supplied by the foundation appropriately. The simple Winkler model is incapable of determining pipeline displacements and therefore would have to be abandoned to achieve this goal. The combination of random field generating software with commercially available FE software would most likely be the quickest route to accomplishing this. Even a small problem would most likely require parallelization and a super computer however. This fact emphasizes the importance of developing simple models to represent complex problems.

#### 5.3 Concluding Remarks

Lateral buckling was identified as a poorly understood phenomenon after a series of catastrophic failures. This acknowledgment prompted the inception of the SAFEBUCK JIP which continues to produce a wealth of knowledge on the subject. The research presented in this thesis uses the knowledge accrued by the JIP, as well as other researchers, to develop a novel probabilistic computational tool able to determine the buckling load of a beam on a random elastic foundation. This information can in turn be used to guide buckle initiation which

is the most effective method of combating the high stress concentrations that develop in pipelines as a result of uncontrolled and unplanned buckling.

This tool addresses a desire to incorporate reliability calculations into design. This allows a more rigorous and quantitative lateral pipeline design methodology to be developed. By providing a means of capturing the uncertainty in the lateral breakout resistance this tool has the ability to determine the probability a pipeline will buckle under a given axial load. This information offers insight into the distance buckle initiators should be placed, as well as determining operating conditions.

The approach used to determine the Winkler foundation modulus is unique and unproven. For this computational tool to be implemented in industry the assumptions underlying the determination of the foundation modulus would require verification. An experienced petroleum engineer might be able to determine the validity of the assumptions without experimentation. If this is not possible experimentation would be required.

The work completed during the course of this thesis also includes the statistical comparison of the K-L and LAS RF generation methods and a demonstration of the BOREF program using realistic parameters.

According to Hetényi the Winkler model rigorously describes a multitude of engineering phenomenon. If this model is indeed incapable of capturing the lateral breakout resistance experienced by an un-trenched pipeline this program and its output can easily be adapted to other problems where the foundation's reaction force can be idealized as linearly increasing. Real world problems always involve imperfect materials. Stochastic descriptions of these materials will always result in a more quantitative assessment of their failure states. The range of applications for the BOREF program extends far beyond pipelines.

AMD	=	Axial Mobilization Displacement [L]
С	=	Attenuation Parameter in K-L Covariance Function [dimensionless]
D	=	Pipeline Diameter [L]
EI	=	Beam Bending Stiffness [F·L <sup>2</sup> ]
$F_R$	=	Resultant Force [F]
Н	=	Peak Horizontal Breakout Force [F/L]
I	=	Beam Length [L]
k	=	Spring Constant [F/L]
М	=	Moment [F·L]
n	=	Mode Shape Number [dimensionless]
$N_{cH}$	=	Normalized Lateral Breakout Force [dimensionless]
N <sub>cr</sub>	=	Critical Buckling Load [F]
OD	=	Outer Diameter
р	=	Foundation Reaction Force [F]
q	=	Distributed Load [F/L]
Su	=	Shear Strength [F/L <sup>2</sup> ]
V	=	Weight of Operating Pipeline [F]
k	=	Winkler Foundation Modulus [F/L <sup>2</sup> ]
x	=	Distance along Beam [L]
у	=	Displacement into Foundation [L]

# NOTATION

Z	=	Depth below Mud Line [L]
ξ	=	Uncorrelated Random Variable [dimensionless]
λ	=	Eigenvalue [dimensionless]
$ au_{ij}$	=	Absolute Distance Between Points <i>i</i> and <i>j</i> [L]
${oldsymbol{ ho}_{ij}}$	=	Correlation Coefficient [dimensionless]
σ	=	Standard Deviation [dimensionless]

 $\theta_E$  = Spatial Correlation Length [L]

## GLOSSARY

- 1D: One-Dimensional
- AMD: Axial Mobilization Displacement
- BOEF: Beam on Elastic Foundation
- BOREF: Beam on Random Elastic Foundation
- CPT: Cone Penetrometer Test
- FE: Finite Element
- FEA: Finite Element Analysis
- JIP: Joint Industry Project
- K-L: Karhunen–Loève
- LAS: Local Averaging Subdivision
- Mud Line: The Boundary between Earth and Water
- PDF: Probability Density Function
- RF: Random Field

## REFERENCES

- Ahmed, A., and A.H. Soubra. 2011. "Subset Simulation and Its Application to a Spatially Random Soil." 209–216. Retrieved June 26, 2012 (http://scholar.google.com/scholar?hl=en&btnG=Search&q=intitle:Subset+Simulation+and+ its+Application+to+a+Spatially+Random+Soil#0).
- Ahmed, A., and A.H. Soubra. 2012. "Probabilistic analysis of strip footings resting on a spatially random soil using subset simulation approach." *Georisk* (September):37–41. Retrieved March 8, 2013 (http://www.tandfonline.com/doi/abs/10.1080/17499518.2012.678775).
- Brown, G., M. Brunner, and X. Qi. 2006. "Lateral Buckling Reliability Calculation Methodology Accounting for Buckle Interaction." *Offshore Technology*. Retrieved May 24, 2012 (http://www.onepetro.org/mslib/servlet/onepetropreview?id=OTC-17795-MS).
- Bruton, D., M.D. Bolton, and M. Carr. 2008. "Pipe-Soil Interaction With Flowlines During Lateral Buckling and Pipeline Walking-The SAFEBUCK JIP." *Offshore Technology*. Retrieved May 24, 2012 (http://www.onepetro.org/mslib/servlet/onepetropreview?id=OTC-19589-MS).
- Bruton, D., M. Carr, M. Crawford, and E. Poiate. 2005. "The Safe Design of Hot On-Bottom Pipelines with Lateral Buckling using the Design Guideline Developed by the SAFEBUCK Joint Industry Project." Pp. 1–26 in *Deep Offshore Technology Conference*. Vitoria, Espirito Santo, Brazil.
- Bruton, D., D.J. White, C.Y. Cheuk, M.D. Bolton, and M. Carr. 2006. "Pipe / Soil Interaction Behavior During Lateral Buckling." (May):1–4.
- Bryant, W., and A. Delflache. 1971. "Geotechnical charts of the deep water portion of the Gulf of Mexico." *Offshore Technology Conference*. Retrieved March 6, 2013 (http://www.onepetro.org/mslib/servlet/onepetropreview?id=OTC-1468-MS).
- Cheuk, C.Y., D.J. White, and M.D. Bolton. 2007. "Large-scale modelling of soil–pipe interaction during large amplitude cyclic movements of partially embedded pipelines." *Canadian Geotechnical Journal* 44(8):977–996. Retrieved May 24, 2012 (http://www.nrcresearchpress.com/doi/abs/10.1139/T07-037).
- Craig, I., and N. Nash. 1990. "Upheaval Buckling: A Practical Solution Using Hot Water Flushing Technique." *Offshore Technology Conference*. Retrieved June 1, 2012 (http://www.onepetro.org/mslib/servlet/onepetropreview?id=OTC-6334-MS).
- Dingle, H. R.C., D.J. White, and C. Gaudin. 2008. "Mechanisms of pipe embedment and lateral breakout on soft clay." *Canadian Geotechnical Journal* 45(5):636–652. Retrieved December 3, 2012 (http://www.nrcresearchpress.com/doi/abs/10.1139/T08-009).
- Fenton, G.A., and E.H. Vanmarcke. 1990. "Simulation of random fields via local average subdivision." *Journal of Engineering Mechanics* 116(8):1733–1749. Retrieved September 27, 2012 (http://ascelibrary.org/doi/abs/10.1061/(ASCE)0733-9399(1990)116:8(1733)).

- Griffiths, D.V., and G.A. Fenton. 2001. "Bearing capacity of spatially random soil: the undrained clay Prandtl problem revisited." *Geotechnique* 51(4):351–360. Retrieved April 12, 2012 (http://www.engmath.dal.ca/rfem/papers/prandtl\_inline.pdf).
- Griffiths, D.V., and G.A. Fenton. 2004. "Probabilistic slope stability analysis by finite elements." *Journal of* 130(5):507–518. Retrieved June 5, 2012 (http://link.aip.org/link/?JGGEFK/130/507/1).
- Griffiths, D.V., and G.A. Fenton. 2009. "Probabilistic settlement analysis by stochastic and random finite-element methods." *Journal of geotechnical and* 135(November):1629. Retrieved April 18, 2012 (http://link.aip.org/link/?JGGEFK/135/1629/1).
- Griffiths, D.V., Jinsong Huang, and G.A. Fenton. 2011. "Probabilistic infinite slope analysis." *Computers and Geotechnics* 38(4):577–584. Retrieved April 12, 2012 (http://linkinghub.elsevier.com/retrieve/pii/S0266352X11000474).

Griffiths, D.V. 2013. Lecture notes: "The Finite Element Method - EGGN 542." Spring, 2013

- Guijt, J. 1990. "Upheaval Buckling of Offshore Pipelines: Overview and Introduction." *Offshore Technology Conference*. Retrieved May 24, 2012 (http://www.onepetro.org/mslib/servlet/onepetropreview?id=OTC-6487-MS).
- Helvoirt, L., and A. Sluyterman. 1990. "A dedicated finite-element model for analyzing upheaval buckling response of submarine pipelines." *Offshore Technology*. Retrieved May 24, 2012 (http://www.onepetro.org/mslib/servlet/onepetropreview?id=OTC-6333-MS).
- Hetényi, Miklós. 1946. *Beams on elastic foundation*. Vol. 1, no. Ann Arbor, Michigan: University of Michigan Press.
- Huang, S.P., S.T. Quek, and K.K. Phoon. 2001. "Convergence study of the truncated Karhunen–Loeve expansion for simulation of stochastic processes." *International Journal for Numerical Methods in Engineering* 52(9):1029–1043. Retrieved April 4, 2012 (http://doi.wiley.com/10.1002/nme.255).
- Li, C.C. 1993. "Optimal discretization of random fields." *Journal of engineering mechanics* 119(6):1136–1154. Retrieved July 18, 2012 (http://scholar.google.com/scholar?hl=en&btnG=Search&q=intitle:OPTIMAL+DISCRETIZA TION+OF+RANDOM+FIELDS#0).
- Li, S., and R.C. Batra. 2007. "Thermal Buckling and Postbuckling of Euler-Bernoulli Beams Supported on Nonlinear Elastic Foundations." *AIAA Journal* 45(3):712–720. Retrieved October 3, 2012 (http://arc.aiaa.org/doi/abs/10.2514/1.24720).

McCarron, William O. 2011. Deepwater Foundations and Pipeline Geomechanics. J. Ross Pub.

Merifield, R.S., and D.J. White. 2008. "The Effect of Pipe-Soil Interface Conditions on the Undrained Breakout Resistance of Partially-Embedded Pipelines." *Proceedings of the 12th* 1–6. Retrieved May 24, 2012 (http://www.civil.iitb.ac.in/~dns/IACMAG08/pdfs/O54.pdf).

- Merifield, R.S., D.J. White, and M.F. Randolph. 2008. "The ultimate undrained resistance of partially embedded pipelines." *Géotechnique* (6):461–470. Retrieved December 5, 2012 (http://www.planyourcareer.com.au/Resources/Research Centres/CGMM/Publications/Richard Merifield/The ultimate undrained resistance of partially embedded pipelines.pdf).
- Phoon, K.K., H.W. Huang, and S.T. Quek. 2005. "Simulation of strongly non-Gaussian processes using Karhunen–Loeve expansion." *Probabilistic Engineering Mechanics* 20(2):188–198. Retrieved September 27, 2012 (http://linkinghub.elsevier.com/retrieve/pii/S0266892005000123).
- Phoon, K.K., and Fred H. Kulhawy. 1999. "Characterization of geotechnical variability." *Canadian Geotechnical Journal* 36(4):612–624. Retrieved (http://www.nrc.ca/cgibin/cisti/journals/rp/rp2\_abst\_e?cgj\_t99-038\_36\_ns\_nf\_cgj36-99).
- Schaminee, P., and N. Zorn. 1990. "Soil Response for Pipeline Upheaval Buckling Analyses : Full-Scale Laboratory Tests and Modelling P \*." *Offshore Technology Conference* 563– 572. Retrieved May 24, 2012 (http://www.onepetro.org/mslib/servlet/onepetropreview?id=OTC-6486-MS).
- Smith, I.M., and D.V. Griffiths. 2004. *Programming the finite element method*. John Wiley & Sons, Inc.
- Spanos, PD, and Roger Ghanem. 1989. "Stochastic finite element expansion for random media." *Journal of engineering mechanics* 115(5):1035–1053. Retrieved September 24, 2012 (http://ascelibrary.org/doi/abs/10.1061/(ASCE)0733-9399(1989)115:5(1035)).
- Taylor, Neil. 1993. "Prop-imperfection subsea pipeline buckling." *Marine structures* 6(October 1991):325–358. Retrieved May 24, 2012 (http://www.sciencedirect.com/science/article/pii/095183399390001J).
- Taylor, Neil, and Aik Ben Gan. 1986. "Refined Modelling for the Lateral Buckling of Submarine Pipelines." 6:143–162.
- Wang, D., D.J. White, and M.F. Randolph. 2010. "Large-deformation finite element analysis of pipe penetration and large-amplitude lateral displacement." *Canadian Geotechnical Journal* 47(8):842–856. Retrieved May 24, 2012 (http://www.nrcresearchpress.com/doi/abs/10.1139/T09-147).
- Westgate, Z.J., M.F. Randolph, D.J. White, and S. Li. 2010. "The influence of sea state on aslaid pipeline embedment: A case study." *Applied Ocean Research* 32(3):321–331. Retrieved December 5, 2012 (http://linkinghub.elsevier.com/retrieve/pii/S0141118709001151).
- White, D.J., and A.J. Barefoot. 2000. "Centrifuge modelling of upheaval buckling in sand." 314(September 2000). Retrieved May 24, 2012 (http://www-civ.eng.cam.ac.uk/geotech\_new/publications/TR/TR314.pdf).

- White, D.J., and M.F. Randolph. 2008. "Upper-bound yield envelopes for pipelines at shallow embedment in clay." *Géotechnique* 58(4):297–301. Retrieved May 24, 2012 (http://www.icevirtuallibrary.com/content/article/10.1680/geot.2008.58.4.297).
- Zhang, Dongxiao, and Zhiming Lu. 2004. "An efficient, high-order perturbation approach for flow in random porous media via Karhunen–Loeve and polynomial expansions." *Journal of Computational Physics* Volume 194(Issue 2):Pages 773–794.

# **APPENDIX A: SUPPLEMENTAL ELECTRONIC FILES**

The source codes for the programs needed to reproduce the work presented in this thesis are included in this appendix. Included are the 1D LAS and K-L RF generators and the source code for the programs used to analyze them, the BOREF program and one altered subroutine, and the three BOEF programs used to evaluate the buckling load of beams on homogeneous foundations as the foundation stiffness, length and bending stiffness change respectively. The files are organized alphabetically using the language they were written in. A Microsoft Excel spreadsheet helpful in determining program input parameters is also included.

Fortran 95 Programs			
BOREF.f95	This program solves for the critical buckling load of a beam on a random elastic foundation. A RF is used to assign the foundation stiffness. The program performs a user specified number of Monte-Carlo realizations and outputs the critical buckling load of each realizations to a file.		
BOREF.dat	Input format for the BOREF program. Needed to specify the model input parameters.		
Change_EI.f95	BOEF program that solves for buckling load and then increments the beam's bending stiffness with each iteration.		
Change_k.f95	BOEF program that solves for buckling load and then increments the foundation stiffness with each iteration.		
increment_length.f95	BOEF program that solves for buckling load and then increments the beam's length with each iteration. See (Figure 2.9)		
stability.f95	Subroutine used by BOREF program to solve for critical buckling load.		
MATLAB Programs			
1D_Analysis.m	Constructs PDFs of means and standard deviations from 5000 RF samples. See (Figure 2.21) and Figure( 2.22).		
Bisect.m	Subroutine used to find function roots to the decomposed covariance function.		
Buck_Load_Analysis.m	Completes the buckling load analysis of the file produced by the BOREF.f95 program. See (Figure 4.1) and (Figure 4.2).		

How_KL_Works.m	Separates the first 4 terms of a K-L expansion and plots them. Then plots the superposition of the first 4 terms. See (Figure 2.13) and (Figure 2.14)	
KL_RF_Gen.m	Creates a K-L RF and plots a sample of the RF. See (Figure 2.16) – (Figure 2.19)	
Plot_LAS_1D.m	Plots 1D RF produced in Fortran by LAS method. See (Figure 2.20)	
Microsoft Excel		
k_calc.xlsx	Excel Spreadsheet used to determine foundation stiffness and pipeline weight. See (Table 3.2) and (Table 3.3)	