ON THE UPPER BOUND OF THE ENERGY OF A CONNECTED GRAPH

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Abstract. New upper bounds for the energy of a connected graph are presented in this note. The upper bounds involve the independence number of the graph.

Mathematics Subject Classification (2010): 05C50
Keywords: Upper bound, Energy, Eigenvalue.

Article history:
Received 23 November 2015
Received in revised form 23 December 2015
Accepted 28 December 2015

1. INTRODUCTION

All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G be a graph of order n with e edges. We use $\delta = \delta(G)$ and $\Delta = \Delta(G)$ to denote the minimum and maximum degrees of G, respectively. The independence number, denoted $\alpha = \alpha(G)$, is defined as the size of the largest independent set in G. The 2 - degree, denoted $t(v)$, of a vertex v in G is defined as the sum of degrees of vertices adjacent to v. We use $T = T(G)$ to denote the maximum 2 - degree of G. Obviously, $T(G) \leq (\Delta(G))^2$. A bipartite graph G is called semiregular if all the vertices in the same vertex part of a bipartition of the vertex set of G have the same degree. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq ... \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of G are called the eigenvalues of G. The spread, denoted $Spr(G)$, of G is defined as $\mu_1(G) - \mu_n(G)$. The energy, denoted $Eng(G)$, of G is defined as $\sum_{i=1}^{n} |\mu_i(G)|$ (see [7]).

Several authors have obtained the upper bounds for the energy of a graph (see [5], [8], [9], [12], [13]). In this note, we will present new upper bounds for the energy of a connected graph. The results are as follows.

Theorem 1. *Let G be a connected graph with $n \geq 2$ vertices and e edges. Then*

$$Eng(G) \leq 2\sqrt{e} + 2\sqrt{(n - \alpha - 1)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2\alpha}{n - \alpha}\right)}$$

with equality if and only if G is $K_{1,1}$ or $K_{1,2}$.

Obviously, Theorem 1 has the following corollary.

Corollary 1. *Let G be a connected graph with $n \geq 2$ vertices and e edges. Then*

$$Eng(G) \leq 2\sqrt{e} + 2\sqrt{(n - \alpha - 1)\left(e + \Delta\sqrt{\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2\alpha}{n - \alpha}\right)}$$

with equality if and only if G is $K_{1,1}$.

2. LEMMAS

In order to prove Theorem 1, we need the following lemmas. Lemma 1 below is Theorem 3.14 on Pages 88 and 89 in [4].

Lemma 1. *Let G be a graph. If the number of eigenvalues of G which are greater than, less than, and equal to zero are p, q, and r, respectively, then*

$$\alpha \leq r + \min\{ p, q \},$$

where α is the independence number of G.

Lemma 2 below is Theorem 1.5 on Page 26 in [6].

Lemma 2. *For a graph G with n vertices and e edges,*

$$Spr(G) \leq \mu_1 + \sqrt{2e - \mu_1^2} \leq 2\sqrt{e}.$$

Equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $e = 0$ or G is $K_{a,\,b}$ for some a, b with $e = ab$ and $a + b \leq n$.

Lemma 3 below is obvious.

Lemma 3. *If $x \geq 0$ and $y \geq 0$, then $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$ with equality if and only if $x = y$.*

Lemma 4 below is Corollary 3.4 on Page 2731 in [10].

Lemma 4. *Let G be a graph. Then $Spr(G) \geq 2\delta\sqrt{\frac{\alpha(G)}{n-\alpha(G)}}$. If equality holds, then G is a semiregular bipartite graph.*

Lemma 5 is Theorem 1 on Page 5 in [2].

Lemma 5. *Let G be a connected graph. Then $\mu_1 \leq \sqrt{T(G)}$ with equality if and only if G is either a regular graph or a semiregular bipartite graph.*

Lemma 6 follows from Proposition 2 on Page 174 in [3].

Lemma 6. *Let G be a graph. Then $\mu_n \geq -\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$ with equality if and only if G is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

3. PROOFS

Next, we will present proofs for Theorem 1.

Proof of Theorem 1. Let $\mu_1 \geq \mu_2 \geq ... \geq \mu_p$ be the p positive eigenvalues of G and let $\rho_q \geq \rho_{q-1} \geq ... \geq \rho_1$ be the q negative eigenvalues of G. Then G has $n - p - q$ eigenvalues which are equal to zero. From Lemma 1, we have

$$\alpha \leq (n - p - q) + \min\{ p, q \}.$$

Thus $\alpha \leq (n - p - q) + q$ and $\alpha \leq (n - p - q) + p$. Namely, $p \leq n - \alpha$ and $q \leq n - \alpha$. Since $\sum_{i=1}^{p} \mu_i + \sum_{i=1}^{q} \rho_i = 0$, we have that

$$Eng(G) = 2\sum_{i=1}^{p} \mu_i = 2\sum_{i=1}^{q} |\rho_i|.$$

From Cauchy - Schwarz inequality, we have that

$$\frac{Eng(G)}{2} = \sum_{i=1}^{p} \mu_i \leq \mu_1 + \sqrt{(p-1)\sum_{i=2}^{p} \mu_i^2} = \mu_1 + \sqrt{(p-1)\left(\sum_{i=1}^{p} \mu_i^2 - \mu_1^2\right)}.$$

Similarly, we have that

$$\frac{Eng(G)}{2} = \sum_{i=1}^{q} |\rho_i| \leq |\rho_1| + \sqrt{(q-1)\sum_{i=2}^{q} \rho_i^2} = |\rho_1| + \sqrt{(q-1)\left(\sum_{i=1}^{q} \rho_i^2 - \rho_1^2\right)}.$$

Hence we get that

$$Eng(G) = \frac{Eng(G)}{2} + \frac{Eng(G)}{2}$$

$$\leq \mu_1 + \sqrt{(p-1)\left(\sum_{i=1}^{p} \mu_i^2 - \mu_1^2\right)} + |\rho_1| + \sqrt{(q-1)\left(\sum_{i=1}^{q} \rho_i^2 - \rho_1^2\right)}.$$

Then by Lemmas 2 and 3 it follows that

$$Eng(G) \leq 2\sqrt{e} + \sqrt{n-\alpha-1}\left(\sqrt{\left(\sum_{i=1}^{p} \mu_i^2 - \mu_1^2\right)} + \sqrt{\left(\sum_{i=1}^{q} \rho_i^2 - \rho_1^2\right)}\right)$$

$$\leq 2\sqrt{e} + \sqrt{n-\alpha-1}\sqrt{2\left(\sum_{i=1}^{p} \mu_i^2 - \mu_1^2 + \sum_{i=1}^{q} \rho_i^2 - \rho_1^2\right)}.$$

Since $\sum_{i=1}^{p} \mu_i^2 + \sum_{i=1}^{q} \rho_i^2 =$ the trace of $A^2 =$ the sum of diagonal entries of $A^2 =$ the sum of degrees of vertices in $G = 2e$, we get that

$$Eng(G) \leq 2\sqrt{e} + \sqrt{2(n-\alpha-1)(2e - \mu_1^2 - \rho_1^2)}$$

$$= 2\sqrt{e} + \sqrt{2(n-\alpha-1)(2e - (\mu_1 - \rho_1)^2 - 2\mu_1\rho_1)}.$$

Then by Lemmas 4, 5, and 6 we get that

$$Eng(G) \leq 2\sqrt{e} + \sqrt{2(n-\alpha-1)\left(2e + 2\sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{4\delta^2\alpha}{n-\alpha}\right)}$$

$$= 2\sqrt{e} + 2\sqrt{(n-\alpha-1)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2\alpha}{n-\alpha}\right)}.$$

If G is $K_{1,1}$ or $K_{1,2}$, it is trivial to verify that

$$Eng(G) = 2\sqrt{e} + 2\sqrt{(n-\alpha-1)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2\alpha}{n-\alpha}\right)}.$$

If

$$Eng(G) = 2\sqrt{e} + 2\sqrt{(n-\alpha-1)\left(e + \sqrt{T\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor} - \frac{2\delta^2\alpha}{n-\alpha}\right)},$$

then, from the proofs above, we have that $p = q = n - \alpha$ and $G = K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$. Since G is connected, its adjacency matrix is irreducible. From Perron - Frobenius theorem, we have that $p = 1$ (see [11]). Thus $\alpha = n - 1$. Hence G must be $K_{1,1}$ or $K_{1,2}$. □

4. ACKNOWLEDGMENTS

The author would like to thank the referee for his or her suggestions which improve the original manuscript.

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