DEPARTMENT OF ECONOMICS AND FINANCE

COLLEGE OF BUSINESS AND ECONOMICS

UNIVERSITY OF CANTERBURY

CHRISTCHURCH, NEW ZEALAND


# Boundary Algebra: A Simpler Approach to Boolean Algebra and the Sentential Connectives


**Philip Meguire**


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**Department of Economics and Finance**
**College of Business and Economics**
**University of Canterbury**
**Private Bag 4800, Christchurch**
**New Zealand**

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Philip Meguire
Department of Economics
University of Canterbury
Christchurch, New Zealand
philip.meguire@canterbury.ac.nz

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## Abstract

*Boundary algebra* [BA] is a ⟨--,(-),()⟩ algebra of type ⟨2,1,0⟩, and a simplified notation for Spencer-Brown's (1969) *primary algebra*. The syntax of the *primary arithmetic* [PA] consists of two atoms, () and the blank page, concatenation, and enclosure between '(' and ')', denoting the primitive notion of *distinction*. Inserting letters denoting, indifferently, the presence or absence of () into a PA formula yields a BA formula. The BA axioms are A1: ()()= (), and A2: "(()) [abbreviated '⊥'] may be written or erased at will," implying (⊥)=(). The repeated application of A1 and A2 simplifies any PA formula to either () or ⊥.

The basis for BA is B1: *abc*=*bca* (concatenation commutes & associates); B2, ⊥*a*=*a* (BA has a lower bound, ⊥); B3, (*a*)*a*=() (BA is a complemented lattice); and B4, (*ba*)*a*=(*b*)*a* (implies that BA is a distributive lattice). BA has two intended models: (1) the Boolean algebra **2** with base set *B*={(),⊥}, such that () ⇔ 1 [dually 0], (*a*) ⇔ *a*′, and *ab* ⇔ *a*∪*b* [*a*∩*b*]; and (2) sentential logic, such that () ⇔ *true* [*false*], (*a*) ⇔ ~*a*, and *ab* ⇔ *a*∨*b* [*a*∧*b*]. BA is a self-dual notation, facilitates a calculational style of proof, and simplifies clausal reasoning and Quine's truth value analysis. BA resembles C.S. Peirce's graphical logic, the symbolic logics of Leibniz and W.E. Johnson, the **2** notation of Byrne (1946), and the Boolean term schemata of Quine (1982).



**Keywords**: Boundary algebra, boundary logic, primary algebra, primary arithmetic, Boolean algebra, calculation proof, G. Spencer-Brown, C.S. Peirce, existential graphs.

## Dedication.

*Historical*: C. S. Peirce, the first boundary logician.

*Past*: To the memories of my father and of my mentor R.C.K.

*Present*: To George Spencer-Brown, to my spouse and helpmeet Ruth, and to my mother.

*Future*: To my daughter.

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# Preface

"*Logic is better presented as algebra*." Hehner (2004)

This book sets out a simplified variant of the classic two-element Boolean algebra, and argues that this variant is an easy way to carry out the calculations elementary logic requires.

In 1969, a British free-lance intellectual named George Spencer-Brown published a curious short book called *Laws of Form* (*LoF*). It was based on a university extension course in elementary logic he had taught for some years. In 1974, I chanced on the American mass market paperback edition of *LoF*, and over the next quarter century, intermittently struggled to grasp its content. In 2001, despite having no training in algebra or logic, I decided to make explicating *LoF* the primary focus of my academic research; thus began a decade's reflection that culminated in this book. I began this research because I was especially intrigued by Spencer-Brown's provocative claim that "…the calculus published in this text renders [standard university logic problems] so easy that we need not trouble ourselves further with them..." (*LoF*, p. viii).

*LoF* sets out a simplified approach, called the *primary algebra* (pa), to that hoary mathematical chestnut, the two-element Boolean algebra to which Paul Halmos gave the name **2**. At the outset, I thought I was primarily exploring elementary logic, but soon saw that the logic was nothing more than an interpretation of **2**. However, *LoF* confused this picture by claiming to do much more, philosophically as well as mathematically.

This book adopts a notation due to Croskin, one I call *boundary notation*, that is not only more keyboard-friendly than the notation of *LoF,* but is also more in the spirit of C.S. Peirce's graphical logic which the pa startlingly parallels (Kauffman 2001). In the pa, the concatenation of subformulae may be interpreted either as Boolean sum or as Boolean product. Because both interpretations are equally valid, the pa is *self-dual*. Because Boolean sum and product both associate, there is no need to indicate grouping, freeing up parentheses to denote Boolean complementation. *LoF* unwittingly rediscovered an enigmatic fact that Peirce discovered in the late 19th century: complementation with an empty scope (hence "()") can be interpreted as a lattice bound and primitive value.

*Boundary algebra* (BA) combines the pa with *LoF*'s *primary arithmetic*, also notated using boundary notation. Employing conventional Boolean notation for the nonce, the primary arithmetic is grounded in two facts from Boolean arithmetic: $1\cup 1= 1$, and "$1'$ may be written or erased at will." The first fact is very well known; the second is is much less so. *LoF* then invoked two algebraic postulates: the distributive law and $(a'\cup a)' = 0$. That '$\cup$' commutes and associates was asserted true by default. This book sets out a new postulate set for the pa that first makes the commutativity and associativity of $\cup$ explicit, then invokes two very familiar laws, $0\cup a = a$ and $a'\cup a = 1$, and the less familiar $a\cup(a\cup b)' = a\cup b'$. These postulates often simplify proofs (here called "demonstrations," as per *LoF*) of identities. This book also often draws on the fact that proving that $\alpha'\cup\beta = \beta'\cup\alpha = 1$ amounts to a proof of $\alpha=\beta$.

While the pa is the focus of this book, I gradually came to appreciate that boundary notation can be applied to other algebraic structures. Thus chapter 3 also speaks to lattices and groups, and mentions other algebraic structures. The pa postulates I now prefer highlight how near the BA is to an abelian group. Chapter 5 shows by example how the methods of this book greatly facilitate the sort of exercises one does in undergraduate logic courses. It also shows how boundary methods can be employed in first order logic, and sheds light on the hoary syllogism. Chapter 6 lays out the close connection between BA and Peirce's graphical logic. Chapter 7 proposes an explanation for why the pa has had little impact even though *LoF* has never gone out of print. Spencer-Brown intended that

*LoF* be a contribution to philosophy, especially to the philosophy of mathematics and logic. A glance at the reference section for this book reveals that my intentions parallel his.

This book is silent about my professional discipline, economics. More generally, it appears that economic reasoning never invokes Boolean algebra (Ba) in any way (although Boolean logic is fundamental to the computers economists use daily in their professional and personal activities). But starting around 1990, work in economic theory began to appear that drew on the generalisation of Ba called lattice theory. Topkis (1998) shows how the theories of the consumer, firm, general equilibrium, and non-cooperative games can be re-exposited using lattice theory. This is one reason why §3.3 introduces a boundary approach to lattice theory.

This book's preferred pa basis, B1-B4, and its numbering system for the derived consequences differ significantly from those in Meguire (2003). §§3.4, 4.3, 5.6, and 6.3 are all new, as are all appendices except A.6-9 and A.17. A new section on the syllogism, §5.5, replaces my earlier discussion of monadic predicate logic. §§2.3, 3.3, and the balance of §5 and §6 are revised and expanded. The former §3.4 and §6.0 are now §4.1 and §6.1, respectively. I have moved material from §§3.1, 5.0, 5.2, and 6.1 in the earlier version, to §§4.4, 5.4, and 6.2-3 here. There are, of course, revisions of detail everywhere.

**Acknowledgements.**
I thank Taylor & Francis for kindly permitting me to draw freely on Meguire (2003) in writing this book. I am also very grateful to George Klir, the editor of the *International Journal of General Systems*, for not asking me to shorten that paper, even though it filled 63 pages of his journal.

Very soon after beginning this research I discovered Kauffman (2001), which convinced me that boundary notation deserved serious mathematical and philosophical consideration. That article also led me to appreciate the connection between boundary methods and C.S. Peirce's work in algebra and logic. Kauffman has personally encouraged my research in a number of ways, including kindly hosting me for a week at the University of Waterloo. I thank Howard DeLong, Michael Pittarelli, and two anonymous referees for finding errors, Rolf Eberle for a careful reading, and David Glynn, Ivor Grattan-Guiness, and Joao Leao for encouragement. William Bricken led me to see the commutativity and associativity of pa juxtaposition as primarily metalinguistic. Art Collings closely scrutinized my account of the relation between pa and abelian groups, and pointed out that, absent a redefinition, my use of "inverse element" was confusing and mistaken.

I thank Lloyd Kannenberg and Simona Vita for each translating one of the monographs cited. I thank Sergey Brin and Larry Page for keeping access to Google free of charge, because Google made possible many of the intellectual connections discussed in this book. I thank my employer, the College of Business and Economics at the University of Canterbury, for supporting this research despite its unorthodox nature. Finally, I thank my family to its forbearance during the years I spent thinking through the matter of this book.

**About the Author.**
Philip Meguire (Ph.D., University of Chicago) is a Senior Lecturer in Economics at the University of Canterbury in New Zealand (philip.meguire@canterbury.ac.nz). In addition to economics, he has written on logic, philosophy, science, statistics, and property rights. At 13, he discovered the enchanted realm of mathematical logic, when he chanced on the *Encyclopedia Britannica*'s article on Frege, Peano, and *Principia Mathematica*, and on a reprint of Shannon (1938). His formal study of mathematics ended with a 1971 introduction to linear algebra. A few years later he formulated the ambition to learn some modern algebra in his lifetime. This book fulfils that wish. He still has the paperback copy of *LoF* he acquired in 1974. A terse summary of much of *Boundary Algebra* can be found on the author's FaceBook page.

## ***Prologue*: *Boundary Algebra and Theoretical Physics.***

Most of the technical terms appearing in this Prologue are defined elsewhere in this book; see especially the Appendix titled "A Precis of Mathematical Logic."

In a paper exploring a hypothetical ensemble of possible universes, the cosmologist Max Tegmark (1998: 1,3)[1] conjectured that "all structures that exist mathematically exist also physically..." and that "physical existence is equivalent to mathematical existence [which is] freedom from contradiction." In other words, all consistent mathematical structures have a physical model somewhere in the universe, although not necessarily within that part of it we can observe. He (Tegmark 2004: Fig. 21.7) then proposed a hierarchy of scientific theories, building on quantum field theory and general relativity, which he proposed to ground in pure mathematics. Later, Tegmark (2008) posited his "Mathematical Universe Hypothesis" to the effect that the universe itself is no more and no less than an abstract mathematical structure. Hence for Tegmark, all of physical and human reality is grounded in quantum field theory and general relativity. In turn, these theories are no more and no less than the mathematics required to exposit them.

To further his modal realist program, Tegmark (1998: Fig. 1) set out a hierarchy of mathematical structures, positing that all of mathematics (or at least those parts relevant to physics) can be built up from one of three first order theories: semigroups, Peano arithmetic, and axiomatic set theory.[2] A first order theory is just first order logic augmented with some proper axioms and a bit of ontology. In turn, Tegmark (1998; 2008) grounded first order logic in Ba, lingering over the latter in some detail. Hence for Tegmark, the mathematics of physics curiously begins with Ba.

A more conventional path to Ba and the foundations of mathematics begins with first order logic, exposits some form of axiomatic set theory as a first order theory, defines ordered sets and lattices (§3.3), then introduces Ba as a type of lattice. First order logic is viewed as an autonomous formalism, despite its connectives being a model of Ba. Tegmark's approach requires a way of expositing Ba without presupposing a background set theory. I invite the reader to judge whether the formalism set out in this book meets that requirement.

Tegmark's speculative approach to physics has affinities with that of John Wheeler. One of the two axiomatic principles undergirding boundary algebra is A2, which asserts that (()) can be written and erased at will. If enclosure by parentheses signifies a "boundary," A2 holds because, by virtue of the notion of boundary adopted here, (()) signifies "nothing." This usage appears consistent with the meaning of "boundary" in the following passage from Wheeler's talk titled "It from Bit," given at the Santa Fe Institute in 1989:

> "The boundary of a boundary is zero. This central principle of algebraic topology, identity, triviality, tautology though it is, is also the unifying theme of Maxwell's electrodynamics, [general relativity], and almost every version of modern field theory. That one can get so much out of so little, almost everything from almost no-

1. I commend to the reader's attention this sweeping, forceful, and readable essay. However, many of its key ideas are stated more precisely in Tegmark (2008).

2. Tegmark does not articulate just what axiomatic set theory he has in mind. Nor does he explain why he did not ground semigroups (and hence groups, rings, etc.) and Peano arithmetic in ZFC, when doing so is straightforward and conventional. In Metamath (http://us.metamath.org/mpegif/mmset.html ), whose set theory axioms are ZFC, the semigroup axioms are theorems **grpcl** and **grpass**. The axioms of Peano arithmetic are **peano3-5**, **nna0**, **nnmo**, **nnasuc**, and **nnmsuc**. Closure of addition and multiplication are **nnacl** and **nnmcl**.

> thing, inspires hope that we will someday complete the mathematization of physics and derive everything from nothing, all law from no law." Wheeler (1994: 302)

I close by pointing out that Tegmark's repeated invocation of "self-aware substructures" (a category which includes *homo sapiens*) in his speculative writings, and John Wheeler's (1975) "participatory anthropic principle," all bring to mind the following enigmatic passage from *LoF*:

> "Let us then consider, for a moment, the world as described by the physicist. It consists of a number of fundamental particles which, if shot through their own space, appear as waves...
>
> "Now the physicist himself [is] made of a conglomeration of the very particles he describes, no more, no less, bound together by and obeying such general laws as he himself has managed to find and to record. Thus we cannot escape the fact that the world we know is constructed in order (and thus in such a way as to be able) to see itself.
>
> "This is indeed amazing. Not so much in view of what it sees, although this may appear fantastic enough, but in respect of the fact that it can see at all. But in order to do so, evidently it must first cut itself up into at least one state which sees, and at least one other state that is seen...
>
> "It seems hard to find an acceptable answer to the question of how or why the world conceives a desire, and discovers an ability, to see itself, and appears to suffer the process. That it does so is sometimes called the *original mystery*. Perhaps, in view of the form in which we presently take ourselves to exist, the mystery arises from our insistence on framing a question where there is, in reality, nothing to question. However it may appear, if such desire, ability, and sufferance be granted, the state or condition that arises as an outcome is, according to the laws here formulated, absolutely unavoidable." *LoF*, pp. 104-05[3]; emphasis in original.

3. Page numbers and the like refer to the 1972 American paperback edition.

# Chapter 1.
# Introduction.

"*No one should fear that the contemplation of characters will lead us away from the things themselves; on the contrary, it will lead us into the interior of things. For nowadays our notions are often confused because the characters we use are badly arranged, but with the aid of characters we will easily have the most distinct notions, for we will have at hand a mechanical thread of meditation, as it were, with whose aid we can easily resolve any idea whatever into its components*."

Leibniz (1969: 193).[1]

"*…to unfold all truths of mathematics down to their ultimate grounds, and thereby provide all concepts of this science with the greatest possible clarity, correctness, and order, is an endeavour which will not only promote the* thoroughness *of education but also make it* easier."

Bolzano.[2] Emphasis in original.

"*Symbols have the same importance for thought that discovering how to use the wind to sail against the wind had for navigation. Thus, let no one despise symbols! A great deal depends on choosing them properly… without symbols we would scarcely lift ourselves to conceptual thinking*."

Frege (1972: 84), writing in 1882.

*"By relieving the brain of… unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of* [humanity]."

Whitehead (1948: 39), quoted in Roberts (1973: 118).

"*...a proper notation is like a live teacher, gently guiding us into the clear and keeping us from error and wooliness. A real effort should be made to express* [logical] *principles in as perspicuous a notation as possible*." Martin (1978: 41).

In 1969, Spencer-Brown[3] published the first edition of his *Laws of Form*[4] (hereinafter *LoF*), which begins by laying out a minimalist formal system, called the primary arithmetic, arising from the primitive mental act of making a distinction. He then let letters denote, indifferently, a distinction or its absence and obtained the primary algebra, the next rung on his ontological ladder. The primary arithmetic and algebra featured a single primitive symbol —'$\overline{\phantom{x}}|$' in *LoF* and '()' here—indicating the *boundary* between the two states generated by a distinction. Meguire (2003) proposed the name *boundary algebra* (BA) for the union of the primary arithmetic and algebra. BA has two intended interpretations: the Boolean algebra **2** (Halmos and Givant 1998: 55), so called because its base set has cardinality two; and *boundary logic*, an equational variant of the classical bivalent sentential

1. Letter to Tchirnhaus, dated May 1678, quoted in translation by Ishiguro (1990: 44).

2. Bolzano, Bernhard, 1804. *Considerations on Some Objects of Elementary Geometry*. Republished in Ewald (1996: 172).

3. George Spencer-Brown entered Cambridge in 1947, the year Wittgenstein resigned his Chair; hence he cannot have studied under Wittgenstein (who only taught advanced classes), as some claim. He graduated with Honours in philosophy and psychology, then taught philosophy at Oxford, 1952–58. In 1957, he published *Probability and Scientific Inference*. In 1963-68, while writing *LoF*, he taught mathematics in the University of London's extramural program. He has held visiting appointments at Maryland, Stanford, and Western Australia. *Source*: http://www. lawsofform.org/gsb/vita.html .

4. Page numbers and the like refer to the 1972 American paperback edition.

logic (hereinafter the *calculus of truth values* or CTV[5]). The research that culminates in this book began with my admiration for the simplicity and elegance of boundary logic.

The primary algebra, as set out in *LoF*, consists of 2 axioms, 11 consequences in the form of equations, the usual rules governing uniform replacement and the substitution of equals for equals, 9 "canons," and a few informal definitions. There are 18 (meta)theorems, including but not limited to the standard metatheory of the CTV: soundness, completeness, and postulate independence. All this fills less than 55 small pages of large type.[6] The balance of the book consists of 20pp of front matter, a chapter claiming that certain recursive Boolean equations have "imaginary" solutions, and 60pp of notes and appendices relating the primary algebra to the CTV as it was understood circa 1940, and to elementary Boolean algebra and syllogistic logic. Much of this peripheral material is frankly speculative and digressive. This book does not evaluate *LoF*'s claim that it is useful to think of certain recursive Boolean equations as having "imaginary" solutions.

I take as given that notational innovations can facilitate both the teaching of extant mathematics and the invention of new mathematics. Since the dawning of modern logic in 1847, three notations for the truth functors have acquired a significant following:

- The notation begun by Boole, revised by C. S. Peirce, and systematised by Schröder in the 1890s. This became the Boolean algebra **2**, with the following arithmetic/set theory/logic interpretations: $a$+$b$-$ab$ / $a\cup b$ / $a\vee b$, $a\times b$ / $a\cap b$ / $a\wedge b$, and 1-$a$ / $\overline{a}$ / $\sim a$. Boolean algebra can be cast in terms of equations and inequalities ('$\leq$' interprets the conditional) with unknowns. A classic treatment is Lewis (1918: chpts. II, III). Thorough modern expositions include Rudeanu (1974) and Givant and Halmos (2009);[7]
- The binary prefix (Polish) notation introduced by Lukasiewicz in the 1920s. Fully exploiting the fact that formulae are ordered trees, this notation requires no brackets because it is free of the ambiguities plaguing its infix rivals;
- The standard notation for first order logic, originated by Peano and modified by Whitehead and Russell in their *Principia Mathematica* (*PM*). While the notation of *PM* was canonical for much of the 20th century, it has been displaced by a variant notation due to Hilbert, Tarski, and their students, one employing '$\wedge$', '$\rightarrow$', and '$\leftrightarrow$' in place of '**.**', '$\supset$', and '$\equiv$', and brackets instead of the dots of Peano and *PM*. This book makes free metalinguistic use of this notation.[8]

5. Synonyms for the CTV include Bostock's (1997) *logic of truth functors, sentential calculus* (Kalish et al 1980), *logic of connectives* (LeBlanc and Wisdom 1976), *propositional calculus* (Church 1956; Mendelson 1997; Halmos & Givant 1998; Cori and Lascar 2000), *propositional logic* (Smullyan 1968; Epstein 1995; Wolf 1998; Hodges 2001), *statement calculus*, (Stoll 1974), *Propcal* (Machover 1996), *truth functional logic* (Hunter 1971; Quine 1982), the *theory of deduction* (*PM*).

6. Smullyan (1968: chpts. I, II) covers in 27pp as much ground as *LoF*. Nidditch (1962), Goodstein (1963: chpt. 4), and Mendelson (1997: chpt. 1) require 81, 17, and 35 pages, respectively, and include the Deduction Theorem. Hunter (1971: §§15-36), Cori & Lascar (2000: chpt. 1, §§4.1-2), Stoll (1974: §§2.1-4, 3.5), Machover (1996: chpt. 7), Epstein (1995: §§II.J-L), and Schütte (1977: chpt. I) require 79, 62, 43, 40, 31, and 12 pp, respectively. All treatments prove the CTV consist and complete; some also prove it compact.

7. On Boolean algebra, see the references in the Bibliographic Postscript. Boolean algebra is distinct from Boole's "algebra of logic" (on which see Kneebone 1963: 184-88, and Lewis 1918: §I.V), in good part because alternation in Boole was exclusive rather than inclusive. Hailperin (1986: 140) argues that the "algebra of Boole" is a "commutative ring with unit, having neither additive nor multiplicative nonzero nilpotents."

8. On the importance of Peano's notational innovations, see Quine (1995: §28). Kneebone (1963: 49-51, §6.4, 87) discusses Polish notation and the notation of Frege (never emulated). The notations '$\forall x$' and '$\exists x$' descend from the '$\prod x$' and '$\sum x$' of Peirce (*W*5: 162-90) and his student Mitchell, a notation the Poles

The notation for BA is, arguably, a notational innovation of the same order as the above. The notation of *LoF*, however, cannot be reproduced using standard word processing software. Hence I employ an alternative notation Bricken and others have employed to discuss *LoF*, a notation in which Latin letters, '()', and the blank page are atomic. To the best of my knowledge, Croskin (1978: 187) was the first to employ this notation, a mild variant of that in Peirce (4.378-383, 1902).[9] The syntax of the primary arithmetic consists of balanced parentheses strings. Inserting Latin letters anywhere in a primary arithmetic formula yields a primary algebra formula.

The domain of analysis is the real and complex numbers; that of geometry, space, Euclidian or not. Algebra studies mathematical structure without commiting to a specific domain. Logic studies the mathematical structure of statements and deductive systems so that, as Peirce, and Halmos and Givant (1998: §§35-39), have maintained, logic can be viewed as an application of algebra. Boundary logic is very much a case in point.[10]

The content of this book comes under four headings: syntax, semantics, proof theory, and the history of ideas. My purpose is mainly expository, in that much of what I say has already appeared somewhere in the literature. *LoF* was written as if the mathematics it advocates were wholly new; in fact, Spencer-Brown was unwittingly reinventing the wheel. Kauffman (2001) shows how the notation of *LoF* was anticipated by C. S. Peirce —in papers written in 1886 but not published in full until 1993—and Nicod (1917). Kauffman also touches on Peirce's *alpha* existential graphs, discussed in §6, whose semantics and proof theory are very much in the BA spirit. In §7, I give possible reasons why the mathematics and logic of *LoF* have made no headway since *LoF* was first published nearly 40 years ago.

This book includes an appendix, titled "A Precis of Mathematical Logic" (hereinafter "Precis"), intended for readers lacking prior exposure to formal logic. I assume the reader has an intuitive grasp of elementary set theory, including the notion of function. *LoF*, however, is quite innocent of set theory, other than a brief mention, near the end and free of all rigor, of the Boolean algebra of sets.

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adopted. While Prior (1962) and Zeman (1973) adopted Polish notation, and texts still mention it, it appears to have died out. The syntactic ('⊢') and semantic ('⊨') turnstiles should be seen as part of the standard metalanguage.

9. To Bricken I owe my awareness of Croskin's work. Here and elsewhere, I cite Peirce's *Collected Papers* (Peirce 1931-35) in the following standard manner: *x.y, z* refers to section *y* of volume *x* of the *Collected Papers*, first published or written in year *z*.

10. Bricken (2001), building on James (1993), uses boundary notation to explore analysis (e.g., by representing $e^x$ as (*x*) ) and the integers, rationals, reals, etc. Kauffman's (2001) term *boundary mathematics* refers to formal systems, mainly algebraic structures, having one or more syntactical boundaries and mathematical as well as logical interpretations. §3.4 shows how to notate a number of common algebraic structures using boundary notation.

# Chapter 2.
# The Primary Arithmetic (PA).

"*The theme of* [LoF] *is that a universe comes into being when a space is severed or taken apart… By tracing the way we represent such a severance, we can begin to reconstruct… the basic forms underlying linguistic, mathematical, physical, and biological science, and can begin to see how the familiar laws of our experience follow inexorably from the original act of severance*." *LoF*, p. v.

"*A common* image schema *of great importance in mathematics is the* Container schema [having] *three parts: an Interior, a Boundary, and an Exterior. This structure forms a gestalt, in the sense that the parts make no sense without the whole. There is no Interior without a Boundary and an Exterior, no Exterior without a Boundary and an Interior, and no Boundary without sides. The structure is topological in… that the Boundary can be made larger, smaller, or distorted and still remain the Boundary of a Container schema*." Lakoff & Núñez (2001: 33).[1]

In the beginning there is a featureless *space*, normally a plane surface, upon which *symbols* (a primitive notion because undefined), especially '(' and ')', may be inscribed. A pair of '(' and ')', in that order, divide the space into two parts, one "inside" or "between" '(' and ')', and the balance being outside '(' and ')'. A *sign* is one or more symbols representing some human intention. The sign '()' marks the *boundary*[2] between these two parts of the space. Letting $x$ be a token or marker, $x$ can be inside a boundary, as in '($x$)', or outside, as in '()$x$'. Each side of a boundary forms one of a pair of mutually exclusive and exhaustive entities. '()' can also indicate either member of this pair of entities, in which case '()' signifies a boundary's *content* as well as its *fact*. A boundary '()' results when one wishes to distinguish that which is inside '()' (which may be nothing) from the remainder of the space. 2.0.1 attempts to codify these admittedly enigmatic ruminations:

**2.0.1. Definition.** The sign '()' inscribed in some *space* both signifies a *boundary* and denotes the *marked state*. Any space on which '()' does not appear represents the *unmarked state*.

The unmarked state can also be called "not ()", "nothing", "the void"; I will grant it a symbol shortly. The formal system whose sole atoms are '()' and "the void" is the *primary arithmetic* (*LoF* also employs the term "calculus of indications"), which I abbreviate to PA. *Boundary logic*[3] begins by interpreting '()' as one of *true* or *false*. (I will often refer to the usual notation for first order logic as *conventional logic*.)

Now consider a plane surface, of indefinite extent, on which the four symbols (not necessarily interpreted or related to each other) #, ∴, ϶, and ♦ are inscribed, as in Fig. 1. Note that '∴' appears twice. If one wished to represent Fig. 1 more concisely, one could assert that the symbols in that Figure form a *list* and write '#∴϶∴♦', without separators such as commas. Note that multiple instances of the same object, namely '∴', are allowed. One could also depict Fig. 1 as a *set* and

1. *Container schema* and *image schema* are terms of art in cognitive science, and are defined in Lakoff & Núñez (2001).

2. 'Boundary,' as employed in this book, is unrelated to the use of that term in topology. At the same time, I do not wish to deny that topology and the *mereotopology* of Simons (1987: §2.10) and Casati and Varzi (1999: chpts. 4,5) may shed light on boundary mathematics; see §5, fn. 6.

3. By grounding BA in the "mental act" of drawing a distinction, Spencer-Brown and I distance boundary logic, for good or ill, from Kneebone's (1963: §12.2.1) reading of *PM*, in which "…logic deals with propositions, not with mental acts; and it follows that… mathematics likewise is essentially propositional."

write {#,϶,∴,♦}, keeping in mind that a set cannot have multiple instances of any of its members; hence the set corresponding to Fig. 1 has four members but the list has five elements.

```
#          ∴
     ϶
∴          ♦
```

Fig. 1.

For both lists and sets, the order in which elements are listed is immaterial. Hence when using list or set notation, the elements can be permuted at will without affecting meaning, consistent with the symbols having no necessary relation to each other. BA should be thought of as consisting of spatial arrangements such as Fig. 1. Wishing to defer to typographical custom in mathematics and conventional logic, and in order to save space, I represent such arrangements as lists.

Any closed curve that does not intersect itself, and with a distinguishable 'inside' and 'outside,' can depict a boundary. Boundaries can also be nested at will. The objects of BA should be seen as inscribed on a surface of dimension at least 2. All objects on a given side of a boundary, other boundaries excepted, have equal status. Hence boundary purists deem jejune the algebraic notions of commutativity and associativity when these are applied to BA. I do invoke these notions, but only as metalinguistic manners of speaking, doing so mainly out of deference to readers accustomed to conventional notations for logic and Boolean algebra. I revisit this syntactic curiosity in §3.2.

### 2.1. PA: Syntax.

''*Although some material may be very familiar, ...one of our main themes is the development of new perspectives for familiar concepts. Hence… these concepts* [should] *be re-appraised, and explicit discussion be provided of things that to many will have become second nature*."

Goldblatt (1984: 4).

**2.1.1. Definition**. A PA *symbol* is an instance of '(', ')', '⊥', and '='. 'Symbol' is otherwise undefined. The symbol '⊥' is *improper*; the remaining symbols are *proper*.

**2.1.2. Definition**. A PA *string* consists of a single symbol, or of two or more symbols juxtaposed.[4] A PA *formula* is a string constructed recursively as follows:

*Base case.* Any *atomic formula*: the string '()', the blank page, any space between symbols.
*Recursive rule*. If 'α' and 'β' are formulae, '(α)' and 'αβ', i.e., 'α' and 'β' juxtaposed, are formulae. This rule may be repeated any finite number of times.[5]

2.1.2 introduces an important notational convention: *Greek letters are metalogical symbols standing for arbitrary formulae or strings*. (N.B. *LoF* also invokes this convention, albeit silently.) I revisit the blank page as atomic formula in §2.2.

---

4. I intend 'juxtaposition' (a term I appropriated from Hehner 2004) and 'concatenation' to be synonymous. String concatenation is mathematically nontrivial, by virtue of its being a model of a semigroup (monoid, if the notion of *empty string* is accepted) over a base set whose members are *atomic strings* (Halmos and Givant 1998: §12).

5. I crafted 2.1.2 so as to resemble the recursive formula definitions in conventional treatments of logic (e.g. Bostock 1997: 21; Machover 1996: §7.1.2) and other formal systems. In linguistics and computer science, 2.1.2 defines a *Dyck language of order 1* with a *null alphabet*, the simplest instance of a Chomskian *context-free language* (Davis et al 1994: §10.7).

**2.1.3. Definition**. If the formula ‘α’ can be obtained by applying the recursive rule in 2.1.2 to the formula ‘β’ one or more times, then ‘β’ is a *proper sub-formula* of ‘α’. A *subformula* of ‘α’ is either ‘α’ itself or a proper subformula of ‘α’.

*LoF* has no synonym for subformula, a lacuna giving rise to occasional awkward periphrases. In no way are ‘string’ and ‘formula’ peculiar to BA. I employ these words mainly to facilitate expositing BA to those who have studied formal systems in the usual way. ‘String’ and ‘formula’ are synonyms for ‘arrangement’ and ‘expression,’ which *LoF* employs without defining. I will write ‘arrangement’ only when quoting *LoF*.

In everyday language, a formula must satisfy the rule: reading from left to right, *any left parenthesis must eventually be paired with a right parenthesis*. A string satisfying this rule is known as a *balanced parenthesis string*. The following algorithm determines whether a given string is balanced and hence a formula.

**2.1.4. Algorithm.**
a) Let $d$ be a counter variable, and initialise it to 0.
b) Starting from one end of the string and working towards the other end, increase $d$ by 1 for each ‘(‘ and reduce $d$ by 1 for each ‘)’.
c) IF $d$ is ever positive [negative], it must always be nonnegative [nonpositive].
d) ELSE the string is not a formula. STOP
e) ELSE IF $d$ is nonzero when the end of the string is reached, the string is not a formula. STOP
f) ELSE the string is a formula.
*End of Algorithm*

2.1.4 is required only because not all possible strings involving only ‘(‘ and ‘)’ are well-formed. This is a drawback (and, I trust, a minor one) of the notation I propose. This problem does not arise with the notation of *LoF*, its signal virtue. That notation is based on the symbol ‘⌉’, called the *mark*, and placed to the *right* and *over* that which ‘()’ encloses. E.g., ‘$\overline{\overline{\urcorner\urcorner}\urcorner\urcorner}$’ in *LoF* corresponds to ‘((()())())’ here. All possible concatenations and nestings of ‘⌉’ are well-formed, as long as the upper part of any ‘⌉’ extends over the left extremity of all ‘⌉’ under it.

**2.1.5. Definition**. When applying 2.1.4 to ‘α’, the absolute value of the counter $d$ at any point inside ‘α’ is the *depth* of ‘α’ at that point. Henceforth, $d$ refers to this absolute value. The place in ‘α’ where $d$ attains its largest value is the *greatest depth,* $d^*_\alpha$, of ‘α’.[6]

**2.1.6. Definition**. Corresponding to every value of $d$ is a *subspace* $s_d$. The subspace $s_d$ *pervades* any sub-formula situated at depth $d$. Given some formula ‘α’, the subspace of depth 0, $s_0$, is the *pervasive space* of ‘α’. Let ‘β’ be a sub-formula occurring at depth $d$ of ‘α’. Then the subspace $s_d$ *pervades* ‘β’ (is the *pervasive space* of ‘β’).

‘()’ marks the boundary between the subspace “inside” and the pervasive space “outside.” Each subspace $s_j$ contains all subspaces $s_k$, for which $i<k\leq d^*$. Hence a formula creates a system of $d^*$ nested subspaces. (The terms *space* and *subspace* are also fundamental to analysis and linear alge-

6. A BA formula can be seen as a *finite ordered tree* (Smullyan 1968: 4-5), whose *level* corresponds to depth in the sense of 2.1.5. *LoF* operationalizes formula depth in a different manner.

bra, but this is coincidental.) A subspace can pervade more than one sub-formula, a fact giving rise to the following definition:

**2.1.7. Definition**. Given a formula of the form '…((β)(γ)…)…', the sub-formulae '(β)', '(γ)', etc. constitute *divisions* of the subspace containing '(β)', '(γ)', etc, and these sub-formulae are said to *divide* the subspace.

I now explicate '⊥'.

**2.1.8. Definition**. The symbol '⊥' represents the *null formula*. It designates "the unmarked state", "nothing", "the void".

'⊥' is a sub-formula of every formula, so that the null formula is a formula in the same sense that the empty set is a set.[7]

**2.1.9. Definition. () and** ⊥ are the *primitive values* of BA. *B*={(),⊥} is the *base set* of BA. In Ba, a synonym for base set is *carrier*.

*LoF* invokes the Principle of Relevance (cf. §3.1) to argue that there is no need for a symbol to denote the unmarked state or the null formula: "...a recessive value is common to every [PA formula] and... by this Principle, has no necessary indicator there" (*LoF*, p. 43). Although *LoF* is silent about the null formula, it (pp. 15-18, 37, 47-48, 56-57) repeatedly employs '*n*' to refer to the unmarked state; the semantics of '*n*' and '⊥' are identical. The counterpart to '*n*' is '*m*', denoting the marked state. *n* and *m* are presumably metalinguistic.

By including the symbol '⊥' in BA, I defer to established usage in logic and Boolean algebra, which all feature a symbol akin to '⊥'. Moreover, '=' with nothing to one side (and strings of this form do occur in *LoF*) leaves the mind guessing at a possible typographical problem. '⊥' is a placeholder like the number 0; I created it in good part simply out of respect for the dyadic character of '='. The controversy surrounding the role of '⊥' in BA is analogous to that surrounding a possible role for the null individual in mereology, the formal theory of the relation of part to whole. For a review of this controversy, see the Appendix titled "The Controverted Ontology of the Null Individual." It would seem that both controversies stem from assuming that a name necessarily denotes some thing, when in fact the null individual and '⊥' denote not things but "nothing" and "the unmarked state," respectively.

## 2.2. PA: Axiomatics, Simplification, Semantics.

"*I have for a long time been urging… the importance of demonstrating all the secondary axioms… by bringing them back to axioms which are primary, i.e., immediate and indemonstrable…*"

Leibniz (1996: 408).

The *LoF* axioms are:

A1. ()() = () A2. (()) = ⊥.

7. 'T' ('top') and '⊥' ('bottom') are standard notation for the bounds of a bounded lattice (3.3.6), and are the primitives of Hehner's (2004) binary algebra. '⊥' is also analogous to Bostock's (1997: 12-13) *empty sequent*, false by definition.

In Spencer-Brown's inimitable Zen-like words (*LoF*, pp. 1-2):

"A1. The value of a call made again is the value of the call. *Calling*
A2. The value of a crossing made again is not the value of the crossing. *Crossing*"

A2 is arguably self-evident, A1 perhaps less so. I ignore the distinction *LoF* draws between "axiom," meaning Calling and Crossing (also referred to as *Number* and *Order*) stated in natural language, and the corresponding "arithmetic initials" ()()=() and (())=⊥.

A1 and A2 reveal that '()' has an "inside" distinguishable from its "outside" by virtue of what each does to another instance of '()' with which it is in contact. A1 lays down that the exterior is idempotent; A2, that the interior is nilpotent. A1 and A2 can also be seen as defining '(' and ')' as the two halves of a single operator, with A1 [A2] being the defining property of the convex [concave] side of a parenthesis. When a pair of parentheses enclose a sub-formula, the pair functions as an (unary) operator; the subformula— which may be no more than an instance of '()'—is the corresponding operand. Note that '(())' has an interior while the constant '⊥' does not. This is the sense in which '⊥' is synonymous with '(())' but not redundantly so.

Pp. 104-06 of *LoF*, perhaps the most sweeping and poetic pages of the book, were intended to lead the reader to a deeper understanding of the plausibility of A1 and A2. These pages suggest Spencer-Brown saw A1 and A2 as ineluctable features of not only the abstract realm, but also of the physical universe and how humans perceive it.

A1 and A2 do not make explicit what value to assign to a formula containing both '⊥' and parentheses. *LoF* is not to blame for this lacuna; having no null formula, it states A2 with nothing to the right of '=', namely as "(())= ". I propose to remedy this omission in either of two ways:

1. Invoke a third axiom making explicit that when '⊥' is combined with '()' in any way, nothing is altered:

A3. ⊥() = (⊥) = ()⊥ = (); ⊥⊥ = ⊥.

Or, in the Zen-like spirit of *LoF*:

A3. The void is perfect inaction.

2. Restate A2 as follows: *An instance of* '(())' *or* '⊥' *may be written anywhere or erased at will*. I submit that a generous reading of *LoF* points to this definition. The four cases covered by A3 above then all follow.

Under either approach, '⊥' can appear anywhere in a formula without affecting its meaning or value. '⊥' is a synonym for '(())' and as such is, in all essentials, optional. Hence parentheses alone suffice to build any PA formula. A2 as restated above has a curious and deeper consequence. Since '(())' aliases with the blank page, and since by 2.1.9 '(())' is a primitive value, the following three things stand for the same atomic formula and can denote the same primitive value: the "space" between any two juxtaposed symbols, an entire blank page, and any blank part thereof. We shall see in §6.1 that Peirce reached a related conclusion at the end of the 19th century while devising his graphical logic.

Table 2-1 summarises the discussion thus far, with each cell in that Table giving one of the six possible ways of forming pairs from '()' and '⊥', keeping in mind that '()' has both an "interior" and "exterior". Table 2-1 and Definition 2.1.2 essentially define the PA. A1 and A2 each yield the value of one cell. The remaining four cells contain the string "A3", which stands for either of the two paths proposed above: either invoke a new axiom, A3, or alter A2 to allow '⊥' to be erased at will. Henceforth, I will take the latter course, so that A2 includes the four equalities in the cells labelled A3. The cell '⊥⊥ =⊥' implies that strings consisting of iterated instances of '⊥' designate the null formula. Once I define Boolean algebra (§3.3), it will be clear that Table 2-1 defines the corresponding Boolean arithmetic. A numerical interpretation of that arithmetic is $1 \Leftrightarrow ()$, $1-\alpha \Leftrightarrow (\alpha)$, and $\max(\alpha,\beta) \Leftrightarrow \alpha\beta$.[8]

**Table 2-1.**
Axiomatic PA operation table,
With a numerical interpretation.

| 1a. Juxtaposition | | | 1b. Enclosure |
|---|---|---|---|
| ()() = () A1<br>0×0 = 0 | ⊥() = () A3<br>1×0 = 0 | | (()) = ⊥ A2<br>1−0 = 1 |
| ()⊥ = () A3<br>0×1 = 0 | ⊥⊥ = ⊥ A3<br>1×1 = 1 | | (⊥) = () A3<br>1−1 = 0 |

A1, A2, and Table 2-1 may appear trivial. However, in 3.3.2 we shall see that A1 and A2 imply that *B* is a partially ordered set. Ordered sets are rich in mathematical (cf., e.g., Davey and Priestley: 2002) and logical (Curry 1963: chpt. 4) content.

Table 2-1 implies that a pair of parentheses can serve as either an operator or an operand. In a sub-formula of the form '·(·)·', the parentheses can be seen as denoting a three-place operator (functor), such that one or more of '·' can be left blank. Leaving all three places blank takes us back to the boundary sign '()', a primitive value and hence an operand. Chapters 3 and 4 will say more about '()' as operand and operator. In the PA, the distinction between operator and operand is purely contextual and has effectively degenerated, a situation to which *LoF* (p. 88) refers as the "partial identity of operator and operand."[9] Any notation proposed for the PA must do justice to this degeneracy. I chose parentheses with this degeneracy uppermost in mind.

Definitions 2.2.1, 2.2.2, and 2.2.4 lay out the principle use of Table 2-1, and the meaning of the symbol '='.

**2.2.1. Definition**. A *step* is any alteration of a formula justified by invoking the contents of a cell in Table 2-1. Steps are of two kinds: slimming and expansion. Let $\alpha$ be some PA formula. To replace '()()' with '()', or to erase an instance of '(())' occurring in $\alpha$ is to *slim* $\alpha$. To replace an instance of '()' with '()()', or to insert '(())' anywhere in $\alpha$, is to *expand* $\alpha$.

---

8. Since the two cells in Table 2-1b form a *dual pair* (cf. 4.1.3), duality (§4.1) suggests that only one of these cells is strictly necessary. Note that $\max(\alpha,\beta) = (\alpha+\beta)-(\alpha\times\beta)$. The dual interpretation is $1 \Leftrightarrow \perp$ and $\min(\alpha,\beta) \Leftrightarrow \alpha\beta \Leftrightarrow \alpha\times\beta$. Another numerical interpretation of Table 2-1 is $\perp \Leftrightarrow -1 < 1 \Leftrightarrow ()$, $\alpha\beta \Leftrightarrow \max(\alpha,\beta)$, and $(\alpha) \Leftrightarrow -\alpha$ (dually, $() \Leftrightarrow -1 < \perp \Leftrightarrow 1$, $\alpha\beta \Leftrightarrow \min(\alpha,\beta)$). A Boolean interpretation of the PA is $() \Leftrightarrow 1$, $\perp \Leftrightarrow 0$, $(\alpha) \Leftrightarrow -\alpha$, and $\alpha\beta \Leftrightarrow \alpha\cup\beta$ [$\alpha\cap\beta$]. In this case, A1 is $1\cup1=1$; A2, $-1=0$, $0\cup1=1\cup0=1$, $0\cup0=0$, $-0=1$. Under this interpretation, A1 and A2 are (3b) and (7b) in Shannon (1938). For more re arithmetical axiomatizations of Ba and sentential logic, see §6, fn. 17.

9. *LoF* (p. 88) asserts that this partial identity characterizes Ba as well, albeit in disguised form.

A1 (A2) justifies the first (second) action mentioned in the last two sentences of 2.2.1. Slimming and expansion can be thought of as an inverse pair of operations, whose arguments are formulae.

**2.2.2. Definition**. To *simplify* a formula ‘α’ is to slim ‘α’ one or more times until the result is a member of *B*. That member is the *value* of ‘α’, denoted ‘|α|’.

**2.2.3. Algorithm.** The following algorithm operationalises what is meant by simplifying the formula ‘α’ with greatest depth $d^*_\alpha$:

1. Go to the subspace of ‘α’ whose depth is $d^*_\alpha - 1$.
2. IF this subspace pervades one or more sub-formulae of the form ‘()()’, THEN invoke A1 and replace every ‘()()’ with ‘()’. Repeat this step until only one ‘()’ is left.
3. ELSE go to the subspace at depth $d^*_\alpha - 2$, which must pervade one or more sub-formulae of the form ‘(())’. Invoke A2 and eliminate all instances of ‘(())’.
4. IF what remains of α is ‘()’ or ’⊥’, THEN STOP.
5. REPEAT 2 through 4, decrementing the depth of the subspace by 1 each time.[10]

*End of Algorithm*

*Remark*. The *simplification* of ‘α’ is the value of α when the algorithm 2.2.3 terminates. For more about on simplification and expansion see §2.3, specifically T3, T4, and the Hypothesis of Simplification.

I now define the symbol ‘=’ (the ’equal sign’) as follows:

**2.2.4. Definition**. The string ‘α=β’, called an *equation* or *identity*, signifies that the formulae ‘α’ and ‘β’ have the same simplification and hence are *equivalent*.[11]

Corresponding to the adjective ‘equivalent’ is the noun ‘equivalence’ which I , following *LoF*, denote by the mathematician’s common-garden ‘=’. The equation ‘α=β’ implies nothing whatsoever about the literal appearance of ‘α’ and ‘β’. I revisit equivalence in 2.3.8-10.

The semantics of the PA are an elusive aspect of *LoF*, and perhaps the greatest obstacle to a wider appreciation of BA. I intend by “semantics of the PA” no more than some asserted *interpretation* of the PA, defined as:

**2.2.5. Definition**. An *interpretation* of the PA is a one-to-one correspondence between *B* and another two-member set.
One such two-member set is {1,0}, containing the Boolean primitives in Table 2-1. Other possibilities include {**V**,**Λ**} (set theory), {top ‘T’, bottom ‘⊥’} (lattice theory), {True,False} (logic), and the everyday meaning of {On,Off}. Hehner (2004) even proposes the numerical reading {∞,-∞}. I do not claim that these minimalist semantics necessarily do justice to the philosophical intent of *LoF*.

*LoF* leaves the PA uninterpreted, saying little more about the possible semantics of the PA than what can be found in the first few paragraphs of its Appendix II, and barely hinting at what I say in §2.0 above. Could the degeneracy of the PA extend even to the distinction between syntax and semantics?

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10. Compare this algorithm to that on p. 13 of *LoF*. After devising 2.2.3, I discovered the following related definitions in Machover (1996): *degree of complexity* of a formula (§§7.1.7, 8.1.8), *weight* of a string (§§7.1.9, 8.2.1), *parity* of a formula (§8.8.3).

11. Synonyms for ‘α=β’ include ‘α↔β’ (common in conventional logic), and ‘α ⫤⊨ β’ (Bostock 1997: 36).

Croskin (1978) concludes that A1 and A2 form a one-to-one mapping between *B* and '()()' and '(())', the two simplest nonatomic formulae. I prefer to see A1 and A2 as arbitrary defining choices of the sort that necessarily ground any formal system. Mapping each of '()()' and '(())' onto any member of {(), ()(), (())} gives rise to 2x3=6 possible pairs of distinct axioms, one of which is (A1, A2). §3.4 shows that (A1, (())=() ) hold in group theory. I leave to future research possible interpretations (e.g., Peano arithmetic, sets, multisets) of the four other possible axiom pairs.

PA *Semantics*: *A Technical Digression*.
Given some standard notions from mathematical logic, the following argument renders plausible interpreting '()' and '(())' as the classical bivalent truth values. Let the *extension* of an *n*-place atomic formula be the set of ordered *n*-tuples of individuals that satisfy it (i.e., for which it comes out true) (Carnap 1958: §10b). Let a sentential variable be a 0-place atomic formula; its extension is a classical truth value by definition. An ordered 2-tuple is known as an *ordered pair*, whose standard set theoretic definition is $\langle a,b\rangle =_{\mathbf{df}} \{\{a\},\{a,b\}\}$, where *a*,*b* are individuals. Ordered *n*-tuples for any $n>2$ may be constructed from ordered pairs in a well-known recursive way (cf. Stoll 1963: §1.6). Bostock (1997: 83, fn. 11, 12) states that Scott (whom he does not cite) has argued that the extension of a sentential variable can also be seen as the empty ordered pair (ordered 0-tuple), $\{\{\},\{\}\}$, equal to $\{\{\}\}$ by set extensionality. Hence T interprets $\{\{\}\}$. Reading $\{\}$ as F follows naturally, if the curly braces of set notation are read as a boundary notation.

One can go much further. Angell (1960) showed how to notate denial, conjunction, quantified variables, and set membership using only parentheses. Angell's notation requires setting A1 and A2 aside. Angell codes the first quantified variable as '()', the second as '()()', and so on. Letting $\phi$ and $\varphi$ be metalogical notation for formulae, the truth functional part of Angell's notation is $(\phi) \Leftrightarrow {\sim}\phi$ and $(\varphi\phi) \Leftrightarrow \phi\wedge\varphi$. The outer parentheses of $(\varphi\phi)$ are needed only because of peculiarities of Angell's notation for quantification and set membership, not described here. Because his notation allows for membership, Angell unwittingly showed that PA syntax suffices for set theory. Angell supplied no axioms or proof theory, as he only wanted to show that his notation was capable of expressing the system of Quine (1951), whose primitives were the Sheffer stroke, universal quantification, and set membership.

## 2.3. PA: Canons and (Meta)theorems.

"*The more important structures of command are sometimes called canons. They are the ways in which the guiding injunctions appear to group themselves in constellations, and are thus by no means independent of each other. A canon bears the distinction of being outside (i.e., describing) the system under construction, but a command to construct (e.g., 'draw a distinction'), even though it may be of central importance, is not a canon. A canon is an order, or set of orders, to permit or allow, but not to construct or create*." *LoF*, p. 80.

"*…the primary form of mathematical communication is not description but injunction… Music is a similar art form, the composer does not even attempt to describe the set of sounds he has in mind, much less the set of feelings occasioned through them, but writes down a set of commands which, if they are obeyed by the performer, can result in a reproduction, to the listener, of the composer's original experience*." *LoF*, p. 77.

**The Six Canons.**
*LoF* includes nine canons, which Spencer-Brown intended to serve mainly as *injunctions*, i.e., directives (see quote above). The PA *canons* and *theorems* establish protocols for altering, and rea-

soning about, PA formulae. They are *about* the primary arithmetic, hence metamathematical. The PA is too elementary for logical/mathematical proof as conventionally understood to apply to its formulae. In a sense, the canons and theorems stand in for the absent PA proof theory.

As best as I can determine, the mathematical and philosophical literatures include no counterpart to *LoF*'s concept of canon. I list below the PA canons in the order in which they appear in *LoF*, referring to them by the names *LoF* gives them. In what follows, I have taken the liberty of replacing the *LoF* term "expression" by the term "formula." Letting *X* be some word or phrase, any sentence below of the form "*X* is undefined" (or words to that effect) is shorthand for "*LoF* does not define *X* with the precision that generally characterizes mathematics, mathematical logic in particular."

*Convention of Intention*
What is not allowed is forbidden.

*Remark*: I trust that no reader has so misunderstood my purpose as to take this Convention as a political or ethical assertion.

*Contraction of Reference*
Let injunctions be contracted to any degree in which they can still be followed.

*Remark*: "Injunction", "contract", and "degree" are not defined. *LoF* (p. 8) states that this canon is shorthand for the following list of instructions:
1. Write '()' in some space.
2. Mark '()' with a name, eg, *a*.
3. Let *a* be the name of '()'.
4. Let the name *a* indicate '()'.

*Expansion of Reference*
Let any form of reference be divisible without limit.
*Remark*: I take "form of reference" to mean "space or subspace" in the sense of 2.1.6, and "divisible without limit" to mean that divisions of a subspace in the sense of 2.1.7 can be created at will, using A1. More generally, this canon permits expanding a formula, with each step justified by A1 or A2.

*Convention of Substitution*
In any formula, any subformula can be replaced by an equivalent subformula.

*Remark*: This canon is:
- An important example of what is meant by a "step";
- The first *LoF* canon or theorem to mention "equivalent," a term *LoF* does not discuss until 13 pages later. I shall revisit "equivalent" below when discussing T5-T7 and 2.3.8.

*LoF* wrote "arrangement" and "changed" where I write "subformula" and "replaced," respectively.

*LoF* distils the sense of 2.2.1–3 into the following canon:

*Hypothesis of Simplification*
Suppose the value of a formula, $|\alpha|$, to be its simplification.

*Remark*. Thus *LoF* defines "simplification."

*Rule of Dominance*
If a formula $\alpha$ shows a *dominant value*, then $|\alpha|=()$. Otherwise, $|\alpha|=\perp$.

*Remark*: This canon introduces "dominant value," which is nowhere defined.

The canons would seem to be assertions of the sort requiring proof; in fact, they are informally motivated at best.[12] The canons sometimes serve as definitions; e.g., the Hypothesis of Simplification and the Rule of Dominance effectively define the *value* of a formula. Curiously, for a work of logic/mathematics, *LoF* contains only one sentence preceded by the word "definition". That sentence, the third one in the body of *LoF*, simply reads: *Distinction is perfect continence.* What this sentence purports to define is less than obvious. §3.1 presents three more canons, bearing on the primary algebra.

### The Seven Meta-Theorems of the PA.

**2.3.1. Definition**. A *theorem* is metalinguistic statement asserted true because it is the last of an ordered finite sequence of metalinguistic statements known as an *informal proof*.

All theorems about the PA are proved informally in the metalanguage, the academic dialect of contemporary written English, using devices that tacitly draw on the reader's previous mathematical experience. (Those lacking such experience will find *LoF* and PA challenging.) An informal proof may draw on concepts that, strictly speaking, are not defined or proved within PA as of the point at which they are invoked. In particular, an informal proof relies strongly on natural language, and may invoke informal reasoning and mathematical concepts that are not part of PA. "Informal proof" is in contrast to "formal proof," defined in the Precis.

Following *LoF*, I number PA theorems consecutively, with the *m*th theorem denoted T*m*. The proofs are freely adapted from *LoF*.

*Establishing Consistency.*

**2.3.2 (T1)**. Any string composed of finite instances of '(' and ')', and satisfying the formation rule 2.1.2, is a formula.

*Remarks*.
1. A formula must be finite in order for the algorithm 2.1.4 to terminate.
2. I state T1 only out of loyalty to *LoF*; the formula formation rule 2.1.2 renders it unnecessary. T1 in *LoF* (p. 12) says that, starting from '()', "any conceivable arrangement" can be constructed by repeated application of A1 and A2. This version of T1 sounds trivial because it is predicated on *LoF*'s '$\urcorner$' notation, in which all possible strings are formulae. *LoF* does not articulate the operational meaning of "conceivable." *LoF* (p. 22-24) unaccountably invokes T1 in the proofs of J1 and J2, to justify asserting that a pa variable can only take on the values '()' and '⊥'.

**2.3.3 (T2)**. If any space pervades the formula '()', the value indicated in the space is the marked state. Notation: $()\alpha=()$.

*Proof.* If $|\alpha|=()$, then $()\alpha$ is $()()$, which simplifies to $()$ by A1. If $|\alpha|=\perp$, then $()\alpha$ simplifies to $()$ by A2. □

*Remark*. T2 is the PA version of the primary algebra consequence C2 (§3.1). *LoF* makes frequent use of T2, which arguably defines the "marked state."

12. *LoF* (pp. 40-41) calls T14 and T15 'canons,' thereby sowing terminological confusion.

**2.3.4 (T3)**. The simplification of a formula is unique.

*Proof.* Review the algorithm 2.2.3. This algorithm systematically reduces a formula, starting from its greatest depth. Each step has only two possible outcomes: ()α or (()). By T2, ()α reduces to (); by A2, (()) can be erased. Given each outcome, the next step is unambiguous. Hence there is only one possible simplification. □

*Remarks.* Let $A$ be the set of all possible PA formulae. Simplification can be represented by the mapping $f\colon A \to B \subset A$. T3 implies that $f$ is a *homomorphism* (Halmos and Givant 1998: §27) and an *isotone* function (Rudeanu 1974: §11.3), whose *fixed points* are '()' and '⊥'.

**2.3.5 (T4)**. The value of a formula constructed by taking steps starting from a primitive value is that same primitive value.

*Proof.* Let α be a formula constructed by taking steps starting with the primitive value $x$. The steps can be retraced back to $x$, so that $x$ is *a* possible simplification of α. By T3, all possible simplifications of α must yield $x$, hence $x$ is also *the* simplification of α. Hence we can write $|\alpha|=x$. □

*Remark.* T3 [T4] says that the value of a PA formula is invariant under simplification [complication]. T3 and T4 together imply that every PA formula has a unique value. Hence the PA is *consistent* (the preferred term nowadays is *sound*) and *LoF* refers to T1-T4 as the "theorems of consistency."

*Procedural Theorems*

**2.3.6 (T5)**. Identical formulae express the same value. Notation: α=α.

*Proof.* Use 2.2.3 to simplify the formula α to some member of $B$; call that member $x$. By T3, $x$ exists and is unique. Hence α is equivalent to $x$, so that we write $\alpha=x$. Beginning with $x$, we reverse each step in the simplification of α, recreating α. By T4, the value of this recreated α will also be $x$, so that we can write α=α. □

*Remark.* The verb "express" in T5 is undefined.

**2.3.7 (T6)**. Formulae having the same value can be equated. Notation: Let $x \in B$. If $\alpha=x$ and $\beta=x$, then α=β.

*Proof.* Identical to the proof of T5, except that we proceed by steps from $x$ to β rather than α, by reversing the simplification of β. □

*Remark.* T6 in effect means "if $|\alpha|=|\beta|$, then α=β."

T7 requires some preliminary definitions.

**2.3.8. Definition** (Wolf 1998: §§6.1-2; Stoll 1974: §1.7). Let $A$ be a set. A binary *relation* $R$ is a subset of $A\times A$, the *Cartesian square* of $A$. $A$ is the *field* of $R$. Hence $R$ is a set whose members are all ordered pairs. The notation $xRy$ denotes that the ordered pair $(x,y)$ is a member of $R$. $R$ is *Euclidian* iff $(aRc \wedge bRc) \to (aRb)$. $R$ is an *equivalence relation* iff $\forall a,b,c \in A$, $(aRa) \in R$ ($R$ is *reflexive*), $aRb \leftrightarrow bRa$ (*symmetric*), and $(aRb \wedge bRc) \to aRc$ (*transitive*). If $R$ is an equivalence relation whose field is $A$, an *equivalence class* is a set $A^* \subset A$, such that $\forall x,y \in A^*$, $xRy$ comes out true.[13]

13. A slightly more general definition of relation goes as follows. If $A,B$ are sets, a *binary relation* is a subset of $A\times B$ and its *field* (or *carrier set*) is $A \cup B$. The term *Euclidian* honors the first of Euclid's "common no-

**2.3.9 (T7).** Formulae equivalent to the same formula are equivalent to one another. More formally, if $\alpha=\nu$ and $\beta=\nu$, then $\alpha=\beta$.

*Proof*. Let $|\nu|=e$. Then $|\alpha|=e$ and $|\beta|=e$, by hypothesis. Now simplify $\alpha$ to $e$, then retrace the simplification of $\beta$, starting from $e$ and ending with $\beta$. Since by T3 and T4, no allowed step alters value, $|\alpha|=|\beta|$, so that $\alpha=\beta$. □

*Remark*. T7 is T6 with $\nu$ replacing $x$. T7 can be recast as "the relation of logical equivalence is Euclidian." *LoF* invokes T7 repeatedly, but invokes T5 and T6 only to prove the pa initials J1 and J2.

I now invoke a result from the logic of relations.

**2.3.10**. **Theorem**. $R$ is an equivalence relation iff $R$ is reflexive and Euclidian.

*Proof*. Even though the proof is neither long nor difficult, I relegate it to §A.2 as it employs features of BA not yet explained in this book.

Let $A$ in 2.3.8 be the set of all possible PA formulae, and let $R$ be logical equivalence, '='. Then '=' is reflexive (by T5) and Euclidian (by T7). Hence by 2.3.10, '=' is an equivalence relation. T3 says that logical equivalence partitions $A$ into two equivalence classes, each corresponding to an element of $B$. Hence T3 is but an instance of the very well-known result that *an equivalence relation partitions its field into equivalence classes* (Wolf 1998: Th. 6.6). Let $[\alpha]$ denote the equivalence class of which $\alpha$ is a member. T4 can then be restated more formally as: $\forall\alpha\in A$, there exists a one-to-many relation $g$: $B\rightarrow A$, corresponding to expansion, such that $[f(g(f(\alpha)))] = [f(\alpha)] = [\alpha]$. When one of True or False interprets '()', '=' denotes *logical equivalence*.

I denote equivalence by '⇔' when one or both formulae linked by '⇔' are not BA formula. The sign '⇔' is part of the metalanguage, and two formulae linked by '⇔' form a *sequent*, a metalingu istic term. More generally, '⇔' can be read as denoting a translation from one syntax to another.

**2.3.11. Recapitulation**. The *primary arithmetic* (abbreviated PA) is a very elementary formal system whose *primitive basis* (cf. Précis) consists of:

- The symbols '(', ')', '⊥', and '=';
- The operator-operand '()', which can have itself as argument, resulting in the formula (()). The defined constant ⊥ is a synonym for (());
- '()' and the blank page as primitive values;
- The definitions of a formula (2.1.2) and the null formula '⊥' (2.1.8), and the algorithms for verifying (2.1.4) and simplifying (2.2.3) formulae;
- Table 2-1, taken as axiomatic;
- Six procedural "canons";
- Equivalence of formulae, an equivalence relation by virtue of T5 and T7, denoted by infix '='. Two formulae linked by '=' form an *equation*.

By virtue of T1-T4, the PA is sound; its intended interpretation is Boolean arithmetic.

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tions" (Eves 1990: 35). Introductory logic texts usually do not mention relations; Carnap (1958: §§29-38) and Suppes (1957: chpt. 10-11) are exceptions. For a nice exposition of how relations are grounded in axiomatic set theory, see Suppes (1960: chpt. 3).

# Chapter 3.

## The Primary Algebra (pa): Syntax and Algebra.

"*It is… valuable to meditate on algebraic notation; the whole of the formal and symbolic part, having gradually broken away and developed immensely, is of great interest.*"

Paul Valéry, quoted in Le Lionnais (1948: 10).[1]

At any point in a PA formula, one can insert a marker that can take on either primitive value. Latin letters, termed (sentential) *variables*, will serve as such markers. The set of possible values a variable can assume is its *domain*; the domain of a pa variable is *B.* Thus the *primary algebra* (hereinafter abbreviated pa, by analogy with the abbreviation PA for the primary arithmetic) is born. Like the PA, the pa consists of formulae and equations, and includes canons, rules, and theorems. We begin by setting out the pa symbols:

**3.0.1. Definition**. The notation of the pa consists of *proper* and *improper symbols*. The proper symbols are:

- The PA proper symbols '(', ')';
- Lower case Latin letters, '*a*', '*b*', etc., often called *statement letters* or *sentential variables*. A letter may have a positive integer subscript, so that the number of possible variables is denumerable.

The improper symbols are '⊥', the prime '′', and the ellipsis '…' combined with the subscript *i* ranging over some range of the positive integers. Improper symbols are merely convenient notational shorthand. Symbols are concatenated into formulae:

**3.0.2. Definition**. The recursive definition of a pa formula is identical to 2.1.2, except that the atomic formulae include any single Latin letter.

Definitions similar to 3.0.2, e.g., Bostock (1997: 21), are standard in the literature. Synonyms for formula include *well-formed formula* (*wff*) and *schema* (Quine 1982: 33).

Because '()' is an atomic formula**,** 3.0.2 implies that a PA formula is also a pa formula. The pairing rule for parentheses, and the algorithm 2.1.4, both hold in the pa as well as in the PA. A nonobvious implication of 3.0.2 is that the result of inserting strings of Latin letters anywhere into a PA formula is a pa formula. Subformulae, proper and otherwise, are defined by obvious analogy with 2.1.3. Informally speaking, a subformula is any "part of" a pa formula that is itself a formula. An atomic formula has no proper subformulae other than '⊥', which is a proper subformula of all formulae other than itself.

In this book, '$=_{\mathbf{df}}$' is part of the metalanguage and serves to define a new notation or concept. Let the string *x* contain an instance of some new symbol, and let the string *y* contain only familiar symbols. The notation '$x =_{\mathbf{df}} y$' defines the new symbol by asserting that the strings *x* and *y*, however they differ in appearance, have the same meaning *by definition*. Let *a, b*, and *r* be pa formulae. I now define the improper symbols the prime, '′', the ellipsis '…', and the letter *i* subscript as follows:

$$a' =_{\mathbf{df}} (a) \qquad a_i\ldots =_{\mathbf{df}} a_1a_2\ldots \qquad a'_i\ldots =_{\mathbf{df}} a'_1a'_2\ldots \qquad (a_ir)\ldots =_{\mathbf{df}} (a_1r)(a_2r)\ldots$$

1. From a letter to Pierre Honnorat, dated February 1932. The translation is mine.

‘$a'$’ is nothing more than a synonym for ‘($a$)’, in which case ‘$a$’ is said to be *primed*. Using ‘$a'$’ in place of ‘($a$)’ is purely a matter of convenience and aesthetics. The letter $i$ subscript and the ellipse always appear in tandem. The improper symbols are not mentioned in 3.0.2, and hence play no essential role in the syntax of the pa. Foremost among the virtues of the pa is its succinct syntax. The notation I propose for BA is more compact than that of *LoF* and requires only standard typographic symbols.

Observant readers will have noticed that thus far, I have enclosed symbols and formulae between single quotation marks. I have done so hoping to steer clear of Quine’s *bête noire*: metadiscourse confusing *use* of a symbol with the *mention* thereof. Henceforth, I will rely on the following general rule, adapted from Conventions (I) and (II) in Suppes (1957: 125-26): *all symbols and formulae from* BA *and conventional logic are to be taken as names of themselves*. I deferred invoking this convenient rule until the syntax of BA was fully set out.

### 3.1. Consequences, Canons, Theorems.

“*Algebra is a science of the eye*.” Peirce (1.34).[2]

**3.1.1. Definition**. T (*true*) and F (*false*) are the possible *truth values*. A *statement* is an object language formula, or piece of metalinguistic discourse, that can be assigned a truth value.

*Remark*. An individual sentential variable is a trivial statement. The last paragraph of §4.0 operationalises the assignment of truth values to pa statements.

**3.1.2. Definition**. An $n$-ary *truth functor* [or simply *functor* when ‘truth’ can be omitted without ambiguity] is a symbol combining $n$ statements into a single statement (Bostock 1997: §2.2; Quine 1982: §§20, 45). A *connective* [*operator*] is a truth functor such that $n \geq 2$ [1]. Constants such as () and $\perp$ are 0-ary (or ‘medadic’) functors by convention.[3]

Truth functors, such as $\sim$, $\rightarrow$, $\wedge$, $\vee$, and $\leftrightarrow$, make up the core of the CTV. BA consists of one unary functor, enclosure by parentheses, and one binary functor, juxtaposition. Table 4-2 gives BA translations of the usual CTV truth functors.

**3.1.3. Definition**. Given some formula in which $n$ sentential variables appear, an *atomic valuation* assigns a member of $B$ to each of the $n$ variables.

The PA definition of *equation*, 2.2.4, carries over to the pa, *mutatis mutandis*.

**3.1.4. Definition.** If $\alpha$ evaluates to () or $\perp$ for a given atomic valuation, that valuation *satisfies* $\alpha$, and $\alpha$ is *satisfiable*. If all $2^n$ possible atomic valuations satisfy $\alpha$ in the same way, then $\alpha$ and the equation $\alpha$=() [or $\alpha=\perp$] are both *tautologies*. If $\alpha\leftrightarrow\beta$ is a tautology, $\alpha$ and $\beta$ are *tautologically equivalent* so that we may write $\alpha=\beta$.

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2. Sylvester wrote “mathematics” not “algebra” (Ewald 1996: 515), but Peirce’s misquotation is perhaps more apt.

3. “A functor is a sign that attaches to one or more expressions of a given grammatical kind or kinds, to produce an expression of a given grammatical kind. [A functor] is grammatical in import but logical in habitat…” (Quine 1982: 129). ‘Statement’ is the only grammatical kind that concerns us here. Carnap (1958: §18) employs ‘functor’ in a sense not used in this book, namely to denote what others call a “first order logic operator.”

The definition of tautology in 3.1.4 differs from the standard one, which defines a tautology as a formula evaluating to T for all possible atomic valuations. $\alpha=\beta$ says nothing about the truth value of $\alpha$ or $\beta$ taken in isolation. It does say that, given any atomic valuation, $\alpha$ and $\beta$ have the same value. Translating '=' as '$\leftrightarrow$' assumes that the biconditional can be seen as an equivalence relation. This is indeed the case; see §A.5. On occasion, I will refer to an equation as a tautology, but strictly speaking, this is an *abus de language*.

The preceding can be put a bit more formally. Let $\aleph$ be the set of possible PA formulae, and let $f$ be the simplification function defined in 2.2.3. T3 and T4 assure us that $f$ maps every member of $\aleph$ uniquely onto one of () or $\perp$. We can then say the following about $f$:

- The image of $\aleph$ under $f$ is $B$;
- $f$ is order preserving;
- $f$ partitions $\aleph$ into two equivalence classes.

The pa is a bit more complicated. Let $A$ be the set of possible pa formulae. Letting the domain of $f$ be $A$, $f$ then partitions $A$ into two non-empty subsets. The subset of $A$ whose image under $f$ is () or $\perp$ consists of tautologies; the complement of that subset consists of the satisfiable formulae.

I now turn to proof and related notions, beginning with the following definitions.

**3.1.5. Definition**. A *consequence* is a tautological equivalence. An *initial* is a consequence proved via a decision procedure; hence an initial is a PA theorem. An *identity* is either an initial or a consequence. A *demonstration* formally verifies ("proves") a consequence.

A decision procedure can verify any consequence. My variant of the *LoF* decision procedure is exposited in §5.1 and named *truth value analysis* (TVA). An initial is *not* an axiom, but can be invoked just like an axiom or consequence. Again, A1 and A2 in §2.2 are the only pa axioms. In conformity with standard mathematical practice, a demonstration consists of a sequence of steps, each relying on one or more BA axioms and theorems (especially the initials), canons, the rules of substitution/replacement (3.1.7-8), and consequences already proved.

Each step in a demonstration is justified by an *annotation*, enclosed in square brackets and formatted as follows:

$\alpha$ [*annotation*] = $\beta$ [*annotation for next step*] = ...

If a step requires more than one consequence, these and the substitutions they may require are listed sequentially, separated by semicolons. If a step includes a rearrangement of subformulae, 'TR' annotates that fact. If symbols appearing in $\alpha$ are absent from $\beta$, I indicate that fact by underlining the relevant parts of $\alpha$. If $\beta$ contains subformulae absent from $\alpha$, the additions to $\alpha$ are shown in bold. When a subformula in $\alpha$ is moved or copied to a greater or lesser depth in $\beta$, the part of $\beta$ that is freshly moved or copied is also printed in bold. For more re demonstration, see §5.0 and the *Precis*.

*LoF* numbers its initials, consequences, and theorems consecutively, with a letter (J for initials, C for consequences, T for theorems) followed by an integer. I do likewise in order to facilitate cross-references to *LoF*.[4] The *LoF* initials are:

**3.1.6.** J1. $(a'a) = \perp$ J2. $((ar)(br)) = (a'b')r$

The verification of J1 and J2 proceeds as follows:

4. *LoF*, Bricken, and Kauffman, following the example of *PM* (p. xii), give Latinate names to the pa identities. I decline to follow their example, as the names they propose are not mnemonic.

**3.1.7. Theorem**. J1 and J2 are tautological equivalences.

*Proof*:
**J1:** Let $a$=(). Then the lhs of J1 is $(\underline{(())}())$ [A2] = $\underline{(())}$ [A2] = ⊥. Now let $a$=⊥, so that the lhs of J1 becomes $((\underline{\perp})\underline{\perp})$ [A2] = $\underline{(())}$ [A2] = ⊥. By T1, () and ⊥ are the only possible values of $a$. Hence J1 always holds.

**J2:** By T2, α()=() for any formula α. Begin by setting $r$=(), in which case the lhs of J2 evaluates to (($a$())($b$())) [T2,2x] = ((())(())) [A2,2x] = (). The rhs evaluates to ($a'b'$)() [T2] = (). If $r$=⊥, then simply erase $r$ from J2. Both sides of J2 then amount to the same thing, namely ($a'b'$), for all possible values of $a$ and $b$. By T1, () and ⊥ are the only possible values of $r$. Hence J2 always holds.
□

*Remark*. The proof of J1 is trivial. That of J2 is, in all essentials, a trivial instance of the method of truth value analysis explained in §5.1. An immediate consequence of J1 is that the formula (α′α), α being any subformula, can be inserted at will anywhere. The repeated application of J1 allows the maximum depth of a formula to be increased at will, without affecting its value under any atomic valuation.

The BA initials used in this book will not be J1 and J2, but B1-B4. I defer discussion of B1 to §3.2. B2 is ⊥$a$ = $a$, an algebraic equivalent of A2. B3 is $a'a$ = (). Taking the complement of both sides of B3 yields J1. B3 is a trivial variant of J1 and, when combined with B2, is a bit more natural than J1. Finally, B4 is ($ba$)$a$ = $b'a$. B4 is C2 in *LoF*, and will be called C2 when I wish to refer to the *LoF* consequence of that name. §A.1 includes demonstrations of B2 and B3 from *LoF* identities. B2-B4 make easy most demonstrations needed in this book.

**Table 3-1. The Identities Invoked Here and in *LoF*.**

| BA | *LoF* | | *How employed in* LoF *or in this book.* |
|---|---|---|---|
| | | | |
| B1 | --- | $abc = bca$ | Juxtaposition commutes & associates. Tacit in *LoF*. |
| B2 | J1 | $\perp a = a$ | Trivial consequence of J1. |
| B3 | --- | $a'a = ()$ | Complement of J1. |
| B4 | C2 | $(ba)a = b'a$ | B2-B4 are the essence of *calculation* (5.0.1). |
| C1 | C5 | $aa = a$ | Algebraic form of A1. Helps prove T13. |
| C2 | C3 | $()a = ()$ | Algebraic form of T2. |
| C3 | C1 | $((a)) =_{\mathbf{df}} (a') = a$ | Invoked in many demonstrations. |
| C4 | C4 | $(a'b)a = a$ | The absorption law of lattice theory follows trivially. Also helps demonstrate C5 in *LoF*. |
| C5 | J2 | $((ar)(br)) = (a'b')r$ | Distributive law; defining Ba property. |
| C6 | C6 | $(a'b')(a'b) = a$ | Helps demonstrate C8. |
| C7 | C7 | $((a'b)c) = (ac)(b'c)$ | Helps prove T14; crucial for normal form. |
| --- | C8 | $(a'r')(b'r') = ((ab)r')$ | Syntactic dual of J2. Invoked in the *LoF* demonstration of C9 and the proof of T15. |
| C8 | C9 | $((a'r')(b'r)) = (ar')(br)$ | Helps prove T17. |

*LoF* invokes the nine consequences C1-C9; this book requires one less, C1-C8. But C$n$ here and in *LoF* usually do not refer to the same consequence; see Table 3-1. Moreover, C8 in *LoF* is the dual of J2 (cf. §4.1) and by 4.1.4, the dual of an identity requires no demonstration. Moreover, C8 also finds no application in this book. Table 3-1 restates C1-C9 from *LoF* in the notation of this book, along with a very brief indication of how they prove useful in *LoF* and here. The purpose of C7 in this book will become clearer in §4.4. C8 here is simpler than its *LoF* equivalent, because the simpler form suffices to prove T17, the only use I (and *LoF*) have for C8. For CTV interpretations of B1-C8, see Table 4-3.

### pa: **Canons.**

The pa features three canons in addition to the PA canons described in §2.3. As before, I deviate from the *LoF* wording of the pa canons in an attempt to reduce their Zennish ambiguity and enhance their perspicuity.

*Principle of relevance*
If a property is common to every indication, it need not be indicated.

*Remark*. Could this be reworded as: "That which characterizes everything distinguishes nothing"?

*Principle of transmission*
Let $\alpha$ be a sub-formula of the formula $\beta$, and let $a$ be a variable appearing in $\alpha$. Let the depth, relative to $\beta$, of $\alpha$ [$a$] be $d_\beta(\alpha)$ [$d_\beta(\alpha)+1$]. When the value of $a$ changes, the value of $\alpha$ either changes or does not change. If it changes, the pervasive subspace of $\alpha$ is said to be *transparent* relative to $a$. Otherwise, the pervasive subspace is *opaque*.

*Remark*. The wording of this Principle differs from that in *LoF*, if only by employing "sub-formula" and "depth." This canon is also closely linked to T16.

*Rule of demonstration*
A demonstation rests in a finite number of steps.

*Remark*. Why did *LoF* say "rests in" rather than "consists of"? On "demonstration" and "steps," consult §5.0 and the Précis. Chapter 11 of *LoF* shows how formulae with infinite depth may violate T1-T4. I do not elaborate on this, nor do I explore formulae with infinitely many symbols but finite depth, as I wish to steer clear of all infinities and Cantorian paradoxes.

### pa: **Substitution**

"*…there is* [no] *need for any other kind of proof than one which depends on the substitution of equivalents*." Leibniz.[5]

An *equational* formal system is one whose axioms and consequences consist of pairs of formulae linked by equality, denoted by '='. The inference rules for an equational system are the *substitutivity of equivalents*, R1 below, and the *uniform replacement of subformulae by subformulae*, R2 below. *LoF* (p. 26) states that "[R1 and R2] are commonly accepted as implicit in the use of the sign '='." Hence BA is equational, as is nearly all of mathematics, with conventional numerical algebra being paradigmatically so. The vast majority of extant formal logics, on the other hand, are *ponential*, so-called because their fundamental inference rule is *modus ponens* (cf. §5.3).[6] R1 and R2 are

5. Letter to Placcius dated 16.11.1687, translated and quoted in Ishiguro (1990: 17).

6. Equational logic has a following among contemporary computer scientists; see Gries and Schneider (1994) and Tourlakis (1998). The equational-ponential dichotomy builds on Curry's (1963: §2.D.1) *relational-assertional* dichotomy. A *relational* system consists of formulae linked by some dyadic relation; if that relation is an equivalence relation, the system is *equational* in the sense of Curry. Curry contrasted relational to *assertional*, by which he meant a system characterized by formulae prefixed by the syntactic turnstile, the defining characteristic, in his view, of conventional logic. Other advocates of equational methods include Meredith and Prior (1968) and Tarski and Givant (1987).

3.1.7 and 3.18 below. In the interest of clarity, I restate these rules in a manner that deviates somewhat from *LoF*. As always, Greek letters are a metalogical device.

**3.1.7. R1**, **Substitution**. Let $\alpha\langle\varepsilon\rangle$ denote that the sub-formula $\varepsilon$ appears at least once in the formula $\alpha$. Let $\phi$ be a formula such that $\phi{=}\varepsilon$. Let $\alpha\langle\phi//\varepsilon\rangle$ be the formula formed by substituting $\phi$ for *any* (possibly 0) instance of $\varepsilon$ in $\alpha$. Then $\phi{=}\varepsilon \rightarrow \alpha\langle\varepsilon\rangle = \alpha\langle\phi//\varepsilon\rangle$.

*Proof* (adapted from Mendelson 1997: Prop. 1.4). The contribution of a subformula $\varepsilon$ to the truth value of any formula $\alpha$ containing $\varepsilon$ is fully determined by the truth value of $\varepsilon$. Hence the truth value of $\alpha$ is not affected by replacing $\varepsilon$ by $\phi$, whose truth value, by assumption, is identical to that of $\varepsilon$. □

*Remark*. $\alpha$ need not be a tautology but if it is, $\alpha\langle\phi//\varepsilon\rangle$ is also a tautology. *LoF* (p. 26) does not prove R1, instead justifying it as an algebraic version of the PA Convention of Substitution, and as an "inference from" T1-T4. Because R1 is essentially Leibniz's "identity of indiscernables," R1 and the reflexivity of '=' suffice to show that '=' is an equivalence relation (Kneebone 1963: §4.1). Some authors refer to R1 as the "substitutivity of the biconditional *or* of equivalents." R1 also answers to the familiar "substitution of equals for equals" of numerical algebra.[7]

**3.1.8. R2, Replacement**. Let $\alpha\langle v\rangle$ be a tautology. Let $\alpha\langle\omega/v\rangle$ be the formula formed by replacing *every* instance of $v$ in $\alpha\langle v\rangle$ by the formula $\omega$ ($v$ and $\omega$ are not necessarily equivalent). Then $\alpha\langle v\rangle{=}\alpha\langle\omega/v\rangle$.

*Proof* (adapted from Mendelson 1997: Prop. 1.3). By definition, the value of a tautology is not affected by the value of any (or all) of its statement letters. Hence any statement letter $v$ can be replaced by some formula $\omega$ (with a value under any interpretation) without affecting the value of $\alpha$, as long as this replacement applies to every instance of $v$. Hence R2 follows from $\alpha\langle v\rangle$ being a tautology. □

*Remark*. The result of applying R2 to an indentity is an *instance* (more pedantically, *substitution instance*) of that identity, making possible the following concise rewording of R2: "a tautology yields tautologous instances." With R2 in hand, all pa identities can be taken as *schemata*; that is, all letters in such identities are taken as schematic variables. Most formal systems dispense with R2 by simply taking the axioms and theorems as schemata from the outset. Some authors refer to Replacement as (variable) "Substitution."[8]

R1 and R2 warrant close scrutiny in light of the careful treatment of Substitution by other authors. The reader should also ponder why R1 says "any" while R2 says "every". *LoF* explicitly invokes

7. R1 is nearly the "Leibniz" inference rule of Gries and Schneider (1994) and Tourlakis (1998). R1 also follows from '=' being a congruence relation (Stoll 1963: 260). For other proofs of R1, see Quine (1982: 64), Mendelson (1997: Prop. 1.4), and Cori and Lascar (2000: Th. 1.24). After devising the '//' notation, I encountered it in Simons (1987: 49). Quine (1982: 63f) refers to Substitution as "interchange," for which he proposes three laws, which I condense to two: (1) R1 holds if '$\leftrightarrow$' replaces '=', and (2) R1 preserves (in)equivalence, (un)satisfiability, (non)validity, and (non)implication. On '$\leftrightarrow$' and '=', see §2, fn. 10. A version of R1 is crucial to the system of Cole (1968).

8. *LoF* does not prove R2, instead asserting (p. 26): "R2 derives from the fact, proved with J1 and J2, that we can find formulae, [equivalent yet not identical,] which, considered arithmetically, are not wholly revealed." For other discussions of Replacement, see Prior (1962: 24-25), Quine (1982: 44), Gries and Schneider (1994: Substitution rule), Bostock (1997: §2.5.D), Halmos and Givant (1998: §§13, 36), Wolf (1998: 86-7), and Cori and Lascar (2000: Cor. 1.23). Replacement is also a commonplace of ponential logic, because it preserves satisfiability and implication (Carnap 1958: T7-1).

R1 and R2 only in its "pedantic" demonstration of C1 and the worked examples on pp. 44-47. In this book, almost all use of R1 and R2 goes unremarked. But regardless of whether R1 and R2 are invoked tacitly or explicitly, the pa would be useless without them.

**Some pa Theorems.**

All PA theorems carry over to the pa; thus the pa inherits soundness and the notion of tautological equivalence from the PA. An additional 11 theorems, T8 through T18, bear on the pa only. I defer discussion of T14-T18 to §4.4. T8 and T9 merely restate J1 and J2. T10-T13 generalize certain consequences:

**3.1.9. (T10)**. C5 generalizes to $((a_i r)\ldots) = (a'_i \ldots)r$.

T10 is needed only to prove T15. T11 and T12 are straightforward generalizations of *LoF*'s C8 and C9 that have no application in this book and so are omitted.

**3.1.10 (T13)**. Let $\phi\langle\varepsilon\rangle$ be a formula containing one or more instances of the subformula $\varepsilon$, and let $\phi$ stand for $\phi\langle\varepsilon\rangle$ with all instances of $\varepsilon$ erased. Then $\varepsilon(\phi\langle\varepsilon\rangle)=\varepsilon(\phi)$.

*Informal Proof.* Let the deepest instance of $\varepsilon$ lie in depth $k$. Invoke B4 $k$-1x to create a copy of $\varepsilon$ in each subspace of depth 1,…,$k$-1. One more instance of B4 then eliminates the instance of $\varepsilon$ at depth $k$. Invoke to C1 to erase any multiple instances of $\varepsilon$ at each of depths 1, …,$k$-1. Then invoke B4 $k$-1 times to undo the process described in the second sentence of this paragraph. Doing so erases all instances of $\varepsilon$ in $\phi\langle\varepsilon\rangle$. □

*Remarks*.
1. T13 nicely generalizes B4 and is an example of a *theorem schema*. A formal proof of T13 would require induction on formula depth. The exact form of 3.1.10 is due to Bricken (2002), who named it Pervasion.
2. Viewing a formula as an ordered tree, any two instances of any subformula can be seen as connected by a *path*. A path is *monotone* if all parentheses crossed along the path are of one type, left or right. Repeated application of B4 allows $\varepsilon$ to be copied into, or erased from, any subspace of $\phi$ whose depth exceeds that of $\varepsilon$, as long as a monotone path connects $\varepsilon$ and its copy.

## 3.2. Order Irrelevance, Tacit and Explicit.

"*…commutation…may be dispensed with by not recognizing any order of arrangement as significant. Associative transformations…will be dispensed with in the same way…*" Peirce (4.374, 1902).

We may think of $a$ and $b$ in the pa formula $ab$ as linked by a tacit connective called juxtaposition, even though no explicit symbol separates $a$ from $b$. Because this connective has been merely tacit thus far, I have said nothing about it. In particular, I have not assumed it commutative or associative, or restricted its scope to the binary. Nor have I assumed that the order or grouping of variables within a pa formula affects its value. I now show how the properties of the PA imply that variable order and binary scope are indeed irrelevant for juxtaposition. Let $\alpha$, $\beta$, and $\gamma$ be arbitrary pa formulae. Then:

- $\alpha\beta$ *commutes*. This is a trivial implication of Table 2-1a. I became aware of the need for a broader axiomatic treatment of $\perp$ when attempting to justify why juxtaposition commutes. More generally, all objects within a given subspace can be reordered at will. Hence a formula can be rearranged so that any multiple instances of a subformula in a given subspace can be juxtaposed. By virtue of C1, these multiple instances reduce to a single one. Hence a BA formula behaves not only like a list, as in Chapter 2, but also like a set.[9]

- $\alpha\beta\gamma$ *associates*. () and $\perp$ are the possible values that $\alpha$, $\beta$, and $\gamma$ can each take on. (By T3 and T4, $\alpha$, $\beta$, and $\gamma$ can each stand for any formulae whatsoever.) Then the value of $\alpha\beta\gamma$ is the value of a PA formula that is some concatenation of () and $\perp$. According to Table 2-1a or T2, this concatenation simplifies to () if at least one of $\alpha$, $\beta$, and $\gamma$ has value (); otherwise, it simplifies to $\perp$. Now T3 assures us that the simplification of any pa formula is unique. Hence the simplification of $\alpha\beta\gamma$ cannot depend in any way on the order in which $\alpha$, $\beta$, and $\gamma$ are paired. Thanks to associativity, BA has no need for bracketing, and brackets are free for another use.

From the preceding, I conclude that the variables and sub-formulae that make up any formula may be reordered at will. *LoF* fully acknowledges this useful and important fact, but does not sufficiently highlight it. Juxtaposition indeed commutes and associates, but this fact should not be seen as a fundamental mathematical property. Rather, it merely offsets a metalinguistic typographic convention. The upshot, perhaps, is that I am merely restating in a bit of algebraic dress what I asserted about the BA at the beginning of Chapter 2.

If a binary algebraic operation commutes and/or associates, it is conventional to either postulate that fact, or to derive it from other postulates. §A.1.1 derives the commutativity and associativity of juxtaposition in two ways:

- The postulates *ab.c=ac.b*, B2, and B3. Dilworth (1938) was the first to propose *ab.c=ac.b*;
- The postulate *ab.c=bc.a* and C1. This approach is due to Byrne (1946: IIB).

Because *ab.c=bc.a* is easily derived from *ab.c=ac.b* (and vice versa; see §A.1.1), they both have the same name, B1.

Using parentheses to denote an algebraic operation would appear to have two drawbacks. First, parentheses are already widely employed in mathematics, e.g., to denote ordered pairs and open intervals. Second, parentheses now cannot serve as brackets, i.e., as devices for resolving ambiguity in formulae or for overriding operator precedence. But the discussion in this chapter shows that the pa requires neither parentheses for grouping nor the notion of operator precedence.[10] Hence the pa is free of all ambiguity arising from infix notation, an ambiguity from which the conventional notation for logic and Boolean algebra both suffer. Specifically, the notations $(x,y)$, denoting the ordered pair consisting of $x$ and $y$, and $(x,y)=\{t|x<t<y\}$, denoting an open interval, are unambiguous because the comma plays no role in BA. More generally, expressions that mix the pa and conventional mathematical or logical notation (and I do not wish to rule out such expressions) should not give rise to ambiguity.

That '()' appears to have a two-sided "exterior" is a drawback of my preferred notation. A geometric analogy may clarify this. Fold a circle upon itself along any diameter. The two resulting half circles will coincide, meaning that a circle is symmetric about any line going through its diameter. Now carry out this same exercise on '()'. Although '()' is not a regular polygon, it definitely has a geometric centre. But the only lines through that centre that preserve folding symmetry are the two lines parallel to the Cartesian axes.

9. Peirce's (4.372-584, 1902) logical graphs, discussed in section 6 below, clearly illustrate the irrelevance of order and grouping for the CTV. The first formal treatment of logic, Frege (1879), also employed a two dimensional notation.

10. Polish notation dispenses with brackets by a prefix notation for the truth functors. Is the persistence of infix notation a manifestation of path dependence?

*Moral*: A boundary can be thought of as a circle, because a circle appears the same from all angles. I intend "circle" as a metaphor, to suggest that a formula mixing statement variables and boundaries commutes and associates. The sign '()', not being symmetric from all perspectives, regrettably does not highlight this key property of boundaries.

### 3.3. The Lattice Road from Antisymmetry to BA.

"*When logically analyzed, order turns out to be… inconceivable and incomprehensible to us unless we had the idea… expressed by the term 'negation'. Thus it is that negation, which is always also something intensely positive, not only aids us in giving order to life, and in finding order in the world, but logically determines the very essence of order*." Royce (1917: 540)

I now modify the definition of an equivalence relation (2.3.8) in a crucial way.

**3.3.1. Definition**. Let $A$ be a set with typical members $a$ and $b$. Let an infix '=' denote an equivalence relation whose field is a superset of $A$ and whose intended reading is 'equality.' Let $aRb$ denote a reflexive and transitive binary relation whose field is $A$. If $(aRb \wedge bRa) \rightarrow a{=}b$, the relation $R$ has the *anti-symmetric* property and is a *partial ordering*. In this case, $R$ is distinct from = and we write $a{\leq}b$. $R$ is said to *partially order A*, and $A$ is a *partially ordered set* (*poset*).

Equivalence relations and partial orderings are both reflexive and transitive, and recur in mathematical logic and foundational mathematics. (The only relation in common use that is neither reflexive nor transitive is set membership itself.) Equivalence relations are also symmetric, but a partial ordering is anti-symmetric, meaning that symmetry holds only for those members of $A$ that are also members of some equivalence class under equality, '='. Anti-symmetry does not require that $a{\leq}b$ be defined for all ordered pairs $(a,b)$. If one or both of $a{\leq}b$ or $b{\leq}a$ is the case, then $a$ and $b$ are *comparable*. If any two members of $A$ are comparable, then $A$ is *linear* or *totally ordered*. Partial order is of interest because of the following theorem:

**3.3.2**. **Theorem**. $B$ is partially ordered in two ways.

*Proof. Case 1*. Let $(a)b \Leftrightarrow a{\leq}b$ and *true* $\Leftrightarrow$ (). Then $\bot{\leq}() \Leftrightarrow (\bot)()$ [A2] = ()() [A1] = ().
*Case 2*. Let $(a(b)) \Leftrightarrow a{\leq}b$ and *true* $\Leftrightarrow$ (()). Then $(){\leq}\bot \Leftrightarrow ((\,)(\bot))$ [A2] = (()()) [A1] = (()). ☐

That A1 and A2 assure that $B$ is a partially ordered set grounds much of the mathematical substance of BA/Ba. That $B$ can be ordered in two ways grounds duality, a topic I revisit in §4.1. Posets have a rich mathematical structure, beginning with the next three definitions.

**3.3.3. Definition** (Machover 1996: 4.2.23). Let ≤ partially order the set $A$, and let $B{\subseteq}A$. If $\forall x[x{\in}B \wedge x{\leq}a]$ holds for some $a{\in}A$, then $a$ is an *upper bound* of $B$. If $b$ is an upper bound of $B$ such that $b{\leq}a$, where $a$ is any upper bound of $B$, then $b$ is the *least upper bound* (l.u.b.) of $B$. Replacing '≤' in the preceding three sentences by '≥' requires changing *upper* to *lower* and *least* to *greatest* (g.l.b.).

The algebraic structure arising from partial order and least/greatest upper bound is that of lattice. Lattice syntax is near-trivial. Let Latin letters denote atomic formulae ranging over some poset $L$, and let $\alpha$ and $\beta$ be arbitrary lattice formulae. Then '$\alpha\beta$', the *concatenation* of $\alpha$ and $\beta$, and '[$\alpha$]', the *dual* of $\alpha$, are both lattice formulae.

**3.3.4a. Definition** (Donnellan 1968: 49). $L$ is a *lattice* if $ab$ denotes one of the pair (least upper bound, greatest lower bound) of $a$ and $b$, and $[ab]$ denotes the other member of the pair.

A matched pair of '[' and ']' simply toggles between l.u.b and g.l.b. Lattices also have the following algebraic definition:

**3.3.4b. Definition**. A *lattice* is a $\langle L,\cdot\cdot,[\cdot]\rangle$ algebra of type $\langle 2,1\rangle$[11] such that $\forall a,b\in L$, the axioms L0, B1, and L1a,b in Table 3-2 hold. If $ab$ is the *meet* of $a$ and $b$, then $[ab]$ is their *join*, and vice versa.

While the concepts in 3.3.4 are conventional, the notation and universal algebra bits are not; the conventional universal algebra definition of a lattice is a $\langle L,\cap,\cup\rangle$ algebra of type $\langle 2,2\rangle$.[12] For a derivation of 3.3.4a from 3.3.4b (conventionally notated) and conversely, see Donnellan (1968: §8).

**Table 3-2. Lattice Axioms.**

| | *Type of lattice* | *Signature* | *Axiom* | *Name* |
|---|---|---|---|---|
| L0 | All | $\langle\cdot\cdot,[\cdot]\rangle$; $\langle 2,1\rangle$ | $ab\in L$ and $[ab]\in L$. | Closure |
| B1 | " | " | $abc = bca$. | Order Irrelevance |
| L1a | " | " | $a[ab] = a$. | Absorption law |
| L1b | " | " | $[a[ab]] = a$. | " |
| L5 | Modular | $\langle\cdot\cdot,[\cdot]\rangle$; $\langle 2,1\rangle$ | $a[b[ac]] = [[ab][ac]]$ | Modular law |
| L6 | Distributive | " | $a[bc] = [[ab][ac]]$ | Distributive law |
| L7a | Bounded | $\langle\cdot\cdot,[\cdot],()\rangle$; $\langle 2,1,0\rangle$ | $()a = ()$ | Bounds exist |
| L7b | " | " | $[()]a = a$ | " |
| L8 | Complemented | $\langle\cdot\cdot,[\cdot],(\cdot),()\rangle$; $\langle 2,1,1,0\rangle$ | $(a)a=()$ | Tautology |
| L9 | BA | $\langle\cdot\cdot,(\cdot),()\rangle$; $\langle 2,1,0\rangle$ | $(a'b'(cd))=[ab[cd]]$. $(())=[()]$ | |
| B4 | " | " | $a(ab) = a(b)$ | Exclusion |

L0 follows from concatenation and dualization being operations defined over $L$. Hence L0 will not be invoked explicitly in this book, and Table 3-2 includes it only for the record.

From L1a-b we can derive the consequences L2 and L3a,b, analogs of C3 and C1:

**L2**: $[[ab]] = ab$. *Dem*. $[[ab]]$ [L1a] $= [[ab][[ab]c]]$ [L1b] $= [ab]$ [L1a] $= [ab[abc]]$ [L1b] $= ab$. ☐

**L3a**: $aa=a$. *Dem*. $aa$ [L1b] $= a[a[aa]]$ [L1a] $= a$. ☐
**L3b**: $[aa]=a$. *Dem:* $[aa]$ [L1b] $= [a[a[aa]]]$ [L1b] $= a$. ☐

§A.1.1 derives the order irrelevance of concatenation from B1 and L3a. L3a,b have the corollary $[a]=a$: *Dem*. $[a]$ [L3a] $= [aa]$ [L3b] $= a$. This corollary has an important implication:

**L4**. $a=b \leftrightarrow [a] = [b]$.

---

11. On the universal algebra notation employed here, see, e.g., Abbott (1969: §2-5) or Burris and Sankappanavar (1981).

12. After having decided on the notation of 3.3.4b, I discovered that Dilworth (1938) denoted the meet of $a$ and $b$ by $[a,b]$, and the join by $(a,b)$. Dilworth did not see that (i) his infix commas are unnecessary; (ii) there is no need to restrict the scope of brackets to be binary, once meet and join are proved associative; and (iii) by (ii) and duality, one type of bracket suffices.

L2 and L4, which derive from L1a,b, distill the operational content of the duality pervading lattice theory.

**3.3.5. Definition.** A *distributive* lattice satifies L6 in addition to L0, B1, and L1a,b.

Any lattice satisfying L5 is a modular lattice. All distributive lattices are modular; the converse does not hold. Modular lattices will not concern us any further.

**3.3.6. Definition**. A *bounded* lattice has two distinguished elements, () and [()], called *bounds*, governed by the axioms L7a,b.

The conventional notation for lattice bounds is ⊥ (*least element* or *bottom*) and T (*greatest element* or *top*). () has two symmetrical intended interpretations. If concatenation is read as meet, ()⇔⊥, [()]⇔T, and [()*a*]⇔T***a***. If concatenation is read as join, simply interchange ⊥ and T in the preceding sentence.

| *Concatenation read as:* | () | [()] | L7a | L7b |
|---|---|---|---|---|
| *Meet* | ⊥ | T | T*a*=T | ⊥*a*=*a* |
| *Join* | T | ⊥ | ⊥*a*=⊥ | T*a*=*a* |

L7a,b is equivalent to $\forall a \in L$, either $\perp \leq a \leq$ T or T$\leq a \leq \perp$.

Now let enclosure by '(' and ')' denote *complementation*, as in the BA, with $(a)=a'$ denoting the *complement* of *a*. $\perp$,T$\in L$. We then define:

**3.3.7. Definition**. A lattice is *complemented* iff:
- *L* is closed under complementation;
- The axiom L8 holds.

L9 is the bridge from lattices to BA, the "Rosetta stone" linking the syntax of complemented lattices to that of BA.

3.3.8 shows that given L6 and L8, L7 is redundant.

**3.3.8. Theorem**. A complemented lattice is *bounded*.

*Dem*. **()a** [L8] = $(a)aa$ [L3a] = $(a)a$ [L8] = **()**.

**a(())** [L8] = $a(a'a)$ [C3] = $a((a)(a'))$ [L7b; L9] = $a((a)(a')(()))$ [L8] = $a((a)(a')(b'b))$ [L9] = $a[aa'[()]]$ [L7b] = $a[aa']$ [L1a] = **a**. ☐

*Remark*. The converse does not hold; a bounded lattice is not necessarily complemented. Because this demonstration invokes L8, the result holds only for complemented lattices. This demonstration also requires a lattice demonstration of C3, shown in §A.3.

**3.3.9. Definition**. A *Boolean algebra* is a complemented and distributive lattice (Donnellan 1968: §27).

**3.3.10. Theorem**. The pa is a complemented distributive lattice and hence a Ba.

*Proof*. Let $L=B$, ()⇔T, and assume L9. (The interpretation (())⇔T, *ab*⇔[*ab*], and (*a′b′*)⇔*ab* is equally valid, because (*a′b′*) and *ab* constitute a *dual pair*, a notion explicated in §4.1.) I now verify that the pa satisfies L0, L1, L6, and L8.

L0. pa juxtapostion and complementation always yield a pa formula (T1). A pa formula has a simplification (T3), one that is necessarily a member of *B*.
B1. pa concatenation commutes and associates (§A.1.1).
L1. Substitute *b′* for *b* in C4. Dually, see §A.1.2.

If T⇔(), then: L6⇔C5, and L8⇔B3.
If ⊥⇔(), then: L6⇔ dual of C5, and L8⇔ complement of B3. □

*Remark*. B3 (C5) assures that BA is complemented (distributive). L1 is C4, which *LoF* derives from J1 and J2. Hence the pa initials in *LoF* get to the heart of definition 3.3.9.[13] For a more conventional proof that the pa is a Ba, see §A.10.

L6 [L8] is evidently analogous to C5 [B3] of Ba. Hence B3 and C5 nicely distill why there are lattices that are not Bas. As argued above, L6 and L8 are independent of L0–L1. But the following argument shows that C5 and B4 are not independent of B1, L1, L6, and L7b. §A.1.2 derives L1 from B2-B4. Moreover, L1 follows trivially from C4, which *LoF* derives from J1 and J2. Hence L1 is redundant, given B2, B3, and one of C5 or B4.

A Ba postulate set fully consistent with 3.3.10 would consist of the general lattice postulates B1 and L1, the distributive law L6, and the complementation law L8. Surprisingly, the only postulate sets of this nature I have found are Koppelberg (1989) and Mann (2003).

**3.3.11. Conjecture**. The Ba postulates B1, L1, L6, and L8, are independent.

*Discussion*. I do not have a proof that these postulates are independent, but their independence is plausible as follows. Each of the following facts is true of one postulate and of no other. B1 contains no boundaries and changes no variable instances. L1 creates or erases a nested pair of boundaries, and creates/erases a variable. L6 creates/erases one instance of a variable. L8 creates/erases all instances of a variable.

In Ba, the relation denoted by '=' is not only an equivalence relation but also a congruence relation, defined as:

**3.3.12. Definition**. A Ba *congruence relation* is an equivalence relation *R* satisfying $aRb \rightarrow a'Rb' \wedge \forall c{\in}B[acRbc]$. (Stoll 1963: 261). For a proof that '=' is a congruence relation, see §A.5.

A Boolean algebra is any algebra defined over a finite poset *B*, such that *B* is closed under:
- A binary operation '·' that commutes and associates;[14]

13. *LoF* (Appendix 1) proves that Ba is a model of the pa, by showing that Sheffer's (1913) postulates for Ba (Table 6-2) are pa consequences. Sheffer formulated Ba with a single binary functor, the *Sheffer stroke*, whose pa representation is (*ab*) (dually, *a′b′*; Table 4-2, bottom row). Sheffer's first two postulates are, in effect, C3 and J1. His third postulate is an easy pa consequence: *Dem*. **((a(bc)))** [C3,2x] = ((*a*((*b′*)(*c′*)))) [C5] = ((((*b′**a*)(*c′**a*)))) [C3] = **((b′a)(c′a))** □. The pa representation of the Sheffer stroke makes it easy to see how it commutes but does not associate: (*ab*)=(*ba*) but ((*ab*)*c*)=(*a*(*bc*)) is not necessarily the case. By contrast, in §3.4 we will see that the sole binary operation of group theory associates but does not necessarily commute. Engineers know the Sheffer stroke as NOR, its dual as NAND.

14. These properties can be demonstrated; see §A.1.1.

- A unary operation ‘′’, such that $\forall c \in B$, $c' \cdot c = infB$ or $supB$,

and such that ‘=’ is a binary dyadic relation. Table 2-1 further implies that juxtaposition commutes and associates, and that $B$ is closed under both juxtaposition and complementation. If $a \cdot b \Leftrightarrow ab$, $a' \Leftrightarrow (a)$, and T $\Leftrightarrow$ (), the pa can be seen as a minimalist notation for Ba. Ba and BA are both $\langle B$, --, (-), ()$\rangle$ algebras of type $\langle 2,1,0 \rangle$. That $B=\{\perp,()\}$ is partially ordered is evident if $a \leq b \Leftrightarrow (a)b$. From Table 2-1 we may conclude that $(a)b$=() for all possible values of $a$ and $b$ except $a$=() and $b=\perp$, as desired if $\perp\leq$() is to be the case. [15]

Since $B$ has only two members, $c'c$ designates the greatest element of $B$. Let $a$, $b$, and $c$ range over $B$. Then $ab=a$ and $ab'=c'c$ both imply that $B$ is partially ordered. Specifically, let $\perp\leq$(). Then $b \leq a$, $ab=a$, and $ab'=c'c$ are all equivalent. More generally, the following theorem shows that $b'a$, $ab=a$ and $(a'b')=b$ all assert the same thing:

**3.3.13. Theorem** (*Consistency Principle*). $a \leq b$, $a \cup b = b$, and $a \cap b = a$ are equivalent Ba statements, and these in turn have the pa equivalents $a'b$=(), $ab=b$ and $(a'b')=a$.

*Proof.* See §A.3.

BA complementation can have an empty scope and the result is a lattice bound; that is the key fact that distinguishes BA from Ba. By virtue of A2, the other lattice bound is, in effect, the blank page. Bricken, in a personal communication, has argued that to grant the blank page a symbol, such as $\perp$, renders BA indistinguishable from Ba. To fasten on $\perp$ in this manner, however, overlooks the signal contribution of BA to our understanding of **2**: the members of the base set need be no more than empty complementation and the blank page.[16] The improper symbol $\perp$ can be seen as just a convenient way to refer to the blank page as a lattice bound. (§4.1 gives another justification for $\perp$.) In all other respects, 3.3.10-13 and the adjacent discussion show that BA and **2** are isomorphic. This fact is independent of the ontological status of $\perp$ or its denotatum.

The near-isomorphism of BA and Ba does not demean BA in any way, because Ba is a quite rich formal system; see the references under “Boolean Algebra” in the Bibliographic Postscript, especially Givant & Halmos (2009), abbreviated below as “GH.” The balance of this chapter gives a quick taste of this richness, starting with a famous deep result about Ba, the *Stone Representation Theorem* (SRT). First some definitions:

**3.3.14. Definition.** Let $a,b \in B$, where $B$ is some base set. $a$ is an *atom* iff $b \leq a \rightarrow (b=\perp \vee b=a)$. A Ba is:

- *Atomic* if for any $b \neq \perp$, there exists an atom $a$ such that $a \leq b$;
- *Complete* if every nonempty subset of $B$ has a least upper bound.

**3.3.15. Theorem (SRT)**. A Ba is isomorphic to the algebra of all subsets of its set of atoms iff the Ba is complete and atomic. (GH §14, Cor. 1).

*Remark*. The SRT holds unconditionally when $B$ is finite and hence bounded, in which case the associated Ba is always complete and atomic (Stoll 1963: Th. 6.5.1). For a short and elegant proof for finite $B$, see Davey and Priestley (2002: §5.5). When $B$ is infinite (a situation that never arises in this book), see Stoll’s Th. 6.5.4. Because a Ba with infinite $B$ is not necessarily atomic, the notion

15. If $a=\perp$ or $b$=(), $(a)b$=() by C3 and A2. Substituting the only remaining valuation, $a$=() and $b=\perp$, into $(a)b$ yields $(())\perp=\perp\perp=\perp$.

16. Here *LoF* unwittingly walked in C.S. Peirce’s footsteps, namely his graphical logic; see §6.1.

of atom is replaced by that of *maximal ideal* (GH: §20; Stoll 1963: 263, 269), whose existence is assured by Zorn's lemma.[17]

The SRT enables the proof (not shown) of a valuable theorem. First some more definitions. Let a Ba whose $B$ has cardinality ≥2 be *nontrivial*. A *subalgebra* of a Ba with base set $B$ is a (proper) subset of $B$ closed under meet, join, and complementation. We then have:

**3.3.16. Theorem**. A Boolean identity (tautological equivalence) holds in every possible Ba iff it holds in **2**.

*Remark*: GH, Th. 9. All Ba⇒**2** is easy to show, because for every nontrivial Ba, $\{\perp,\mathrm{T}\}\subseteq B$. Hence **2** is a subalgebra of every nontrivial Ba. Koppelberg's (1989: Prop. 2.19 (a) and (d)) concise proof of **2** ⇒All Ba, invokes the SRT and ultrafilters, a notion beyond the scope of this book.

Since **2** is a model for BA, a consequence of 3.3.16 is that the advantages of BA as a calculation tool apply to any nontrivial Ba. Unfortunately, BA as it stands cannot notate nontrivial Bas other than **2**, because of the following theorem:

**3.3.17. Theorem**. The cardinality of $B$ in BA is necessarily 2.

*Proof*: §A.3 establishes that $(x=() \vee x=\perp) = ()$.

Modifying BA so that its models include any Ba with a finite base set would be a worthy endeavour. Models of such "large" Boolean algebras include certain types of *mereology* (Casati and Varzi 1999), the formal theory of part and whole.

### 3.4. Boundary Notations for Other Algebraic Structures.

"*Algebra includes many formal calculations drawing consequencs from axioms, so the notation should be chosen to make these calculations efficient. The device of juxtaposing two letters… is so efficient that it is used in many different senses…*" Birkhoff and MacLane (1998: 70).

Boundary notations for other algebraic structures are possible. Let concatenation denote an unspecified binary operation, but now let enclosure in parentheses simply denote grouping in the way that is customary in mainstream mathematics. This notation suffices for *semigroups*, for which the single axiom is $(ab)c = a(bc)$. Add the axiom $aa=a$ to a semigroup and a *band* results. Add $ab=ba$ to a band and a *semilattice* results. Since parentheses are no longer needed to indicate grouping, we are free to reinterpret enclosure so that if $ab$ denotes the meet of $a$ and $b$, then $(ab)$ denoted the join of $a$ and $b$. Or vice versa; both interpretations are equally valid. The result, as shown in §3.3, is a *lattice*.

Algebras with a single binary operation are *magmas* (an older term for which is "groupoid"). BA syntax is adequate for all magmas whose binary operation associates, and having one unary operation and at least one distinguished element. BA is an example of each of the following magmas:

- *Commutative semigroup* because BA juxtaposition is order-irrelevant;
- *Semilattice* because C1 is a BA consequence;
- *Commutative monoid* with *identity element* ⊥, by virtue of B2**;**

17. Zorn's lemma, an equivalent of the axiom of Choice, is widely invoked in contemporary mathematics, especially algebra (Machover 1996: chpt. 5). For an exposition of the SRT with a more topological flavour, see Cori and Lascar (2000: §2.6) or Givant and Halmos (2009).

- *Logic algebra*, a commutative monoid with a unary operation, complementation, such that, by B2 and B3, the inverse element () is the complement of the identity element (()).

That Ba (and all lattices) are also magmas is seldom remarked.

The most studied type of magma is the group, defined as follows:

**3.4.1. Definition.** A *group* is a set $G$ closed under a binary operation, called *product* and denoted by juxtaposition, and a unary operation, called *inverse*. The product of $a$ and $b$ is denoted $ab$, and the inverse of $a$ is denoted $(a)$. (This is just BA notation, of course!). Then the following axioms hold:

| | |
|---|---|
| G1: $ab.c = a.bc$. | Product associates. This established, the period is no longer necessary; |
| G2: $ea = a$. | $e \in G$ is the *identity element*; |
| G3: $(a)a = i$. | $i \in G$ is the *inverse element*; |
| G4. $i$=$e$. | The identity element and inverse element are identical. |

*Remark*. Both $e$ and $i$ are provably unique. Given some $a \in G$, conventional group theory defines the inverse element of $a$ as the $b \in G$ such that $ba$=$e$.

Lettng $i$=() yields a boundary notation for groups isomorphic to the BA notation this book advocates for Ba. An instance of G3 is $(i)i = i$. Since the $e$ in G2 is unique, $(i) = e$, so that $e$ = (()). G4 then implies that (())=(), so that A2 does not hold in group theory. Moreover, both () and (()) may be erased at will. A1, on the other hand, does hold because $ii$ [G4] = $ei$ [G2]] = $i$.

An *abelian group* is a group whose product commutes as well as associates. Hence G1 can be replaced by a single axiom implying commutativity and associativity. Incorporating G4 into G2 and G3, the axioms for abelian groups then are:

| | |
|---|---|
| GA1: $ab.c = ac.b$. | In §A.14, I derive commutativity and associativity from G1-G3; |
| GA2: $()a = a$. | () is the identity element; |
| GA3: $a(a) = ()$. | () is also the inverse element. |

See §A.14 for more re abelian groups.[18] Note that GA1 is one form of B1, and GA3 is B3. An instance of G3 is $(e)e$=$i$. Then by G2 and commutativity, we have $(e)$=$i$. Since we proved $e$=$(i)$ above, $e$ and $i$ are mutual complements, a fact holding for all complemented lattices.

The BA initials are GA1, G2, G3, and B4, but not G4. In BA, B3 establishes that $i$=(), and B2 lays down that $e$=$\perp$. Two arithmetical facts that characterize BA are $(e) = i$, just proved, and $e \leq i$, equivalent to $ei = i$ and thus a trivial instance of G2. Letting $i$=() and $e$=(()), these arithmetical facts become ((())) = () and ()(()) = (), both trivial consequences of A2. Logic algebras require $e \neq i$, and this is the precise point where logic algebras and group theory part company. Hence all that distinguishes BA from abelian groups is that B4 holds only in BA, and (()) = () holds only for abelian groups. Alternatively, the axioms for an abelian group are B1-B3 plus the arithmetical identity () = (()). The unconventional use of $i$ and $e$ in 3.4.1 merely serves to highlight the closeness of BA and abelian groups described in this paragraph.

The key fact distinguishing BA from abelian groups is B4, which has no group theory counterpart. B4 assures that BA is a:

- *Lattice*, by making demonstrable the lattice properties L1 and C1;

18. A name in bold type is that of the Metamath ZFC theorem that corresponds to the abelian group axiom preceding it: associativity (**grpass**), commutativity (**ablcom**), G2 and G3 (**grpidinv**). On magma, semigroup, monoid, group, and semilattice, see Burris and Sankappanavar (1981: §2.1). On other connections between Boolean and other algebras, see Rudeanu (1974: §§12.3-7) and Burris et al (1981: §II.1).

- *Distributive lattice*, by making C5 demonstrable.

Given the crucial role B4 just encunciated, it is natural to ask what $a(ab)$ evaluates to in other algebraic structures. The answer to this question proves surprisingly rich and will take up the balance of this section. It turns out that a fair number of algebraic structures betray themselves by what $a(ab)$ simplifies to, given the axioms for the structure in question. For semigroups, we have simply that $a(ab) = (aa)b$. For a band, $a(ab)$ [Ass.] = $(aa)b$ [L3a] = $ab$. This result also holds for semilattices, with the added proviso that $ab=ba$. For lattices, $a(ab)$ [L1a] = $a$.

What $a(ab)$ evaluates to in group theory will require a bit of work. We now demonstrate an elementary fact about group theory:

**3.4.2. Lemma**: The inverse of $ab$ is $(b)(a)$.

*Proof. Dem*. $ab(b)(a)$ [G3] = $a()(a)$ [G2] = $a(a)$ [G3] = $()$. □

This proves that $(b)(a)$ is *an* inverse. It can be proved that the inverse of any element is unique.[19] Hence $(ab) = (b)(a)$.

*Remark*. Note that the demonstration does not invoke commutativity in any way. Hence this lemma holds for all groups, not just abelian ones.

Hence in group theory, $a(ab)$ [3.4.2] = $a(b)(a)$. In abelian group theory, however, $a(ab)$ [3.4.2] = $a(b)(a)$ [TR, Ass.] = $(a)a(b)$ [G3] = $()(b)$ [G2] = $(b)$. Note that the same result holds for nonabelian groups if we start from $(ab)a$. $a(ab)$ drives a wedge between abelian and non-abelian groups.

We now touch on a few less known magmas that do not associate, so that parentheses are needed to indicate grouping. Parentheses are available for this purpose because these magmas lack a unary operation. $a(ab)=b$ is an axiom for *Steiner magmas*. A *rack* satisfies the axiom $(ab)c = (ac)bc$. Substituting $a$ for $c$ yields the axiom instance $(ab)a = (aa)ba$. An idempotent rack is a *quandle*, in which case $(ab)a = (aa)ba = (a)ba$. The axioms for the *implicational calculus* (Wolfram 2002: 803) are $(ab)a = a$, $(ab)b = (ba)a$, and $a(bc) = b(ac)$. An instance of this last axiom is $a(ab) = a(ab)$. The axioms for an *equivalence algebra* include $aa = a$, $(ab)a = a$, and $a(bc) = ab(ac)$. An instance of the last axiom is $a(ab) = aa(ab) = a(ab)$.

The following structures have more than one binary operation.

*Quasigroups.* A quasigroup can be formulated in terms of three binary operations. Let $(ab)$ denote one of those operations, and $[ab]$ denote another. Then two of the quasigroup axioms are $[a(ab)] = b = (a[ab])$. A quasigroup can also be formulated like a group, with a binary product denoted by concatenation and a unary inverse denoted $(a)$. A *Bruck loop* is a quasigroup with an added inverse such that $(ab) = (a)(b)$. Then $a(ab) = a(a)(b)$.

*Ring*. Let $(ab)$ denote the ring sum of $a$ and $b$, and $[ab]$ denote the ring product of $a$ and $b$. Then $[a(bc)] = ([ab][ac])$ says that product distributes over sum. $a(ab)$ is not defined in ring notation, but $[a(ab)]$ is and it equals $([aa][ab])$ by the distributive law. Moreover, it is the case that $(a(ab)) = ((aa)b)$, because ring addition is associative.

*Relation algebra* (Givant 2006) is a proper extension of Ba, with letters ranging over all possible binary relations whose field is some given set (see 2.3.8). There are two additional operations, binary composition[20] denoted here by concatenation, and unary converse which will not detain us. En-

19. Theorem 1.4, http://en.wikipedia.org/wiki/Elementary_group_theory .

20. Let $A$ be some set, and let $a,b \subseteq A \times A$. Then the *composition* of the binary relations $a$ and $b$, denoted $ab$, is the set of ordered pairs $(x,z)$ such that there exists a $y \in A$ with $(x,y) \in a$ and $(y,z) \in b$. If $a$ and $b$ are func-

**Table 3-3. $a(ab)$ as Algebraic Discriminant.**

| *Algebraic Structure*: | a(ab) *reduces to*: | | ()() = | (()) = |
|---|---|---|---|---|
| 1 *binary nonassociative operation.* | | | | |
| Steiner magma | $b$ | A | --- | --- |
| Rack | $(ab)a = (aa)ba$ | A | | |
| Quandle | $(ab)a = (a)ba$ | T | | |
| Equivalence Algebra[21] | $a(ab)$. $(ab)a = a$ | T | | |
| Implicational Calculus | $a(ab)$. $(ab)a = a$; $(ab)b = (ba)a$ | A | | |
| 1 *binary associative operation;* 0 *unary operation.* | | | | |
| Semigroup | $(aa)b$ | A | --- | --- |
| Equivalential Calculus[22] | $b$ | T | | |
| Band | $ab$ | T | | |
| Semilattice | $ab=ba$ | T | --- | --- |
| 1 *binary associative operation;* 1 *unary operation.* | | | | |
| Bruck loop | $a(a)(b)$ | T | ()() | --- |
| Group | $a(b)(a)$ | T | ()() | () |
| Abelian group | $(b)$ | T | ()() | () |
| Logic algebra | $a(ab)$ | T | ()() | () |
| Lattice | $a$ | A | --- | --- |
| pa | $a(b)=(b)a$ | A | () | (()) |
| >1 *binary operation* | | | | |
| Quasigroup[23] | $[a(ab)] = (a[ab]) = b$ | A | --- | --- |
| Ring | $(a(ab)) = ((aa)b)$ <br> $[a(ab)] = ([aa][ab])$ | T <br> T | ()() | () |
| Relation algebra | $(ab)$ (if $a$ an equivalence) | T | | |

*Note*. The reduction shown in col. (2) is (A) an axiom/postulate or a substitution instance thereof. Otherwise it is a theorem (T). I know of no printed source covering the diversity of known algebraic structures. For a concise listing of a number of structures and their axiomatic properties, see: http://en.wikipedia.org/wiki/List_of_algebraic_structures . Also see Wolfram (2002: 803, 1171).

closure in parentheses denotes Boolean complementation, as before. If $a$ is an equivalence relation (as per 2.3.8) and $b$ is any binary relation, then $a(ab) = (ab)$ (Tarski and Givant 1987: 50, xxv).

tions, $ab$ is also a function. Because composition associates, the set of all possible binary relations on $A$ forms a monoid, with composition interpreting product and the identity relation interpreting the identity element.

21. Jezek and McKenzie (2001: 212, Prop. 1.1).

22. Wolfram (2002: 803).

23. http://en.wikipedia.org/wiki/Quasigroup . See also Steiner magmas (Wolfram 2002: 1171).

The upshot of this discussion is that a number of algebraic structures reveal themselves by what the simple expression $a(ab)$ (or in the case of quasigroups and rings, a close variant) evaluates to. Hence $a(ab)$ can thought of as sort of algebraic *discriminant*. The interpretation of the parentheses varies by structure. For a band, equivalence algebra, equivalential calculus, magma, semigroup, or semilattice, parentheses merely serve to indicate how the sole binary operation is to be grouped. For a quasigroup or ring, $(ab)$ denotes a binary operation over $a$ and $b$. Otherwise, $(ab)$ denotes a unary operation applied to the result of combining $a$ and $b$ via the binary operation appropriate for that structure. Table 3-3 summarizes these results.

# Chapter 4.
## pa Semantics: From BA to Boundary Logic.

"*Yet logic is nothing more than the properties of the act of distinction*!" (Kauffman 2001: 90).

*LoF* (pp. 113-17) shows how the CTV and the elementary Boolean algebra of sets are possible interpretations (models) of the pa. Before showing how the pa translates the CTV, I first sketch some facts about the key players in the CTV, the truth functors (functors for short). A functor has an arity $n \in N$. Because $B$ has cardinality 2, there are $2^{2^n}$ possible functors with arity $n$; in particular, there are 16 binary functors. Six of these map $a$ and $b$ into one of $\{a,\sim a,b,\sim b,\mathsf{T},\mathsf{F}\}$ and will not detain us. The remaining 10 binary functors are $\{\wedge,\vee,\rightarrow,\leftarrow,\leftrightarrow\}$ and their negations (see Table 4-2). There are $2^{2^1}=4$ possible unary functors; of these, only $\sim a$ need be considered. There are $2^{2^0}=2$ 0-ary functors, T and F by convention. All functors of arity>2 are redundant, because any formula employing such functors is tautologically equivalent to a formula whose functors all have arity≤2 (Epstein 1995: §II.J.3).

It would seem that there are 5+1+2=8 essential truth functors. In fact, there is ample redundancy among these, in that starting from 2 or 3 functors, the remaining 5 or 6 can be defined. If for any CTV formula, there exists an equivalent CTV formula in which only a subset of these 8 functors appears, the members of that subset are termed *expressively adequate* (abbreviated EA) or *truth-functionally complete* (Bostock 1997: §§2.7, 2.9).[1]

For the purpose at hand, the primitive basis of CTV (e.g., DeLong 1971: 107) consists of the primitive values T and F, and any EA set of functors. *Boundary logic* results from a one-to-one correspondence between the BA and an EA set of functors. Among the binary functors, $\vee$, $\wedge$, and $\leftrightarrow$, commute and associate, just as juxtaposition does in the pa. Denial, ~, is a unary functor whose scope is set by brackets, which is exactly the way (·) works in the pa. Many authors, including Meguire (2003), denote the denial of $a$ by $\neg a$ rather than $\sim a$. Here I reserve '¬' for intuitionist negation. We shall see in §4.3 that $\{\vee,\sim\}$ and $\{\wedge,\sim\}$ are EA; the upshot is two of the three interpretations of the pa shown in Table 4-1, which summarizes this chapter.

I now establish a correspondence between the PA and conventional logic, beginning with the assumption ()⇔T. Table 2-1b immediately reveals that the semantics of ($\alpha$) are identical to those of $\sim\alpha$, namely ~T=F (A2) and ~F=T. Thus emerges the most salient fact about boundary logic: *a truth value interprets a negation with an empty scope*, just as an empty boundary denotes a BA primitive value. ()() = () (A1) and $\perp\perp = \perp$ in Table 2-1a imply that juxtaposition is idempotent. The two remaining cells of Table 2-1a reveal that juxtaposition commutes, as discussed in §3.2. Hence by virtue of the PA, $\alpha\beta$ can interpret either $\alpha\vee\beta$ or $\alpha\wedge\beta$ and the road to a CTV translation of Table 2-1 is now clear.[2]

In the 1880s, Frege and Peirce laid down that the preferred primitive CTV connective should be the conditional, nowadays notated by infix '→'.[3] The well-known equivalence $a\rightarrow b \Leftrightarrow \sim a\vee b$ suggests the interpretation $a\rightarrow b \Leftrightarrow a'b$. Then note that $a\rightarrow\mathsf{F} \Leftrightarrow a'\perp = a'$; in this fashion the pa happily ac-

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1. For more on connectives, axiomatics, etc., see the references under *Calculus of Truth Values* in the Bibliographic Postscript.

2. 3.3.4 and 3.3.6 also suggest that the CTV with primitive $\{\vee,\neg\}$ is a model for the pa.

3. Quine (1982: §3) rightly prefers 'conditional' to the 'material implication' of *PM*, because $a\rightarrow b$ does *not* translate "$a$ implies $b$", but arguably does translate "if $a$ then $b$". I prefer reading $a\rightarrow b$ as a synonym for $a\leq b$, where $a$ and $b$ are members of some ordered set.

commodates the EA set {→,F}. Table 2-1b now translates as T→⊥ and its converse, ⊥→T. §4.2 discusses other possible CTV interpretations of the pa. Henceforth, "Prior *m.n*" refers to axiom set *m.n* in Appendix I of Prior (1962).[4]

**Table 4-1. Some Interpretations of the pa.**

| *Interpretation* | *Primal* | | *Dual* |
|---|---|---|---|
| *Key Binary Functor* | Alternation | Conditional | Conjunction |
| *Implied EA Functor Pair* | {∨,~} | {→,F} or {→,~} | {∧,~} |
| pa *Equivalent* | {((··)),(·)} | {(·)·,(())} or {(·)·,(·)} | {((··)),(·)} |
| *Representation of*: | | | |
| Alternation | *ab* | | (*a′b′*) |
| Conjunction | (*a′b′*) | | *ab* |
| Conditional | *a′b* | | (*ab′*) |
| *Antecedents* | 1910: *PM*, 6.11 | 1879: Frege, 1.1<br>1885: Peirce, 3.11<br>1956: Church $\mathbf{P_1}$, 1.4c | 1892: Johnson<br>1897: Peirce EG |
| *Recent Examples* | Halmos & Givant (1998: §§8,13) | Machover (1996: §7.6)<br>Bostock (1997: §5.2) | Quine (1982: §1) |

**Note**. *m.n* refers to a numbered system in Prior's (1962) Appendix I.

### 4.1. Duality.

Let *S* be a set partially ordered by the relation *R*, and let *a*,*b*∈*R*. Then there exists a relation *R′* that also partially orders *S*, such that *bR′a*=T ↔ *aRb*=T; this is the *duality principle* for posets (Donnellan 1968: Th. 13; Davey & Priestley 2002: 1.19-20). Let *S*=*B*, [{T,F}] *Rab* ⇔ *a*≤*b* [→], and *R′ab* ⇔ *a*≥*b* [↚], and the duality of Ba [CTV] follows. Ba is typically formulated such that *B*= {0,1}, 1′≡0, and 0≠1. BA duality follows from (())≠() being a trivial consequence of A2, and from (())≠() ⇔ 0≠1.

By 3.3.2, *B* is partially ordered. *B* must also be connected, since *B* has only two members, so that one of ⊥≤() or ()≤⊥ must be the case. ()=⊥ leads to triviality; hence the inequalities must hold strictly. Thus far, I have tacitly assumed ⊥<(). The nearest *LoF* gets to the content of this paragraph is the first complete paragraph on p. 113.

**4.1.1. Definition**. The BA semantics that flow from assuming ⊥<() [()<⊥] make up the *primal* [*dual*] *reading*. Each reading is the *semantic dual* of the other. *Duality* refers to the fact that Ba, BA, and logic can all be carried out under either reading. To switch from one reading to the other, *mutatis mutandis*, is known as *dualization*.

Duality is little more than an interesting consequence of *B*'s being an ordered set. By an interprêtation of BA I mean a one-to-one correspondence between *B* and another set, and there are two possi-

4. For more on systems with {→,F} primitive, cf. Prior (1962: §I.III.1, 3.11-13). Systems based on {→,¬} are quite standard, e.g., Prior 1.1-5, Church's (1956: §20) $\mathbf{P_2}$, Epstein's (1995: 408) **PC**, and Mendelson's (1997: 35) **L**. Systems with {∧,¬} primitive include that of Johnson (1892) discussed in §6.2 below, those of Rosser and of Sobocinski (Prior 6.3), and the modal logics of C. I. Lewis (Prior 11.1). Peirce's existential graphs are the subject of §6.1. For more on historical CTV axiom systems, see §6.2-3 below, Prior (1962: Appendix I), and Epstein (1995: 407-9).

ble such correspondences between $B$ and {T,F}. Thus far, I have assumed T⇔(). The dual reading begins with T⇔⊥. Conjunction now interprets juxtaposition, and the conditional interprets ($ab'$). Thus the dual of {∨,~} is {∧,~}.

Under the dual reading, Table 2-1a is now the table for Boolean multiplication, and Boolean and numerical multiplication yield the same result when the base set is assumed to be {0,1}. Perhaps surprisingly, Table 2-1 and T1-T7 hold under both interpretations. Likewise, the rules defining and simplifying PA and pa formulae do not change. Hence the syntax of BA is invariant under dualization; the BA is a *self-dual* formalism. PA duality is merely a semantic affair, namely switching the two possible one-to-one correspondences between $B$ and some interpretive set such as {T,F}.

Matters are a bit more involved for the pa, because dualization alters the semantics of juxtaposition. The semantics of a formal system are known as its *truth definition* or *Boolean valuation* (Smullyan 1968: §I.2, Def. 1). Recall that an atomic valuation (3.1.3) assigns one of () or ⊥ to every variable. The pa then has the following trivial truth definition:

**4.1.2. Definition**. *A Boolean valuation for* BA. Let $\phi,\delta$ be metalogical notation for BA formulae, and let the value of $\phi$ be $|\phi|$, given some atomic valuation. All molecular formulae then evaluate to either () or ⊥ by recursive application of two elementary rules: $|(\phi)|=(|\phi|)$, and $|\delta\phi|=\max[|\phi|,|\delta|]$, where max[(),⊥]=() [=⊥] under the primal [dual] reading.[5]

This truth definition follows trivially from Table 2-1. A tautology can now be defined as a formula whose value is invariant to the choice of atomic valuation. Moreover, $\phi=\delta$ is a tautological equivalence if $|\phi|=|\delta|$ holds for all atomic valuations. Some further definitions:

**4.1.3. Definition** (adapted from Halmos and Givant 1998, §22): Let $\alpha = \alpha\langle a_1,\ldots,a_n\rangle$ be a formula containing the atomic formulae $a_1,\ldots,a_n$. $\alpha$ is the *primal*, $(\alpha\langle a_1,\ldots,a_n\rangle)$ the *complement*, $\alpha\langle a_1',\ldots,a_n'\rangle$ the *contradual*, and $\alpha^D = (\alpha\langle a_1',\ldots,a_n'\rangle)$ the *syntactic dual* (*dual* for short). The dual of the dual is the primal; hence a primal and its dual are known as a *dual pair*.

4.1.4 answers the question: if the equation $\alpha = \phi$ holds under some or all atomic valuations, what is true of $\alpha^D$ and $\phi^D$?

**4.1.4. Duality Theorem**. $\alpha = \phi \leftrightarrow \alpha^D = \phi^D$.

*Proof*. See §A.8.

A corollary of the Duality Theorem is that the dual of a tautology is also a tautology. Keeping in mind that $\alpha$ in 4.1.4 is a tautology if $\alpha$=() or ⊥ for all possible valuations of $a_1,\ldots,a_n$, 4.1.4 says more. If a formula or equation is tautologous under some interpretation, then its contradual and dual are also tautologies under that same interpretation.[6]

Duality, about which *LoF* is silent, is another compelling reason for an explicit symbol denoting the unmarked state. There is no syntactical or proof-theoretic ground for preferring the primal reading

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5. On truth definitions, see Bostock (1997: §§2.4, 3.4) and Hodges (2001: §3).

6. Quine (1982: §12) states five "laws of duality." The first follows from semantic duality; the second, proved in §A.8, defines syntactic duality. His third law is $\alpha$=() ⇔ $\alpha^D$=⊥; the fourth, $(\alpha \rightarrow \phi) \leftrightarrow (\phi^D \rightarrow \alpha^D)$; the fifth, 4.1.4. On duality, also see Bostock (1997: §2.10) and Givant & Halmost (2009: chpt. 4).

over the dual one, or vice versa. Spencer-Brown preferred the primal reading because $a'b$ is a more economical representation of the conditional than $(ab')$ (*LoF*, p. 113-14).[7] However, there is a modest semantic reason for preferring to read concatenation as conjunction to disjunction. I agree with Prior when he wrote:

> "…'and' and 'not' are the only operators which are quite unambiguously truth functional in ordinary speech; truth functional interpretations of other ordinary-speech connectives all wear at times an air of artificiality." Prior (1962: 254).

In §6.1, we shall see that Peirce too came to prefer the dual reading.

## 4.2. Boundary Logic.

"*…everything in pp. 98-126 of* Principia Mathematica *can be rewritten without formal loss in the one symbol* $\urcorner$*… Allowing some 1500 symbols to the page, this represents a reduction of the mathematical noise-level by a factor of more than 40,000.*" *LoF*, p. 117.

Table 4-2 translates the ten nontrivial CTV binary connectives into the $\mathsf{pa}$, assuming the primal reading. Each row of Table 4-2 contains a dual pair; hence the connectives can be grouped into two groups of five, I and II, with each group being the dual of the other. Connectives sharing the same numerical identifier (shown in the two middle columns) can be derived from each other via negation. Let $a^*$ stand for either $a$ or $a'$; $a^*$ is a *literal*. The *simple* connectives are those that can be described by $a^*b^*$ or the duals thereof; these are $ab$, $a'b$, $ab'$, $a'b'$, and their syntactic duals.

Note that assigning () to $\mathsf{T}$ [$\perp\leq()$] is just as arbitrary as assigning it to $\mathsf{F}$ [$()\leq\perp$]. But once a assignment is made, the $\mathsf{pa}$ representation of all connectives is determined. Table 4-2 translates $ab$ as $a\vee b$, and its dual, $(a'b')$, as $a\wedge b$, both as per the first column of Table 4-1. Likewise, either $a'b'$ or $(ab)$ translates the Sheffer stroke, $a|b$. These translations render obvious that | can be read as both "not and" and "if $a$ then not $b$"; the latter reading suggests more strongly, perhaps, the peculiar expressive power of the Sheffer stroke. The duality of $a'b'$ and $(ab)$ points to "not or" as the semantic dual of the Sheffer stroke.[8]

The meaning of CTV duality should now be clear: for any statement $\alpha$, there exists an equivalent statement $\alpha^D$ derived by interchanging $\wedge$ and $\vee$, $\rightarrow$ and $\nleftarrow$, $\leftrightarrow$ and $\nleftrightarrow$, | and $\downarrow$, and $\mathsf{T}$ and $\mathsf{F}$. More generally, under either interpretation, the $\mathsf{pa}$ representation of conjunction is the dual of the $\mathsf{pa}$ representation of alternation, and the same dual relation holds for the conditional and the negation thereof.

Duality reveals that the conventional syntax for $\mathsf{Ba}$ and CTV are uneconomical. A given $\mathsf{pa}$ formula enjoys a multiplicity of CTV interpretations, revealing the ample redundancy inherent in the CTV. For instance, take De Morgan's well-known laws, $\sim(a\vee b)\leftrightarrow(\sim a\wedge\sim b)$ and $\sim(a\wedge b)\leftrightarrow(\sim a\vee\sim b)$. The $\mathsf{pa}$ translation of these laws, $(ab)=(ab)$ and $a'b'=a'b'$, immediately reveals that these laws are artifacts of conventional notations, and as such are trivial.

7. Spencer-Brown makes too much of this, especially if one downplays the conditional in favor of conjunction/alternation. Moreover, while $(ab')$ has more symbols than $a'b$, $(ab')=(())$ is equivalent to $b'a=()$, which has no more symbols than $a'b=()$.

8. The misconception that the $\mathsf{pa}$ is little more than a new notation for the Sheffer stroke (Grattan-Guiness 2000: 557; Wolfram 2002: 1173) may stem from hasty readings of *LoF*'s Appendix 1.

Because the algebra of sets is a model for **2**, it is also a model for the pa. Let U be the universal set, $a,b \in$ U, and $\varnothing$ be the null set. Then the columns headed by “Sets” show how the algebra of sets is a model of the pa.

**Table 4-2.**
The 10 Nontrivial Binary Connectives (Functors).

| **Primal** | | | | | **Dual** | | | | |
|---|---|---|---|---|---|---|---|---|---|
| *Name* | *Logic* | *Sets* | **pa** | | | **pa** | *Sets* | *Logic* | *Name* |
| Alternation | $a \vee b$ | $a \cup b$ | $ab$ | 1 | 5 | $(a'b')$ | $a \cap b$ | $a \wedge b$ | Conjunction |
| Conditional | $a \rightarrow b$ | $a \subseteq b$ | $a'b$ | 2 | 4 | $(ab')$ | $b \sim a$ | $a \not\leftarrow b$ | Difference |
| Symmetric Difference | $a \not\leftrightarrow b$ | $a \Delta b$ | $(a'b)(ab')$ | 3a | 3a | $((a'b)(ab'))$ | $a \subseteq b \subseteq a$ | $a \leftrightarrow b$ | Biconditional |
| | | | $((a'b')(ab))$ | 3b | 3b | $(a'b')(ab)$ | | | |
| Converse | $a \leftarrow b$ | $a \supseteq b$ | $ab'$ | 4 | 2 | $(a'b)$ | $a \sim b$ | $a \not\rightarrow b$ | Difference |
| Sheffer stroke | $a \mid b$ | $\overline{a \cap b}$ | $a'b'$ | 5 | 1 | $(ab)$ | $\overline{a \cup b}$ | $a \downarrow b$ | NOR |

**Note**. Each row contains a *dual pair*. Items with the same number are negation pairs. The six remaining binary connectives are uninteresting as they map $\{a,b\}$ into one of $a$, $b$,$\sim a$,$\sim b$, **T**, and **F**.

Table 4-3 translates the *LoF* consequences into CTV notation, using Table 4-2 as the key. For each *LoF* consequence, Table 4-3 also supplies a name, if the conventional literature provides one, and the number of the corresponding tautology in Kalish et al (1980: §II.11) (KMM), an unusually comprehensive list of tautologies. *LoF* says very little about how its J1-C9 relate to the extant literature on logic and Ba. C1-C3 and C5 should be very familiar. B3 is the Law of Excluded Middle (LEM); B4, Johnson’s (1892: 342) Law of Exclusion; C2 means that () is a lattice upper bound, cf. 3.3.6; C4, the biconditional corresponding to an axiom in conditional form proposed by Peirce (W5: 162-90, 1885). C6 is the Law of Elaboration or Development, so called by Bostock (1997: 41). C7 and C8 are less well known. If $a$ and $b$ on either side of C8 were to trade places, the two sides of C8 would then form a dual pair.[9]

*LoF* invokes J1-C3 104x, C4-C9 15x, and C5-C9 a mere 6x. *LoF* invokes C2 more often than any other consequence, C3 excepted. C2 allows a subformula to be copied into and erased from any subspace deeper than the shallowest instance of itself (with the proviso that a subformula cannot be copied into a part of itself). While not a standard part of elementary logic, C2 is at once a trivial corollary of the Consistency Principle, 3.3.13, and a powerful tool for BA demonstrations (§§5.0, 5.2). §A.7 says more about C2 (here called B4).

Perhaps all I have done thus far is to employ the following elementary reasoning to eliminate all explicit truth functors from the syntax. Alternation and conjunction commute and associate; hence mere juxtaposition suffices to notate either. Brackets are then free to notate negation. It is well known that negation and one of conjunction or alternation are EA. Hence brackets are the only explicitly truth functional notation required. QED. Equivalently, recall that $\{\rightarrow,\perp\}$ is EA and that $(a)b$

9. The identities of this book correspond to the following *PM* (*2–*5) tautologies: B3, 2.08; B4, 2.621 & 2.67; C1, 4.25; C3, 4.13; C5, 4.41; C6, 4.42. Listing of tautologies include Rosser (1953: Theorem VI.6.1), Carnap (1958: T8-2, T8-6), Wolf (1998: Appendix 3), and Cori & Lascar (2000: §1.2.3). In Ba, J1 is known as “complementarity”; C2, “union”; C1, “idempotence”; De Morgan’s laws, “dualization”. For more on the relation between B2-B4 / C1-C4, on the one hand, and conventional logic and Ba on the other, see §A.6.

⇔ $a \to b$ and $(a)\bot$ ⇔ $\sim a$. Hence a single two place functor, (−)−, and the constant ⊥ also suffice to express all truth functors. To express juxtaposition, note that *ab* [C3] = **((**$a$**))**$b$ [B4] = (($a$)$\boldsymbol{b}$)$b$.[10]

| | **Table 4-3. The Standard Reading of the Identities.** | | | |
|---|---|---|---|---|
| BA | ***LoF*** | **Conventional Notation** | **Name** | **KMM** |
| B1 | Tacit | $a\vee b\vee c \leftrightarrow b\vee c\vee a$ | | 24,25,53,54 |
| B2 | --- | $a\vee\bot \leftrightarrow \bot$ | *B* has a lower bound | |
| B3 | --- | $\sim a\vee a \leftrightarrow \top$ | | 1,59 |
| --- | J1 | $\sim(a\to a) \leftrightarrow \bot$ | Contradiction | |
| B4 | C2 | $(b\vee a)\to a \leftrightarrow b\to a$ | Exclusion | 73 |
| C1 | C5 | $a\vee a \leftrightarrow a$ | Idempotence; Tautology | 47 |
| C2 | C3 | $\top\vee a \leftrightarrow \top$ | *B* has an upper bound. | |
| C3 | C1 | $\sim(\sim a) \leftrightarrow a$ | Involution | 110 |
| C4 | C4 | $((a\to b)\to a) \leftrightarrow a$ | Peirce's Law | 23 |
| --- | --- | $(a\wedge b)\vee a \leftrightarrow (a\vee b)\wedge a \leftrightarrow a$ | Absorption | 123 |
| C5 | J2 | $(a\vee r)\wedge(b\vee r) \leftrightarrow (a\wedge b)\vee r$ | Distribution | 62 |
| C6 | C6 | $(a\wedge b)\vee(a\wedge\sim b) \leftrightarrow a$ | Elaboration | 68 |
| C7 | C7 | $[(a\to b)\wedge\sim c] \leftrightarrow \sim[(a\vee c)\wedge(b\to c)]$ | | |
| C8 | C9 | $[(a\to r)\wedge(r\to\sim b)]$ <br> $\leftrightarrow \sim[(a\vee r)\wedge(r\to b)]$ | | |

With the pa and its CTV interpretation in hand, and given our definition of a Ba, we can speak to the algebraic structure of the CTV (cf. Stoll 1963: 267-76). Let $S_0$ be a set of CTV atomic formulae, and let $S$ be the set whose members are all possible formulae constructed from members of $S_0$ by conjunction (or alternation) and denial. Let logical equivalence '=' be the congruence relation (cf. 3.3.12) Ba requires. A congruence relation partitions its field into equivalence classes; let $S/=$ be the set of equivalence classes resulting from '='. Define ⊥ as $\sim a\wedge a$, $\forall a\in S$, and ⊤ as $\sim\bot$. Then $\langle S/=, \wedge, \sim, \bot\rangle$ is a Ba, specifically the "free Boolean algebra generated by $S_0$ under =," more commonly known as a *Tarski-Lindenbaum algebra*.

*A Historical Digression on Notation*.
What I enclose in parentheses, Spencer-Brown places under ⏋, the 'mark'; so that ($a$)$b$ and ($ab$) correspond to $\overline{a}\rceil b$ and $\overline{ab}\rceil$ in *LoF*. (Martin Gardner (*Scientific American* 1980 (2): 14) deemed *LoF*'s notation "eccentric.") Both BA and *LoF* notations have antecedents. In a paper written 1880 but not published until 1933, Peirce (4.12-20) proposed to notate Ba with concatenation, interpreted as NAND, and brackets. This notation is that of this book, except that Peirce limited concatenation to a binary scope. Kauffman (2001), citing an excerpt (Peirce 1976: 106-15) from a manuscript titled "Qualitative Logic," which Peirce wrote in 1886 but that was not published in full until 1993 (as *W*5: 323-71), points out that Peirce fused the overbar (denoting Boolean complementation) to the Boolean '+' (OR) to create the "sign of illation," closely resembling the '⏋' of *LoF* and having the same semantics.[11] Peirce saw that his sign of illation sufficed for Ba and syllogistic logic. Kauffman also notes that Nicod's (1917) '·⌈· ·' notation has the functionality of '. .⏌. ', but does not mention that Nicod sometimes wrote '$\overline{b}\rceil a$' instead of '$a\lceil\overline{b}$'.

10. The pa is thus also a $\langle B,(-)-,(())\rangle$ algebra of type $\langle 2,0\rangle$, a model of which is Church's (1956) $\mathbf{P_1}$; see Table 4-1.

11. Peirce's manyfold contributions to mathematics, logic, and semiotics inform Kauffman's discussion in other ways.

### 4.3. The Enigmatic Degeneracy of BA.

"*Every logical notation hitherto proposed has an unnecessary number of signs*."

Peirce (4.12, 1880).

The expressively adequate (EA) subsets of the functors in common use are {→, F/~} and {∧/∨, ~}. Table 4-1 shows how these four EA functor sets map into BA. This mapping reveals a curious detail: corresponding to each EA functor set is a pair of BA formulae, one involving one boundary, the other two. The question naturally arises as to whether this is true of the Sheffer stroke and its dual, and all nine EA functor sets with two members (Wernick 1942: 132), consisting of four dual pairs and the self-dual set, {→,↛}. Row 1 of Table 4-4 shows how the Sheffer stroke follows from the pair {(--),(())}. Rows 2 through 5 show how seven of the nine two-member EA functor sets can be derived by inserting 1 or 2 letters into (), and 0 or 2 letters into (()). The sixth row reveals that the dual pair of EA functors involving ↔ cannot be represented in this manner, suggesting that ↔ should be seen as a tacit conjunction of conditionals.

**Table 4-4.**
Building the Nine EA CTV Functor Pairs from () and (()).

| Pa | | *EA CTV Functor Pairs* | | | |
|---|---|---|---|---|---|
| | | *primal* | | *dual* | |
| (*ab*), (()) | 2,0 | a↓b | F | a\|b | T |
| (*a*)*b*, (()) | 2,0 | a→b | F | b↛a | T |
| ((*a*)*b*)*b*=*ab*=((*ab*)), (*a*) | 2,1 | a∨b | ~a | a∧b | ~a |
| ((*b*)*a*), (*a*) | 2,1 | b↛a | ~a | a→b | ~a |
| (*a*)*b*, ((*b*)*a*) | 2,2 | a→b, b↛a | | | |
| (*a*)*b*, ((*a*)*b*)((*b*)*a*) | 2,4 | a→b | a↮b | b↛a | a↔b |
| *Source for EA Functor Pairs*: Wernick (1942: 132). | | | | | |

The first five rows of Table 4-4 capture the essence of the correspondence between Ba and the CTV. The reader is may explore further and at leisure the symmetries present in the two leftmost columns of Table 4-4.

The pa suggests that expressive adequacy requires two capabilities, namely a way of:

- Concatenating subformulae. Let these ways be *a***b* and (*a***b*);
- Enclosing subformulae. We may create *a*′ in one of three ways:
  - Invoke it outright;
  - Given (*ab*), set *a*=*b* so that (*a*$\underline{a}$) [C1] = *a*′;
  - Given *a*′*b*, set *b*=(()) so that *a*′$\underline{(())}$ [B2] = *a*′.

Note how (()) follows from (*b*′*a*): either erase *a* and *b*, or let *a*=*b* and invoke B2-B3. Note that the Sheffer stroke is EA by itself. As () and (()()) denote distinct primitive values, () alone suffices for all of truth functional logic.

Tables 4-4 and 4-5 reveal that all possible EA functor pairs can be obtained by inserting letters in certain ways into () alone, or into () and (()). Hence there is a sense in which the two members of *B* encapsulate all EA functor pairs. The members of *B* can be seen as the operators (-)- and ((--)-), where '-' indicates a possible location of a letter. BA does not syntactically demarcate operators from operands; only in context can the operators (-)- and ((--)-) be distinguished from the operands () and (()), the primitive values.

| **Table 4-5.** How $a*b$, $(a*b)$, and $a'$ Yield the Sheffer Stroke and the 9 Expressively Adequate Functor Pairs. | | | | | | |
|---|---|---|---|---|---|---|
| | *Interpretation*: | | *Needed to Obtain* $(a)$: | | | |
| | *primal* | *dual* | *Assume* | *Interpretation* | | $(a)$ = |
| *Commute* | | | | | | |
| $ab$ | a∨b | a∧b | $a'$ | ~a | | --- |
| $(ab)$ | a↓b | $a \mid b$ | --- | --- | | $(aa)$ |
| *Do not Commute* | | | | *primal* | *dual* | |
| $a'b$ | a→b | b↛a | $a'$ | ~a | | --- |
| " | " | " | $(())$ | **F** | **T** | $a'(())$ |
| " | " | " | $(c'a)$ † | c↛a | a→c | $a'(a'a)$ |
| " | " | " | $(c'a)(c'b)$ | a↮c | a↔c | $a'(a'a)(a'a)$ |
| † Self-dual row. | | | | | | |

## 4.4. pa: Metatheory.

Every pa formula has a *normal form*, a fact repeatedly invoked in proofs of pa metatheory.

**4.4.1. Definition.** Let the pa formula α contain *n* variables so that $\alpha = f(a_1,...,a_n)$. The *normal form*, *NF*, is a formula, tautologically equivalent to α, and having the form:

(#) $$(a_i^*...)_j \Leftrightarrow \bigvee_j [\bigwedge_i a_{ij}^*].$$

All variables in (#) appear as literals.

$(a_i^*...)_j$ is the *jth disjunct*. The ranges of the indices *i* and *j* begin with 1 and are finite; otherwise, these ranges are deliberately unspecified, if only because the NF is not unique. Also, either *i* or *j* may in some cases not exceed 1. If the *j*th disjunct is (⊥), then the entire NF degenerates to (); if it is (()), the *j*th disjunct vanishes. The NF can be seen as the analog of a polynomial in ordinary algebra. It is easier to parse a NF if the variables in each disjunct appear in lexicographic order, moving from left to right. This reordering is allowed because the variables in any disjunct can be reordered at will, but is not a mathematical imperative.

Given any Ba/CTV formula, there exists an equivalent formula resembling the rhs of (#), namely a series of subformulae linked by alternation. Each of these subformulae in turn consists of literals linked by conjunction. This is the disjunctive normal form (DNF), whose dual is the conjunctive normal form (CNF). *LoF* is silent about the well known Ba/CTV result that there exists a CNF/DNF dual pair equivalent to any formula. In the pa, the distinction between the DNF and the CNF is merely semantic.[12]

4.4.2–7 lay down the metatheory of the pa. The corresponding proofs are in §A.9.

**4.4.2 (T14).** Let α be a formula such $d_\alpha^*$>2. Then α can be transformed, by taking steps, into an equivalent formula β such that $d_\beta^*$=2.

12. For more on the CNF and DNF see, e.g., Quine (1982: §10), Bostock (1997: §2.6), Halmos & Givant (1998: §38), and Cori & Lascar (2000: §1.3.2). Bostock defines the DNF so that each disjunct includes all *n* variables, in which case *i* in (#) necessarily ranges over 1 to *n*. He does this so that the truth table corresponding to α can be easily recovered from the DNF. This stipulation is unnecessary here because truth tables play no essential role in the pa.

*Remarks*.
**1.** In *LoF*, T14 only serves to help prove T15 and T17.

**2.** Crucial to the proof of T14 is the ability of C7 to transform any subformula of depth 3 into an equivalent sub-formula of depth 2. Invoking C7 repeatedly, beginning at each point in α with depth=$d_\alpha^* - 3$, transforms α into an equivalent formula with maximum depth≤2. The Appendix proof views a pa formula as an ordered tree; the *LoF* proof does not.

**3.** Read from left to right, both C4 and C8 can also be seen as depth reduction tools. C4 [C8] reduces a subformula of depth 2 [3] to one of depth 0 [2].

**4.** Note that no formula in B2-C8 is more than two parentheses deep, the left side of C7 and C8 excepted.

**4.4.3 (T15).** Let the pa formula α⟨*v*⟩ contain more than two instances of the variable *v*. Then α⟨*v*⟩ can be transformed, by taking steps, into an equivalent formula β⟨*v*⟩, such that β⟨*v*⟩ contains at most two instances of *v*.

*Remark*. In *LoF*, T15 only serves to prove T17. T15 is essentially a simple form of the following well-known Ba theorem (Hohn 1966: 229, Lemma 2), recast into pa notation as follows:

*Let f be a truth function whose arguments are* $x_1,...,x_n$. *Then* $f(x_1,...,x_i,...,x_n)$ = $(f(x_1,...,(),...,x_n)'x_i')$ $(f(x_1,...,\perp,...,x_n)'x_i)$, $1 \leq i \leq n$.

T14 and T15 together guarantee that every pa formula has an NF equivalent, whose depth does not exceed 2 and that contains at most two instances of any given variable.

**4.4.4 (T16).** If two or more formulae are equivalent in every case of one variable, they are equivalent.

*Remarks*.
**1.** A more fullsome restatement of this enigmatic theorem is "Let the variable *v* appear in one or both of the formulae α and β, and let $v_i$ =|*v*|. Let $\alpha\langle v = v_i\rangle$ and $\beta\langle v = v_i\rangle$ be $\alpha\langle v\rangle$ and $\beta\langle v\rangle$ with *v* set to $v_i$. If $\alpha\langle v = ()\rangle = \beta\langle v = ()\rangle$, and α⟨v=⊥⟩= β⟨v=⊥⟩, then α=β." The converse is also true.

**2.** *LoF* maintains that T16 justifies the decision procedure described in §5.1.[13]

**3.** Erasing every instance of a variable is equivalent to setting that variable equal to ⊥. Hence T16 has another implication, heretofore unmentioned: erasing every instance of any given variable in a tautology yields another tautology.

**4.** Prior (1962: §I.III.4) shows that the CTV can be derived from a single metalogical axiom. Its pa translation, $(\alpha\langle v = ()\rangle)(\alpha\langle v = \perp\rangle)\alpha\langle v\rangle$ = (), reveals that it is an instance of a clause (§5.3) and is equivalent to T16.

13. T16 is Cole's (1968: 346) rule R2. *LoF* (p. xvii) states that T16 resembles a lemma in Quine's (1938) proof that the CTV is complete. *LoF* is silent about Quine's later invention of TVA, which is essentially identical to the *LoF* decision procedure for which T16 is the main justification. Prior's (1962: 53, (3); 58-60) reexposition of Quine's completeness proof includes a lemma that is essentially T16. The *LoF* proof of T16 (restated in §A.9) is much easier than that of Quine or Prior.

**4.4.5 (T17).** The pa is complete.

*Remark*. A logic is *complete* if for any tautology $\alpha$, one of $\alpha$ or ~$\alpha$ can be proved from the axioms/initials, using the inference rules. The pa inference rules are in fact R1 and R2, although *LoF* does not make this explicit. Moreover, the completeness asserted by T17 is of the strong sort (*LoF*, p. 119), because adding an initial that cannot be proved from the existing initials would render the pa unsound. Finally, if a formula simplifies to a member of *B*, T17 also implies that there exists a corresponding tautology in the pa.

While T17 is arguably the most important (meta)theorem of the BA, it is not an unexpected result because, as we shall see, the CTV and **2** are both models of the pa, and the completeness of these models is well established. The *LoF* proof of T17 requires all *LoF* consequences except C1 and C4; hence T17 can be seen as the culmination of chapters 1-10 of *LoF*. The proof of T17 resembles Quine's (1938) proof (which *LoF* cites) that the CTV is complete, in that both proofs proceed by strong induction on the total number of variables in hypothetical pa formulae in normal form; this is the only explicit instance of an inductive proof in *LoF*. Crucial to this proof are two facts:

- Every pa formula has, by virtue of T14 and T15, an equivalent in normal form;
- A1 is a tautological equivalence because it is an instance of C1. The same is true of A2 and C3.[14]

If a logic is *sound*, then there does not exists a formula $\alpha$ such that $\alpha$ and ~$\alpha$ can both be proved in that logic. If both $\alpha$ and $\alpha'$ were provable in the pa, then $(\alpha'\alpha)$=(), contradicting B3. Hence the soundness of the pa follows from its completeness. If there exists a formula $\alpha$ such that both $\alpha$=() and ~$\alpha$=(), then all formulae can be equated to ().[15] In short, if a logic is both sound and complete, then for any statement $\alpha$ of that logic, $\alpha$ *is a tautology* $\leftrightarrow$ $\alpha$ *is provable*. A simple direct proof of soundness goes as follows:

**4.4.6. Theorem.** The pa is sound.

*Proof*: The pa initials are tautologies. When R1 is applied to a tautology, the result is a tautology. The rule R2 is valid only when applied to tautologies, in which case it yields a tautology. R1 and R2 are the sole rules of inference. Hence any demonstration results in a tautology. □

*Remark*. Any formal system for which a proof of this nature goes through is said to possess the *hereditary property* (DeLong 1971: 134).

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14. §A.1 demonstrates every consequence required for the proof of T17 in §A.9. §A.9 also includes a pa version of Kneebone's (1963: 48) proof that the CTV is complete, perhaps the simplest proof extant. Post (1921) was the first to prove the CTV complete (for a summary, see Hunter 1971: §30). For the completeness proofs of Hilbert and Ackerman, and of Quine (1938), see Prior (1962: §§I.II.3, I.III.2). These proofs require over 20 tautologies apiece. Hunter (1971: §§31, 32) restates the proofs of Kalmar and Henkin. Kalmar's proof is the basis for those of Stoll (1963: Th. 9.2.3), Epstein (1995: §§II.L.2-3, II.M.2), and Mendelson (1997: 1.14). These proofs require the Deduction Theorem and at least a dozen lemmas. The proof of T17 merely requires J1-C6 and C8. Henkin's proof has the advantage of yielding the Compactness and Interpolation theorems as corollaries. Finally, there is Anderson and Belnap's (1959) cryptic proof, restated less tersely in Hunter (1971: §37.4). Nowadays, the preferred approach to proving the CTV complete relies on refutation trees (e.g., Bostock 1997: §§4.6-7; Smullyan 1968: chpt. II). I invite the reader to decide which proof of completeness is the most economical.

15. The proof is in §A.1.3. On soundness, see Hunter (1971: §§24, 25a,b, 28) and Hunter's references to Church (1956). Also see "Inconsistency" in Table 5-3.

**4.4.7 (T18).** The *LoF* initials J1 and J2 are independent.

*Remark*. That is, neither initial can be proved from the other alone. The very concise *LoF* proof of T18 is wholly syntactic and predicated on there being only two initials. Given that J1 and J2 can be demonstrated from C6 alone (cf. §6.2 and §A.4), and that I prefer to make B1 explicit, so that there are in fact three initials, T18 loses some of its luster. On axiom independence, also see Hunter (1971: §36) and Bostock (1997: §5.2).

## Chapter 5.

## pa: Proof.

"*As a material machine economises the exertion of force, so a symbolic calculus economises the exertion of intelligence. ...the more perfect the calculus, the smaller the intelligence compared to the results*." Thus begins Johnson (1892).

What is conventionally termed a proof, *LoF* calls a *demonstration*, meaning a sequence of steps showing that two pa formulae, e.g., $\phi$ and $\gamma$, are equivalent. The consequence $\phi=\gamma$ results. Each step invokes an axiom, initial, or previously demonstrated consequence. R1 or R2 are seldom explicitly invoked in demonstrations. A demonstration is carried out entirely *within* an object language, the pa or other formal system. The correctness of a demonstration can be verified algorithmically, at least in principle. In *LoF*, *proof* applies only to (meta)theorems. A proof is necessarily metalinguistic, may draw on any device from mathematics or logic, and cannot be verified by algorithm.[1]

A demonstration of $\phi=\gamma$ consists of a sequence of formulae, beginning with $\phi$. Each formula in the sequence results from a *step*, inferred from one or more preceding formulae in a manner to be discussed shortly. The demonstration terminates when a step results in the formula $\gamma$. *Hilbert demonstration* is the admittedly pedantic name I propose for an exercise of this nature. A Hilbert demonstration is a variant of common-garden mathematical proof. By virtue of the completeness of the pa (T17), there exists a Hilbert demonstration for any tautology. But T17 gives us no clue on how to find that demonstration; the proof of T17 suggests restating $\phi=\gamma$ in normal form. Hence if both $\phi$ and $\gamma$ are hypothesized from the outset, it is usually easier to verify $\phi=\gamma$ by calculation.

Hilbert demonstrations were once the only verification technique. During the past 50-odd years, however, the reigning fashion among logicians (in contrast to mathematicians doing logic) was first natural deduction and sequent calculi, both derived from Gentzen's work in the 1930s, then refutation trees (Bostock 1997: §§4.1-4, 6.2, 7.4).

A *calculation*,[2] prefaced by *Cal*, is a type of demonstration that works hard the fact that $\alpha=\beta \Leftrightarrow \alpha\leftrightarrow\beta$. From this equivalence, it follows that $\alpha=\beta$ is logically equivalent to the pair of identities $\alpha'\beta = x = \beta'\alpha$, where $x$ is a primitive value. A calculation methodically simplifies $\alpha'\beta$ and $\beta'\alpha$ to the same primitive value, as per the following algorithm:

### 5.0.1. Algorithm.

1. To *simplify* $\alpha'\beta$ is to alter $\alpha'\beta$ in a series of steps, justifying each step using one or more identities, by progressively eliminating variable instances and boundaries. The objective is eliminate all variables from $\alpha'\beta$, so that the result is a primitive value. This is the LR (*left to right*) part of the calculation. If a primitive value cannot be obtained, STOP: $\alpha=\beta$ is not an identity;

   *Remark*. B2-B4 are especially powerful here, and C1-C6 can have considerable value. C7 can be useful as a last resort.

2. Simplify $\beta'\alpha$. This is the RL (*right to left*) part of the calculation. If a primitive value cannot be obtained, STOP: $\alpha=\beta$ is not an identity;

3. If the results of (1) and (2) are identical, then $\alpha=\beta$ by T7.

---

1. Others who distinguish between proof and demonstration are Quine (1951: 319-22), Machover (1996: 120), and Mendelson (1997: 36, fn. †).

2. "Calculation", a word not appearing in *LoF*, shortens Dijkstra and Scholten's (1990: 21) *calculation proof*, meaning a series of steps that transform a given Boolean expression into *True*.

4. Equivalently, $\alpha=\beta$ holds if the biconditional $(\alpha'\beta')(\alpha\beta)$ can be simplified to ().
*End of Algorithm*

### 5.1. A Decision Procedure.

"*An operand in the primary algebra is merely a conjectured presence or absence of an operator*."
(*LoF*, p. 88)

A different proof procedure, very much in the spirit of the PA, follows from T16: *If formulae are equivalent in every case of one variable, they are equivalent, and conversely*. Let $f\langle a\rangle$ and $g\langle a\rangle$ be formulae containing the variable $a$. Let $\phi\langle ()/a\rangle$ denote the uniform replacement of $a$ by (), and so on. If $f\langle ()/a\rangle = g\langle ()/a\rangle$ and $f\langle \bot/a\rangle = g\langle \bot/a\rangle$, T16 concludes that $f=g$, regardless of the values of any other variables appearing in $f$ and $g$. Hence $f=g$ is a tautological equivalence.

Consider the following algorithm for determining the satisfiability of $f$. Evaluate $f\langle ()/a\rangle$ and $f\langle \bot/a\rangle$; let these be two *branches*. Then note the following facts:

- Setting an unprimed [primed] variable to $\bot$ [()] makes the variable vanish;
- Setting an unprimed [primed] variable to () [$\bot$] results in ();
- If both branches result in the same formula, they terminate;
- If a branch simplifies to () or $\bot$, it terminates.

At any stage, a branch may be simplified by invoking a consequence; in this regard, B2, B3, and C2 are especially useful. A branch containing a recognisable tautology terminates; set it to whichever of () or $\bot$ is applicable. Repeat this procedure, each time selecting the remaining variable with the most instances so as to save labor. I find it useful to notate in the left margin of a row the variable being instantiated in that row. The structure of this algorithm is that of a tree; the algorithm terminates when all branches of the tree have terminated.

If all branches of the tree terminate with the same formula, the original formula is a tautology. If the branches terminate in a mixture of () and $\bot$, the formula is satisfiable, with the pattern of () and $\bot$ indicating the satisfying atomic valuations. This algorithm sufficiently resembles Quine's (1982: §5) *truth value analysis* (TVA) that I have appropriated the name.[3]

Fig. 2 gives, by way of example, a TVA proof of Leibniz's (1969: 244) *Praeclarum Theorema*, $[(p\rightarrow r)\wedge(q\rightarrow s)]\rightarrow[(p\wedge q)\rightarrow(r\wedge s)]$. I give three demonstrations of the *Theorema* in §6.1.

**Fig. 2.**
**Verifying Leibniz's *Praeclarum Theorema* via Truth Value Analysis.**

| | | | |
|---|---|---|---|
| | | $(((p'r)(q's)))((p'q'))(r's')$ | |
| $p$ | $(((r)(q's)))((q'))(r's')$ | | $((((()r)(q's)))((()q'))(r's')$ |
| $q$ | $((r's'))(())(r's')$ | $(((r)(()s)))((()))(r's')$ | $((((()r)(q's)))()(r's')$ [C2; B2] |
| | $((r's'))(r's')$ [B2] | $(((r)(()s)))()(r's')$ [A2] | () [C2] |
| | () [B3] | () [C2] | □ |

3. TVA first saw the light of day in the 1950 first edition of Quine (1982). I owe my discovery of TVA to Bostock's (1997: §2.11) elegant treatment thereof, to my knowledge unique among contemporary texts. Prior (1962: 17) gives a good example of a TVA proof in tree form. N.B: the "truth value analysis" in Kalish et al (1980: §§II.8-9) is an unrelated concept.

Verifying both $\alpha'\beta$ and $\beta'\alpha$ by TVA amounts to a TVA verification of the equation $\alpha=\beta$. Iterate the TVA until the set of formulae terminating the branches of $\alpha$ is the same as the set terminating the corresponding branches of $\beta$, in which case the equation if verified. In all other cases, the equation does not hold.

Given a formula with *n* distinct variables, the construction of the corresponding truth table requires evaluating $2^n$ PA formulae. This is always a tedious affair, but not an impractical one when *n* does not exceed three or four. Moreover, T3 assures us that a truth table and a pa demonstration must yield the same result. But thanks to TVA, any such resort to brute force is unnecessary. One round of the above algorithm, applied to the variable with the most instances, often suffices.[4]

## 5.2. More on the Initials B1-B4.

All CTV tautologies can be verified by TVA. Furthermore, the axioms of the CTV are a small subset of the set of all tautologies. From these undisputed facts, Quine (1951: *100; 1982: §13) argues, citing Herbrand, that all CTV tautologies are equally deserving of the honorific title of axiom (concurring voices include Smullyan 1968: 81, Wolf 1998: 79, and Cori & Lascar 2000: §4.1.1). This commendably egalitarian view, however, fails to distinguish the context of *verification*, i.e., determing the satisfiability of formulae, for which decision procedures are indeed adequate, from the context of *discovery*, one requiring proof from axioms or rules, trial, error, and inspiration.[5]

With this in mind, we shall now explore other bases for BA. Because the pa is a Ba, the many postulate sets proposed for Ba (Rudeanu 1963: chpt. 5) are also possible pa bases. Bricken (1986) demonstrated B3 and C5 from B4, C2, and C3 (and, following *LoF*'s example, asserted order irrelevance without granting it an axiom); hence B1, B4, C2, and C3 can also serve as initials. B4, C2 and C3 are very easily verified by a decision procedure, and B4 is more concise than C5. More important is that the examples in §5.4 will show how B4 and C2 alone suffice to justify most calculation steps.

Bricken (2002) proposes a more economical basis consisting of the complement of C2, $(a())=\perp$, and an identity equivalent to B4 and T13 (see §3.1), and tacit order irrelevance. From this basis Bricken calculated C3. Bricken's work, and the many detailed demonstrations in §§5.3-4, shows that adding B4 to B1-B3 is a Ba basis. In §A.1, I derive J1 and J2 from B1-B4, and B2 and B3 from J1 and J2. T18 in §A.9 proves B1-B4 independent.

Table 5-1 shows how the B2-B4 can be seen as insertion/cancellation rules whose tacit goal is to make all variables vanish:

- *B3 Insert*. Any formula may be written on both sides of an empty boundary.
- *B3 Cancel.* If the *entire* content of a boundary also occurs in the pervasive space, both instances of that content may be erased, leaving an empty boundary.
- *B2 Insert/Cancel*. (()) may be inserted/erased anywhere.

---

4. Ascertaining via truth tables the satisfiability of a Boolean formula having *n* distinct variables requires evaluating $2^n$ interpretations. Hence the truth table decision procedure for CTV satisfiability is said to require *exponential time*. Whether there exists a decision procedure for satisfiability that is merely a polynomial function of *n* (i.e., is said to be executable in *polynomial time*) is a major unsolved problem in computational mathematics; see Hodges (2001: 23-24) and references cited therein. While I submit that TVA is quicker and easier than truth tables, especially when *n* is not large, I cannot claim that executing TVA on a computer would require less than exponential time for any *n*.

5. For a defense of Hilbert proof in contexts where a decision procedure is available, see Epstein (1995: §II.K.1).

- *B4 Insert*. Anything outside a boundary may be copied into a boundary.
- *B4 Cancel*. If any *part* of the content of a boundary also occurs in the pervasive space, that part within the boundary may be erased. When a given subformula appears on both sides of a boundary, the deeper instance is always redundant.

**Table 5-1. The pa in a Nutshell.**

| *Initial* | *Content of* () | *Action* | *Notation* | *Antecedents* |
|---|---|---|---|---|
| B1 | na | The content of a subspace may be reordered at will. | $abc = acb$<br>$abc = bca$ | Dilworth (1938)<br>Byrne (1946) |
| *Let there be a boundary, and let* a *appear outside it.* | | | | |
| B3 | *a* | Erase inner and outer *a*, leaving (). | $(a)a = ()$ | Natural deduction |
| B2 | () | Write, erase (()) at will. | $(())a = a$ | Huntington (1904) |
| B4 | *ab* | Only the shallowest instance of a subformula is nonredundant. | $a(\beta\langle a\rangle)=a(\beta\langle\bot//a\rangle)$ | (De)Iteration in Peirce's Existential Graphs; cf. 2i,e in Table 6-1. |

Likewise, B3 and B2 can be seen as paired insertion and elimination (Fitch's term was *intelim*) rules for () and ⊥, respectively. B3 establishes that BA is a complemented lattice. B3 can also be seen as the sole axiom of natural deduction, and as akin to the rule for closing a branch in a tableau (cf. Bostock 1997: chpts. 4,6). B4 assures that BA is a distributive lattice, and can be seen as a natural deduction intelim rule for ¬ and ∨/∧.

It should now be clear why pa demonstrations here and in *LoF* work B2-B4 very hard. It is also a raw fact that consequences beyond these are not often required. Demonstrating C8 requires C6; calculating it requires C7 as well. The proofs of T14-T18 invoke C8 twice and C7 once. The pa demonstrations in §3.3 and §5.4 invoke C1 twice and C4 once.

C5 is one of Huntington's (1904) Ba postulates (cf. §6.2). To my knowledge, all demonstrations of C5 from postulate sets that do not include it are nontrivial. Is this why Spencer-Brown chose J2 as an initial? In any event, by deeming J2 an initial, all *LoF* consequences, C1 and C8 excepted, have easy demonstrations. The demonstration of C5 (equivalent to J2) from B1-B4 (cf. §A.1.2) is a bit involved, but this is amply offset by an easy derivation of C1 and by the calculating power, revealed in §5.4, that B3, C1, and B4 afford.

**5.2.1. The pa Recapitulated**. The primitive basis of the primary algebra (pa) consists of:
- The primary arithmetic, PA;
- Variables (statement letters), with or without subscripts ranging over the natural numbers, inserted anywhere in a PA formula. '′' and '…' are improper symbols;
- The initials (3.1.5) $abc{=}bca$, $\bot a{=}a$, $(a)a{=}()$, and $a(ab) = a(b)$;
- The usual inference rules for equational logics, the substitution of equivalents (R1), and the uniform replacement of variables (R2).

Juxtaposition is a tacit connective that commutes and associates. Hence the contents of a boundary and its pervasive space may be rearranged at will. The pa is well-suited to a decision procedure resembling Quine's truth value analysis. That decision procedure verifies the initials; other tautologies may be demonstrated, *a la* Hilbert, or verified by calculation. The pa is sound and complete, and has two intended interpretations: **2** and the CTV. Boundary logic follows from the interpretation () ⇔ T [F], then αβ ⇔ α∨β [α∧β]. In either case, (α) ⇔ ~α.[6]

**5.3. The Usual Inference Rules of Logic.**

"*…if one could find characters or signs appropriate for expressing our thoughts as neatly and as exactly as arithmetic expresses numbers or geometric analysis expresses lines, one could accomplish in all subjects* in so far as they are amenable to reasoning *all that can be done in Arithmetic and Geometry. For all investigations depending on reasoning would be performed by the transposition of characters and by a sort of calculus, which would render very easy the invention of beautiful results. Hence we would not need to worry our heads as much as we do at present, yet we would be sure that we could execute anything feasible. Moreover, we could convince everyone of what we had found or concluded, since it would be easy to verify the calculation… And were someone to doubt what I was proposing, I would say to him 'Sir, let us calculate' and thus… soon settle the question*." Leibniz (1903: 155-56); emphasis in original.[7]

The BA inference rules R1 and R2 suffice for equational logics. Table 5-2 shows a number of other truth functional inference rules that appear in conventional logic, including the *modus ponens* that defines ponential logics. It turns that that these rules are all special cases of a very general inference rule Wolf (1998: §§3.5, 4.2) calls *propositional consequence* (PC).

**Table 5-2.**
Some Common Instances of Propositional Consequence.

| *Name* | $\phi_1$ | $\phi_2$ | $\phi_3$ | $\chi$ | Source |
|---|---|---|---|---|---|
| Contrapositive | α′ | β | | α′β | 88 |
| Conjunction | α | β | | (α′β′) | 88 |
| *modus ponens* | α | α′β | | β | 79 |
| *modus tollens* | α′β | β′ | | α′ | 88 |
| Biconditional | α′β | β′α | | ((α′β)(β′α)) | 85 |
| Syllogism[8] | α*β | β′γ* | | α*γ* | --- |
| Proof by cases | αβ | α′γ | β′γ | γ | 84 |

**Source**: Corresponding page number in Wolf (1998).

First some definitions:

**5.3.1. Definition**. An *argument* consists of one or more formulae called *premises*, and a formula called a *conclusion*. A *clause* (Cori & Lascar 2000: §§1.3.2, 4.4.1) links the conjoined premises to

6. Kauffman (2001) and Bricken (2002) exposit BA and boundary logic in a manner arguably more philosophically attuned to *LoF*. §A.15 sketches Rosser's (1969: chpt. 2) derivation of Ba from a handful of point set topology notions. Other possible approaches, not pursued here, to a deeper understanding of BA include mereology and mereotopology (Simon 1987; Casati and Varzi 1999), the cognitive approach to mathematics (Lakoff and Núñez 2001), and semiotics (Merrell 1995).

7. From an essay in French titled "Préface à la Science Génerale." The translation is mine, because Wiener's translation (Leibniz 1951: 15) is inaccurate.

8. *Cal*. ($\underline{\alpha*}$β)(β′$\underline{\gamma*}$)α*γ* [B4,2x] = (β)(β′)α*γ* [B4; OI] = (β′α*γ*)β′α*γ* [B3] = (). □ Validity requires that the letter not appearing in the conclusion (β in this case) appear primed in one premise, unprimed in the other.

the conclusion via the conditional. A *valid* argument is one whose conclusion follows from its premises.

PC asserts that an argument is valid iff the corresponding clause is a tautology. More formally:

**5.3.2. Definition of PC**. Let $\phi_1\ldots\phi_n$ be premises, and $\chi$ be the conclusion. Then PC is:

(1) $\phi_1,\ldots,\phi_n \vdash \chi \Leftrightarrow \vdash [\phi_1\wedge\ldots\wedge\phi_n]\rightarrow\chi \Leftrightarrow \underline{((}(\phi_1)\ldots(\phi_n)\underline{))}\chi$ [C3] $= (\phi_1)\ldots(\phi_n)\chi = ()$.

*Remark*. The BA clause associated with an argument simply encloses each premise, then juxtaposes the conclusion and the enclosed premises. I have tacitly assumed that a clause contains a single conclusion but this is inessential; a clause can contain multiple conclusions, all juxtaposed.

The satisfiability of a clause may be ascertained as follows:
- Translate every premise and every conclusion into the pa;
- Enclose each premise, then concatenate the premises and conclusions;
- Invoke C1 to erase all duplicate instances of a subformula within a given subspace;
- Invoke C3 to erase redundant parentheses;
- Invoke B4 to erase redundant subformula instances at different depths;
- Invoke J1 to erase any subformulae of the form $(\alpha'\alpha)$.

If the above results in a primitive value, the clause or its negation is always valid. If the result is a formula, the clause is valid under those atomic valuations satisfying that formula.

**Table 5-3.** Some Common Logical Rules and Their Boundary Derivations.

| *Name* | *Formal Version* | pa *Derivation* | *Source*† |
|---|---|---|---|
| *Bostock's* (1997: 385) *Structural Rules* | | | |
| Basic Sequents‡ | $\Gamma,\varphi \vDash \varphi,\Delta$ | $\Gamma'\underline{\varphi'\varphi}\Delta$ [B3] = (). | p. 285 |
| INTerchange, L | $[\Gamma,\varphi,\psi,\Delta\vDash\Theta] \rightarrow [\Gamma,\psi,\varphi,\Delta\vDash\Theta]$ | $(\Gamma'\varphi'\psi'\Delta'\Theta)\Gamma'\psi'\varphi'\Delta'\Theta$ [TR; B3] = (). | §7.1 |
| “ , R | $[\Gamma\vDash\Delta,\varphi,\psi,\Theta]\rightarrow[\Gamma\vDash\Delta,\psi,\varphi,\Theta]$ | $(\Gamma'\Delta\varphi\psi\Theta)\Gamma'\Delta\psi\varphi\Theta$ [TR; B3] =() | “ |
| CONtraction, L | $[\Gamma,\varphi,\varphi \vDash \Delta] \rightarrow [\Gamma,\varphi \vDash \Delta]$ | $(\Gamma'\underline{\varphi'}\varphi'\Delta)\Gamma'\varphi'\Delta$ [C1; B3] = (). | “ |
| “ , R | $[\Gamma \vDash \varphi,\varphi\Delta] \rightarrow [\Gamma \vDash \varphi,\Delta]$ | $(\Gamma'\varphi'\Delta)\Gamma'\underline{\varphi'}\varphi'\Delta$ [C1; B3] = (). | “ |
| CUT | $[\Gamma \vDash \varphi,\Delta] \wedge [\Phi,\varphi \vDash \Theta]$ $\rightarrow [\Gamma,\Phi \vDash \Delta,\Theta]$ | $\underline{((}(\underline{\Gamma'}\varphi\underline{\Delta})(\underline{\Phi'}\varphi'\underline{\Theta})\underline{))}\Gamma'\Phi'\Delta\Theta$ [C3; B4,4x] = $\underline{(\varphi)(\varphi')}\Gamma'\Phi'\Delta\Theta$ [B3] =() | §2.5.C |
| *Machover's* (1996) *Inference Rules* | | | |
| Indirect proof, *reduction* | $\Gamma,\sim\alpha \vdash \bot \leftrightarrow \Gamma \vdash \alpha$ | $\Gamma'\underline{((}\alpha\underline{))}\bot$ [C3] = $\Gamma'\alpha$ . | §7.8.9, 15 |
| Deduction Theorem | $\Gamma,\alpha \vdash \beta \leftrightarrow \Gamma \vdash \alpha\rightarrow\beta$ | $\Gamma'\alpha'\beta = \Gamma'\alpha'\beta$. | §7.7.2 |
| Inconsistency¶ | $[\Gamma \vdash \bot] \vdash [\Gamma \vdash \beta]$ | $(\Gamma'\bot)\Gamma'\beta$ [B2] = $(\Gamma')\Gamma'\beta$ [B3] = $()\beta$ [C2] = (). | §7.8.6 |

† Section of Bostock (1997) or Machover (1996) where the rule in question is introduced and discussed.
‡ Replacing Bostock's (§2.5) ASSumptions, and THINning from the left and right.
**Note**: L=left; R=right. A lower [upper] case Greek letter denotes a single formula [set of formulae]. A primed upper case letter signifies that each constituent formula is primed.
¶ Cf. §A.1.3 and text related to §4, fn. 15.

Table 5-3 presents the usual inference rules of contemporary logic, taken from Machover (1996) and Bostock (1997), along with their boundary justifications. The boundary variant of the syntactic and semantic turnstiles is: prime all objects to the left of the turnstile, then concatenate everything on both sides of the turnstile. This done, the inference rules in Table 5-3 are all trivial pa consequences.

*Basic Sequents* and *Inconsistency* are all B3 in another guise; ditto for INT and B1, CON and C1, and *Indirect Proof* and C3. The Cut Rule is the only rule whose demonstration invokes B4. Its meaning is simpler than may appear: if both φ and φ′ appear in the premises, φ is irrelevant to the conclusion.[9]

The molecular subformulae making up the BA representation of a clause can be permuted at will. Hence all partitions of these molecular subformulae into premises and conclusions have the same BA translation and hence are equivalent. In particular, $(\phi_n)$ and ⊢ in (1) can be transposed; the result is the boundary logic equivalent of the Deduction Theorem. Moreover, the validity of a clause does not depend on whether any particular molecular subformula is included among the premises or the conclusion, as long as any formula moved from one side of the turnstile to the other is first enclosed. This is presumably why boundary logic dispenses with all turnstiles.

### 5.4. Some Worked Examples from Logic Texts.

"*…the calculus published in this text renders* [*standard university logic problems*] *so easy that we need not trouble ourselves further with them...*" *LoF*, p. viii.

I now give a number of worked examples showing that CTV proofs in undergraduate textbooks can be greatly simplified if the problem is first translated into BA notation, as per Table 4-2, and the proof carried out as a pa demonstration or calculation, as described in §5.0. The following metatheorem justifies this procedure.

**5.4.1. Theorem**. The BA and the CTV have the same expressive power. In symbols, BA ⊢ CTV and CTV ⊢ BA.

*Proof*. See §A.10.

Many of the demonstrations in the rest of this section are in columnar form, with the annotations written to the right of each step. Text about to be deleted in the next step is underlined. As before, a new instance of a subformula or of nested parentheses is shown in bold. B2 usually can do what A2 does; A1 is very seldom required.

If a step invokes one of C5-C8, the annotation may be more complicated, building on the fact that BA formulae can be taken as schemata, in which case they are stated using upper case letters. E.g., C6 is assumed to take the form (A′B′)(A′B)=A. R2 allows the uniform replacement of any upper case statement letter by a subformula. Substitutions are notated as per the following example. If the subformulae α and β are substituted for A and B in C6, the annotation is 'C6, α/A, β/B', with the actual values of α and β written using lower case letters.

9. This is the essence of the *resolution* method (Cori & Lascar 2000: §4.5) built into many theorem proving programs, and of *LoF*'s (p. 123) unproved Interpretive Theorem 1, which nowhere mentions the Cut Rule.

*Example* 1. I now carry out six exercises whose proofs in standard texts are rather involved. The first two are from Nolt *et al* (1998: 4.46, 109). I chose them because the corresponding demonstrations in Nolt *et al* are the longest purely sentential proofs in that text, respectively 18 and 21 lines long.

*Dem*. $(p\rightarrow q) \leftrightarrow \sim(p\wedge\sim q) \Leftrightarrow (p)q = \underline{((}(p)\underline{((}q\underline{))}\underline{))}$ [C3, 2x] $= (p)q$. □

I have taken the liberty of translating '↔' as '='. To anyone experienced in the pa, that $p\rightarrow q$ and $\sim(p\wedge\sim q)$ are tautologically equivalent is evident at a glance.

The next exercise is to verify the clause:

| | |
|---|---|
| $\sim s\leftrightarrow(\sim p\vee\sim v),\ v\wedge p \vdash s$ | From the conjunction of everything to the left of '⊢', infer the alternation of everything to the right of '⊢'. |
| $\Leftrightarrow \underline{((}(((sp'v')((p'v')s'))\underline{))}\underline{((}v'p'\underline{))}))s$ | |
| $(sp'v')((p'v')s')v'p's$ | C3, 3x |
| $\underline{(sp'v')sp'v'}((p'v')s')$ | TR, 4x |
| $()$ | B3. □ |

The next two exercises, from Kalish *et al* (1980: 417, 66f), are tautology verifications.

| | |
|---|---|
| $(a\rightarrow b) \rightarrow [(a\wedge b) \leftrightarrow a)]$ | |
| $\Leftrightarrow (a'b)((\underline{((}a'b'\underline{))a})(a'(a'b')))$ | |
| $(a'b)(\underline{(a'ab')}(a'(a'b')))$ | C3; TR |
| $((a'(\underline{a}'b')))(a'b)$ | J1; TR |
| $((a'\underline{(}b'\underline{)}))(a'b)$ | B4 |
| $((a'b))(a'b)$ | C3 |
| $()$ | B3. □ |

| | |
|---|---|
| $[(\sim a\rightarrow r)\wedge(b\rightarrow r)] \leftrightarrow [(a\rightarrow b)\rightarrow r]$ | |
| $\Leftrightarrow (\underline{((}(ar)(b'r)\underline{))}((a'b)r))(((a\underline{r})(b'\underline{r}))(a'b)r)$ | |
| $((ar)(b'r)((a'b)r))((a'\underline{(}b'\underline{)})(a'b)r)$ | C3; B4,2x |
| $((ar)(b'r)((a'b)r))((a'b)\underline{(a'b)}r)$ | C3 |
| $((ar)(b'r)\underline{((a'b)r)})((a'b)r)$ | C1 |
| $((a\underline{r})(b'\underline{r}))((a'b)r)$ | B4 |
| $\boldsymbol{r}\underline{((}a'b\underline{))}(a'b)r$ | C5 |
| $()$ | C3; B3 □ |

Kalish *et al* require 27 and 32 lines, respectively to verify that these formulas are tautologies. The latter demonstration fills an entire page and is preceded by five pages of discussion.

I have not reproduced here the four demonstrations in Nolt *et al* and Kalish *et al*, because they require a total of 102 lines, and invoke natural deduction techniques that are beyond the scope of this book, typographically as well as logically. The corresponding pa calculations require a mere 26 steps.

The next exercise (MacKay 1989: exercise 9m.4) requires determining the satisfiability of $((p\leftrightarrow\sim q)\leftrightarrow\sim p)\leftrightarrow\sim q$. I translate $\alpha\leftrightarrow\sim\beta$ as $((\alpha'\beta')(\alpha\beta))$. To avoid working with a single long formula, I break up the rightmost biconditional into two conditionals.

| | |
|---|---|
| $((p\leftrightarrow\sim q)\leftrightarrow\sim p) \rightarrow \sim q$ | |
| $\Leftrightarrow (((p'\underline{q'})(\underline{p}q)p)(((p'q')(pq))p'))q'$ | |
| $(((p')\underline{(q)}p)(\underline{((p'q')\mathbf{p'q'}(pq))}p'))q'$ | B4,4x |
| $(((\underline{p'})p)(\ p'))q'$ | J1; B4 |
| $\underline{((p)(\ p'))}q'$ | B4 |
| $q'$ | J1. □ |

| | |
|---|---|
| $\sim q \rightarrow ((p\leftrightarrow\sim q)\leftrightarrow\sim p)$ | |
| $\Leftrightarrow q((p'q')(pq)p)(((p'q')(p\underline{q}))p')$ | |
| $q\underline{((p'q')(pq)p\mathbf{q})}((\underline{(p'q'\mathbf{q})}(p))p')$ | B4,3x |
| $q\underline{(((p))p')}$ | J1,2x |
| $q$ | J1. □ |

As one conditional simplifies to $q'$ and the other to $q$, their conjunction evaluates to $\perp$ by C3 and B3. While this calculation is a bit involved (14 steps), mainly because '$\leftrightarrow$' lacks a concise pa representation, it requires only J1 and B4. By contrast, MacKay's (pp. 368-69) proof is 43 lines long and invokes 11 natural deduction rules.

The most spectacular example of this nature I have saved for last. Leblanc and Wisdom's (1976: 395) proof of $[p\vee(q\rightarrow r)]\leftrightarrow[(p\vee q)\rightarrow(p\vee r)]$ is 42 lines long and invokes eight natural deduction rules. If the single instance of '$\leftrightarrow$' is taken as '=', the pa demonstration is well-night trivial: *Dem.* $pq'r$ [TR] = $q'pr$ [B4] = $(\mathbf{p}q)pr$. □

*Example* 2. Quine (1982: 69) introduces the DNF as a method for determining satisfiability, and builds his exposition of the DNF around a six page discussion of:

(1) $\sim(((p\rightarrow(\sim s\wedge q))\rightarrow\sim((s\wedge q)\rightarrow p))\wedge\sim(\sim(r\wedge p)\wedge\sim(p\rightarrow s))),$

which he deemed "forbidding". Because (1) includes five instances of conjunction and none of alternation, I translate it using the dual reading, so that $x\wedge y \Leftrightarrow xy$, not $(x'y')$.

$$
\begin{array}{rl}
\multicolumn{2}{l}{\Leftrightarrow (((p(s'q))\underline{((}(sq)p'\underline{))})((rp)(ps')))} \\
(((p(s'q))(sq)p')\chi) & \text{C3; let } \chi=((rp)(ps')) \\
((\underline{(\mathbf{p}'p(s'q))}(sq)p')\chi) & \text{B4; TR} \\
(((sq)p')\chi) & \text{B3} \\
(((sq)p')((r\underline{p})(\underline{p}s'))) & \text{Expand } \chi \\
(((sq)p')\mathbf{p}(r'\underline{(}s'\underline{)})) & \text{C5} \\
(((sq)p')\mathbf{p}(sr')) & \text{C3; TR} \\
((\underline{p'\mathbf{p}}(sq))p(sr')) & \text{B4; TR} \\
(\underline{(()(sq))}p(sr')) & \text{B3} \\
(p(sr')) & \text{C2; B2.}
\end{array}
$$
□

Conclusion: (1) is satisfied when $p\rightarrow s\rightarrow r$ is. Note how this technique easily reveals the irrelevance of $q$, and even of all of (1) to the left of the third '$\wedge$'.

*Example* 3. The following two examples are from recent texts. Hurley (2000: 415, exercise 19) asks students to verify the clause:

$$
\begin{array}{rl}
\multicolumn{2}{l}{a\rightarrow(nn')\rightarrow s\vee t,\ t\rightarrow(f\wedge\sim f)\ \therefore\ a\rightarrow s.} \\
\multicolumn{2}{l}{\quad\Leftrightarrow (a'\underline{(nn')}st)(t'\underline{(ff')})a's} \\
(a'st)\underline{(t')}a's & \text{J1,2x} \\
(a'st)a'st & \text{C3; TR} \\
() & \text{B3. □}
\end{array}
$$

Hurley's natural deduction proof (p. 653) requires 19 steps and invokes 13 rules.

Lepore's (2003: 131) exercise 8.5.2 asks whether $((p\wedge q\wedge k)\vee\sim r)$ and $(r\rightarrow(\sim q\rightarrow (p\wedge\sim(v\vee\sim j))))$ are equivalent. Following Lepore, I set up two conditionals then calculate each:

$$
\begin{array}{rl}
\multicolumn{2}{l}{((p\wedge q\wedge k)\vee\sim r) \rightarrow (r\rightarrow(\sim q\rightarrow(p\wedge\sim(v\vee\sim j))))} \\
\multicolumn{2}{l}{\quad\Leftrightarrow ((p'q'k')\underline{r'})r'q(p'vj')} \\
\underline{((}p'q'k'\underline{))}r'q(p'vj') & \text{B4} \\
p'k'r'\underline{q'q}(p'vj') & \text{C3; TR, 2x}
\end{array}
$$

$$
\begin{array}{rl}
\multicolumn{2}{l}{(r\rightarrow(\sim q\rightarrow(p\wedge\sim(v\vee\sim j)))) \rightarrow ((p\wedge q\wedge k)\vee\sim r)} \\
\multicolumn{2}{l}{\quad\Leftrightarrow (\underline{r'}q(p'vj'))(p'q'k')r'} \\
(q(\underline{p'}vj'))(\underline{p'}q'k')r' & \text{B4} \\
(q(vj'))(q'k')p'r & \text{C5.}
\end{array}
$$

$()$ B3; C2. $(qv')(qj)(q'k')p'r$ C7.

The conditional on the right cannot be simplified any further. Hence the two halves of the biconditional do not simplify to the same formula, and the two statements are not equivalent. Note the use of C7 on the right to obtain a NF that is more nakedly revealing of the inability to proceed further. Lepore's worked answer using refutation trees is 25 lines long, invokes 6 rules, and fills all of his p. 389.

*Example 4*. The following detailed example of how the pa simplifies clausal reasoning reworks Stoll's (1963: 184) Example 4.4.3, reproduced in Table 5-4. A lone 'p' in columns 3 or 7 of that Table identifies a row containing a premise. A 't' in these columns signifies that an unspecified tautology has been invoked. The numbers in columns 4 and 8 are the row numbers of the premises upon which the formula in a given row depends. The overall conclusion is in row 13. Readers unversed in natural deduction need only take away from Table 5-4 the relative opacity of its content.

**Table 5-4. Stoll's (1963) Example 4.4.3.**

| *1* | *2* | *3* | *4* | | *5* | *6* | *7* | *8* |
|---|---|---|---|---|---|---|---|---|
| 1 | $\sim C \wedge \sim U$ | p | 1 | | 8 | $(W \vee P) \rightarrow I$ | P | 8 |
| 2 | $\sim U$ | 1,t | 1 | | 9 | $I \rightarrow (C \vee S)$ | p | 9 |
| 3 | $S \rightarrow U$ | P | 3 | | 10 | $(W \vee P) \rightarrow (C \vee S)$ | 8,9,t | 8,9 |
| 4 | $\sim S$ | 2,3,t | 1,3 | | 11 | $\sim(W \vee P)$ | 7,10,t | 1,3,8,9 |
| 5 | $\sim C$ | 1,t | 1 | | 12 | $\sim W \wedge \sim P$ | 11,t | 1,3,8,9 |
| 6 | $\sim C \wedge \sim S$ | 4,5,t | 1,3 | | 13 | $\sim W$ | 12,t | 1,3,8,9 |
| 7 | $\sim(C \vee S)$ | 6,t | 1,3 | | | | | |

Stoll's example can be recast as the following clause:

*Premises*: $(CU) \Leftrightarrow \sim C \wedge \sim U$ *Conclusion*: $(W) \Leftrightarrow \sim W$

$(S)U \Leftrightarrow S \rightarrow U$

$(WP)I \Leftrightarrow (W \vee P) \rightarrow I$

$(I)CS \Leftrightarrow I \rightarrow (C \vee S)$

A pa calculation verifying this clause goes as follows:

$\underline{(}(CU)\underline{)}((S)U)((WP)I)((I)CS)W'$ Enclose premises, concatenate premises & conclusion.

$CU((S)\underline{U})((WP)I)((I)\underline{C}S)W'$ C3

$CU\underline{(}(S)\underline{)}((WP)I)((I)S)W'$ B4,2x

$CUS((WP)I)((I)\underline{S})W'$ C3

$CUS((WP)I)\underline{(}(I)\underline{)}W'$ B4

$CUS((WP)\underline{I})IW'$ C3

$CUS\underline{(}(WP)\underline{)}IW'$ B4

$CUS\underline{W}PI\underline{W'}$ C3

$PI\underline{W'W}CUS$ TR

$()$ B3; C2. □

Deciding where to introduce any given premise into Stoll's proof requires nontrivial reflection. The pa calculation, on the other hand, introduces all premises at the outset, then mechanically prunes redundant instances of variables (boundaries) by invoking B4 (C3). When a primed and unprimed instance of the same variable appears in the pervasive space, B3 and C2 terminate the calculation. I submit that the above calculation is vastly simpler than Stoll's proof. The pa also reveals that any valid argument from Stoll's premises requires that at least one variable appearing in the premises also appear primed in the conclusion.
*End of Examples*

These examples reveal that pa calculations are much easier than conventional proofs. The simplicity of pa calculation stems from:

- A notation that fully embodies the expressive adequacy of $\{\vee/\wedge,\sim\}$;
- Working very hard a mere five rules, TR, B2-B4, and C3.

The 10 demonstrations in Examples 1-4 employ all other resources of the pa a mere four times: C5 twice, and C1 and C7 once apiece. That the pa accomplishes so much with so little reveals that in practice, the pa is more than just a new notation for the CTV and **2**.[10]

### 5.5. Syllogisms as Clauses.

"*If, as I hope, I can conceive all propositions as terms, and hypotheticals as categoricals... this promises a wonderful ease in my symbolism and analysis of concepts, and will be a discovery of the greatest importance*." Leibniz (1966: 66).[11]

The syllogism of traditional logic is the oldest and most intensively studied clausal form. Let $\alpha$ and $\beta$ be metavariables standing for *terms*. Linguistically and intensionally, a term is a *common noun* or a noun phrase. Mathematically and extensionally, a term is a *set*, in which case 'all $\alpha$ are $\beta$' may be seen as shorthand for 'all members of set $\alpha$ are also members of set $\beta$', i.e., $\alpha\subseteq\beta$. A *categorical form* has the structure "[All/Some] $\alpha$ are [Not] $\beta$." A *syllogism* is a clause consisting of two premises and one conclusion, each in categorical form. There are three terms, with one term appearing in both premises. For a modern overview of the syllogism, see Kneebone (1963: 8-22).

Table 5-5 shows how to interpret the pa as a logic of terms and categorical forms, if let ters are reinterpreted as term names. Letting, as before, a '*' after a variable denote a literal, the pa notation $(\alpha'\beta^*)^*$ captures all possible categorical forms. Then monadic logic works as follows. Let A*a*=() if it is indeed the case that *a* is a member of set $\alpha$; likewise for B*b* and the set $\beta$. Quine (1982: §§18-20) designed his *Boolean term schemata* (BTS in Table 5-5) so as to embody the Boolean structure common to the syllogism, the logic of terms, and the monadic predicate calculus. The resulting notation is (unwittingly) very similar to that of the pa.[12]

---

10. I invite the reader to compare the demonstrations in *LoF* and here with those in Nidditch (1962), a book comparable to *LoF* in size and time of writing, also intended for undergraduate instruction, but far more conventional in approach. Deferring to intuitionist logic, Nidditch posits 11 algebraic axioms and the rule *modus ponens*, then proves 4 lemmas and 58 theorems (a category that lumps together what are here called (meta)theorems and consequences).

11. Original in Leibniz (1903: 377, §75).

12. Leibniz algebraized the categorical forms in a manner closely related to the one set out in the text. In paragraphs 83-87 of a paper written in 1686 but published only in 1903, Leibniz (1966: 67-68) wrote $\alpha\beta^*=\alpha$ and $\alpha\beta^*\neq\alpha$ where I write $\alpha'\beta^*$ and $(\alpha'\beta^*)$. Also see Leibniz (1966: xlvii, Scheme III).

**Table 5-5.**
Alternative Notations for the Four Categorical Forms.

| * | *Categorical Form* | pa | *BTS* | *Monadic Logic* | *Set Algebra* |
|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) |
| **A** | All α are β | $\alpha'\beta$ | $-\alpha\beta'$ | $\forall x[Ax\rightarrow Bx]$ | $\alpha\subseteq\beta$ |
| **E** | No α are β | $\alpha'\beta'$ | $-\alpha\beta$ | $\forall x\neg[Ax\wedge Bx]$ | $\alpha\cap\beta=\varnothing$ |
| **I** | Some α are β | $(\alpha'\beta')$ | $\alpha\beta$ | $\exists x[Ax\wedge Bx]$ | $\alpha\cap\beta\neq\varnothing$ |
| **O** | Some α are not β | $(\alpha'\beta)$ | $\alpha\beta'$ | $\exists x\neg[Ax\rightarrow Bx]$ | $\alpha\cap\beta'\neq\varnothing$ |

* These abbreviation are from medieval logic.

The clause $(\alpha'\beta)(\beta'\gamma)\alpha'\gamma$ corresponds to the syllogism medieval tradition named 'Barbara'.[13] This clause is an instance of the more general clause $(\alpha^*\beta)(\beta'\gamma^*)\alpha^*\gamma^*$ (cf. Table 5-2), which admits of 24 possible permutations, and tradition indeed asserts the validity of 24 syllogistic forms. Unfortunately, not all of those forms correspond to one of the 24 permutations of $(\alpha^*\beta)(\beta'\gamma^*)\alpha^*\gamma^*$. Determining the number and form of the possible variants of $(\alpha^*\beta)(\beta'\gamma^*)\alpha^*\gamma^*$ admitted by tradition is a nontrivial combinatoric exercise. There are three meaningful permutations of $(\alpha'\beta)(\beta'\gamma)\alpha'\gamma$, one for each possible position of $\alpha'\gamma$. There are also six possible permutations of $(\alpha'\beta)(\beta'\gamma')\alpha'\gamma'$. Given any of the latter, if both terms in a premise are primed, the terms may commute. Thus $(\alpha'\beta)(\beta'\gamma')\alpha'\gamma'$ gives rise to three permutations, for a total of 15 valid syllogisms thus far.

If we assume that at least one of α, β, or γ is nonempty, i.e., that at least one of $\alpha\neq\perp$, $\beta\neq\perp$, or $\gamma\neq\perp$ holds, then variants of the clause $(\alpha'\beta)(\beta'\gamma^*)(\alpha'\gamma^*)^*$ can be valid, where each '*' stands for the presence or absence of a '′'. $(\beta'\gamma')(\alpha'\beta)(\alpha'\gamma)$ is valid if $\alpha\neq\perp$ or $\gamma\neq\perp$. Moreover, permuting $\beta'\gamma'$ does not affect validity, resulting in four valid syllogisms. $(\alpha'\beta)(\beta'\gamma')(\alpha'\gamma)$ is valid if $\alpha\neq\perp$; $\beta'\gamma'$ can again be permuted, resulting in two valid syllogisms. $(\alpha'\beta)(\beta'\gamma)(\alpha'\gamma')$ is valid if at least one of $\alpha\neq\perp$, $\beta\neq\perp$, or $\gamma\neq\perp$ is the case, resulting in three valid syllogisms. The approach of this paragraph yields nine more valid syllogisms, for a total of 24. These 24 include five pairs whose members differ only in that where one has "all" in the conclusion, the other has "some."

Hence Appendix 2 of *LoF* is mistaken when it asserts that Barbara (by which Spencer-Brown meant $(\alpha^*\beta)(\beta'\gamma^*)\alpha^*\gamma^*$ ) nests *all* 24 valid syllogisms. Barbara nests only the 15 syllogisms not requiring that one or more terms be assumed nonempty. $(\alpha'\beta)(\beta'\gamma^*)(\alpha'\gamma^*)$ is not an instance of Barbara, but can be valid given suitable nonemptiness assumptions. Deriving the necessary and sufficient conditions for a syllogism to be valid, I leave to future research.

The above approach is essentially that of Lukasiewicz (as per Prior 10.11), who formalized the syllogism by adding term variables to the CTV, extending the scope of R2 to such variables. He then introduced four axioms which, when translated into the pa notation of this section, are:

- $\alpha'\alpha$=(). B3 holds for term variables;
- $(\alpha'\alpha')$ [C1; C3] = α = (). In effect, all terms are assumed nonempty;
- Two axioms equivalent to asserting that $(\alpha'\beta)(\beta'\gamma^*)\alpha'\gamma^*$=() holds for term variables.

On this and other attempts to algebraize the syllogism, see Prior 10.11-6. None of these alternatives are as simple as $(\alpha'\beta)(\beta'\gamma^*)(\alpha'\gamma^*)^*$, or reveal that elementary Ba suffices for syllogistic logic if all

13. Barbara follows from the CUT rule (Table 5-3) when both Δ and Φ are empty.

terms are assumed nonempty. At any rate, the pa nicely trivializes what had been a rather involved subject for over 2000 years.[14]

**5.6. Segue to First Order Logic, *or* How to Quantify *sans* Quantifiers.**

"*Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle—they are strictly limited in number, they require fresh horses, and must only be made at decisive moments*."
Whitehead (1948: 61).

Let there be a *domain* composed of one or more *individuals*. Associated with each individual is an alphanumeric symbol string called its *name*. A *variable* now stands for any individual in the domain. Equivalently, a variable is said to *range over* the domain. Variables are *implicitly quantified*, with the scope of quantification determined by the depth of the shallowest instance of a variable. This approach to quantification, by the way, renders moot the whole matter of vacuous quantification. Just what is to be included in the domain can vary from application to application. The resulting mutability of quantified logic is part of what renders it philosophically nontrivial. For more on first order logic, see the discussion of the "calculus of quantified values" (CQV) in *A Précis of Mathematical Logic.*

Let $\varphi\langle\varepsilon/\alpha\rangle$ mean that an instance of the name $\varepsilon$ replaces every instance of the variable $\alpha$ in $\varphi$, and likewise for $\varphi\langle\alpha/\varepsilon\rangle$. If $\xi=\varphi\langle\varepsilon/\alpha\rangle$ or $\varphi\langle\alpha/\varepsilon\rangle$, then $\varphi$ and $\xi$ form a *substitution pair.* The fundamental principles of quantification theory are:

- I1. *The cancellation property inherent in* B3 *holds for substitution pairs*.
- QN. *A stand-alone variable may be written outside any boundary and erased at will*.

These principles result in the axioms for quantification shown in Table 5-6.

| **Table 5-6. Quantification Axioms.** | |
|---|---|
| *Label* | *Axiom* |
| I1 | $(\xi)\varphi = (\varphi)\xi = ()$ |
| QN | $(\varphi) = \alpha(\varphi)^{\dagger}$ |
| † $\varphi$ must have an even number of variables at depth 0; cf. I. | |

QN in conventional notation is $Q\alpha[\neg\varphi\langle\alpha\rangle] \leftrightarrow \neg Q^{*}\alpha[\varphi\langle\alpha\rangle]$.[15]

The cancellation property implied by A2 also applies to variables and substitution pairs, but does not normally come into play. In particular, A2 generalized for quantification is not needed in the following trivial derivations of the four fundamental rules of quantification theory:

14. Other formalizations of the syllogism include Church (1956: Ex. 46.22), using monadic first order logic, and Halmos and Givant (1998: §52), using a proper extension of Ba they call "monadic algebra."

15. Following Quine (1951) and Zeman (1967), the approach to quantification theory advocated here does not allow for open formulae.

UG: $(\varphi\langle\alpha\rangle(\varphi\langle\alpha/\varepsilon\rangle)\alpha) = (\xi(\varphi))$ [QN. This is EG] $= \perp$ [I1].
EI: $\alpha(\varphi\langle\alpha\rangle(\varphi\langle\varepsilon/\alpha\rangle)) = (\varphi(\xi))$ [QN. This is UI] $= \perp$ [I1].

The shallowest instance of $\alpha$ in EI and UG is stand-alone. In these cases the term introduced via substitution must be new to the demonstration. Combining EG and EI yields $\alpha(\varphi(\xi))(\xi(\varphi))$. Interpreting the biconditional as equivalence implies $\xi=\varphi$, with the quantification of $\alpha$ understood to be existential. Combining UI and UG yields $(\varphi(\xi))(\xi(\varphi)\alpha)$, which cannot be interpreted as a biconditional. Hence invoking UI results in a conditional rather than an equation, and UI is the only irreversible step in boundary logic.[16]

There are, as yet, no proofs, using boundary notation, of the well-known metatheory of FOL (undecidable except in special cases such as when the atomic formulae are all monadic, completable, Löwenheim-Skolem property).

*Quine's Main Method*: *An Extended Worked Example*.
I now describe the "main method" of Quine (1982: §30) for verifying quantified formulae and clausal forms.[17] To replace all instances of a variable with a particular name letter is to *instantiate* the variable. When instantiating a variable, begin by erasing any stand-alone instances. A variable can always be instantiated mechanically, by a name letter that is *new to the proof at hand*. If the shallowest instance of a variable has even depth, no other form of instantiation is allowed; this is the substance of EI. If the shallowest instance of a variable has odd depth, it can be instantiated with a name letter previously introduced by EI or UI. The goal of UI is to choose name letters in such a way as to make all atomic formulae disappear using BA rules.

The following worked example demonstrates how boundary notation facilitates the main method. The problem, taken from Quine (1982: 192), requires showing that the following formulae are mutually inconsistent:

$$\exists u\forall yFuy, \qquad \forall x\exists v\forall y\neg(Fvy\wedge Fyx), \qquad \forall x\forall y\exists w[Fxy\rightarrow(\neg Fxw\wedge Fwy)\vee(Fyw\wedge Fwx)].$$

Translate each formula into the dual interpretation, then concatenate the translations:

$$u(y(Fuy))(x(v(FvyFyx)))(Fxy\underline{((}((Fxw)Fwy)(FywFwx))\underline{))}.$$

Note how I indicate that $x$ and $y$ are universally quantified, and that $u$ and $v$ are existentially quantified. Now carry out any obvious truth functional simplifications:

$$= \underline{u}(y(Fuy))(x(\underline{v}(FvyFyx)))(Fxy((Fxw)Fwy)(FywFwx)) \qquad \text{[C3]}$$

Instantiate the existentially quantified variables, recalling the convention that letters early in the alphabet serve as names. Here I introduce another convention: the first time a name replaces a variable, write the name in boldface:

$$= (y(F\boldsymbol{a}y))(x\underline{((}F\boldsymbol{b}yFyx\underline{))})(Fxy((Fx\boldsymbol{c})F\boldsymbol{c}y)(Fy\boldsymbol{c}F\boldsymbol{c}x)) \qquad \text{EI } u/a,\ v/b,\ w/c$$

---

16. The standard axioms for quantification (Bostock 1997: 387) are UI and $(\alpha(\xi\rangle\alpha\langle(\varphi\langle\alpha\rangle))\ ((\xi\rangle\alpha\langle(\varphi\langle\alpha\rangle)))) = (())$, and UG taken as a rule. Note that the approach to quantification taken here works hard the substitution of variables for names and vice versa. There exist quantification axioms dispensing with substitutions (and the rule of generalization); see Tarski and Givant (1987: 8).

17. A good secondary source for the main method is Ullian (2004: 279-80), who says it is similar to the "semantic tableau" method, better known as the method of refutation trees.

(1) $= (y(Fay))(FbyFyx)(Fxy((Fxc)Fcy)(FycFcx))$ C3

Now replace the universally quantified variables *x* and *y* with names just introduced by EI, chosen with the goal of eliminating all atomic formulae from (1). (That this step cannot be wholly mechanical follows from the undecidability of FOL.) Hence I use '⇒' instead of '=' to link the formula created by UI to its predecessor. I first instantiate *x* mechanically:

$\Rightarrow (y(Fay))(FbyFy\boldsymbol{a})(F\boldsymbol{a}y((F\boldsymbol{a}c)Fcy)(FycFc\boldsymbol{a}))$ UI *x*/*a*

A universally quantified variable can be instantiated more than once, and the results concatenated. Moreover, the scope of an instantiation can be confined to what lies between a matched pair of parentheses. These facts, and the parallelism between *FbyFya* and *FycFca*, suggest instantiating *y* twice, once as *b*:

$\Rightarrow Fa\boldsymbol{b}(FbyFya)(Fa\boldsymbol{b}((Fac)Fc\boldsymbol{b})(F\boldsymbol{b}cFca))$ UI *y*/*b*,

and once as *c*:

$\Rightarrow Fab\boldsymbol{Fac}(Fb\boldsymbol{c}F\boldsymbol{c}a)(\underline{Fab}((Fac)Fcb)\underline{(FbcFca)})$ UI *y*/*c*.

Crucial to the final three steps is that depth 0 contains instances of both *Fab* and *Fac*:

$= FabFac(FbcFca)\underline{((}(Fac)Fcb\underline{))}$ B4,2x
$= FabFac(FbcFca)(Fac)Fcb$ C3
$= \underline{(Fac)Fac}Fab(FbcFca)Fcb$ TR
$= ()$ B3; C2.

As the dual interpretation of () is *false*, the formulae are inconsistent. □

While identity is absent from this exercise, dealing with it is trivial. Any instance of the atomic formula $\sigma=\tau$ can be eliminated by replacing one or more instances of $\sigma$ by $\tau$, or vice versa.

Bostock's (1997: 387) axioms of quantification, and his rule of generalization GEN, are but trivial consequences of I1:

A4: $\forall\alpha[\phi]\rightarrow\phi\langle\tau/\alpha\rangle$ *Dem.* $(\phi\alpha(\phi\langle\tau/\alpha\rangle)) = (\phi(\psi)) = \bot$ [I1].
A5: $\forall\alpha[\phi\rangle\alpha\langle\rightarrow\xi]\rightarrow(\phi\rangle\alpha\langle\rightarrow\forall\alpha[\xi])$. *Dem.* $((\alpha\phi(\xi))((\alpha\phi(\xi)))) = \bot$ [I1].
GEN: If $\phi\langle\beta\rangle$ then $\forall\alpha[\phi\langle\alpha/\beta\rangle]$. *Dem.* $(\phi\beta(\phi\langle\alpha/\beta\rangle)\alpha) = (\phi(\psi)\alpha) = (\phi(\psi))$ [QN] $= \bot\bot$ [I1].

I now derive Tarski's quantification axioms (Tarski and Givant, 1987: 8; $\forall,\rightarrow$ primitive), which dispense with the rule of generalization and the notion of substitution:

$\forall x\forall y\phi\rightarrow\forall y\forall x\phi$. *Dem.* $(x((y(\phi))))(y((x\phi)))$ [QN,4x] $= ((((\phi))))(((\phi)))$ [I1] $= ()$.

$\forall x[\phi\rightarrow\psi]\rightarrow(\forall x\phi\rightarrow\forall x\psi)$. *Dem.* $(x(\phi(\psi)))((x(\phi))(x\psi))$ [QN,3x] $= \underline{((}\phi(\psi)\underline{))}(\underline{((}\phi\underline{))}(\psi))$ [C2,2x] $= \phi(\psi)(\phi(\psi))$ [I1] $= ()$.

UI: $\forall x\phi\rightarrow\phi$. *Dem.* $(x(\phi))(\phi)$ [QN] $= ((\phi))(\phi)$ [I1] $= ()$.

UG: $\phi\rightarrow\forall x\phi$ (*x* new). *Dem.* $\phi(x\phi)$ [QN] $= \phi(\phi)$ [I1] $= ()$.
*Historical Antecedents and Contemporary Parallels*.

The approach to quantification theory set out here originated with Peirce's (4.372-509) *beta* existential graphs. While Peirce employed predicate letters and dispensed with variables by means of a graphical device, he effectively understood that the parity of a variable's shallowest instance determined the type of quantification. Zeman (1964) was the first to appreciate that the beta graphs were isomorphic to first order logic with identity. Also see Zeman (1967) and Shin (2002). The "main method" has affinities to refutation trees (e.g., Bostock 1997: chpt. 4; Ullian 2004: 279), and to Lepore's (2003) emphasis on the adequacy of UI, EI, and QN.

# Chapter 6.

# Historical Antecedents and More Axiomatics.

## 6.1. Peirce's *Alpha* Existential Graphs.

"*The spatial relations of written symbols on a two dimensional writing surface can be employed in far more diverse ways than the mere following and preceding in one-dimensional time, and this facilitates the apprehension of that to which we wish to direct our attention. In fact, simple sequential ordering in no way corresponds to the diversity of of logical relations through which thoughts are interconnected.*" Frege (1882: 87)

"[*A thorough understanding of mathematical reasoning*] *is the purpose for which my logical algebras were designed but which, in my opinion, they do not sufficiently fulfill. The system of existential graphs is far more perfect in that respect...*" Peirce (4.429, 1903)

This chapter builds on Kauffman's (2001) finding that the pa has a distinguished ancestry owing nothing to *LoF*, namely the graphical logic to which C.S. Peirce devoted much of the last 20 years of his life. Peirce's logical graphs are planar representations of logic formulae, consisting of ovals, called *seps*, that may be nested, and atomic formulae written anywhere. (Peirce usually referred to seps as *cuts*, a term I avoid because of possible confusion with the Cut Rule of conventional logic, discussed in §5.3.) The graphical logic has but one very simple syntactic rule: seps cannot intersect. Seps and BA boundaries are functionally identical and have a common logical interpretation: denial.[1]

Peirce devised two systems of graphical logic, the *entitative* and *existential* graphs. In the former, the blank page denotes *falsity*, so that alternation interprets juxtaposition. *LoF* (p. 5) unwittingly concurs with this entitative interpretation of the blank page. Hence the entitative graphs and the primal reading of the pa share the same semantics. In the existential graphs (EG), the blank page denotes *truth* and juxtaposition denotes conjunction. Hence the EG are dual to the entitative graphs. Peirce developed the EG at far greater length, even subtitling them "My Chef d'Oeuvre" (4.347-529, 1903); hence I say no more about the entitative graphs. The scope and power of Peirce's graphical logic did not become clear until Roberts (1973).[2] Nevertheless, that logic is a major precursor to boundary logic.[3]

---

1. An advantage of the *alpha* graphs is that they dispense with formula definitions such as 2.1.4: any (finite) nesting or juxtaposition of nonintersecting seps is well-formed.

2. Roberts (1973) evolved out of his 1963 PhD thesis. Zeman's (1964) thesis, never published, likewise saw that *alpha* is isomorphic to CTV, and went beyond Roberts by proving that the *beta* graphs are isomorphic to first order logic with identity. However, Roberts was the first to give this fact wide currency. Shin (2002) includes a thorough exposition of *alpha* and *beta*, and discusses (§§2.4, 2.5) Peirce's view of logic as a form of semiotic. This section has not benefited from Hilpinen's (2004) survey of Peirce's logic.

3. By pointing out parallels between BA and Peirce's graphical logic, I do not wish to suggest plagiarism. *LoF* is an undergraduate text, not a scholarly tract. Moreover, *LoF* predates the publication of Peirce (1976), containing the excerpt from Peirce's 1886 paper on the "sign of illation" (cf. §4.2 above) that is crucial to the historical claims I make here. *LoF* cites Volume IV of Peirce's *Collected Papers* (Peirce 1933), which includes 115pp on the logical graphs, but no aspect of *LoF* hints at the graphical logic in any way. In any event, Spencer-Brown could easily have overlooked this part of Peirce's *oeuvre*, as it was dismissed or ignored until Roberts (1973). Roberts also made extensive use of Peirce's unpublished papers, not accessible to scholars before 1956 and not catalogued until 1967. *LoF* was also written in a time (the 1960s) and place (the UK) where the sparse secondary literature, cited in Roberts, on Peirce's graphical logic was easy to overlook and hard to access.

**Table 6-1**.
Peirce's Existential Graphs and Boundary Logic

| *Name in Roberts* (1973) | *Remarks* | BA |
|---|---|---|
| | | |
| CV0 (4.394, 1903) | What is not forbidden is permitted. *This contradicts the* PA *Convention of Intention*. | |
| CV1 | A blank surface asserts truth. Sometimes referred to as SA, the "sole axiom" of the alpha graphs. | T1 Dual reading |
| CV2 | A graph asserts some truth about the domain. | T2 |
| CV3 | $ab \Leftrightarrow a \wedge b$. | Dual reading |
| CV4 | $\boxed{a\ \boxed{b}} \Leftrightarrow (a(b))$. | " |
| CV5 | $\boxed{a} \Leftrightarrow (a)$; $\boxed{\phantom{a}} \Leftrightarrow () \Leftrightarrow$ *false*. | " |
| **R1i**. Insert Odd | Any subgraph may be written in an odd depth. | 6.1.1 |
| **R1e**. Erase Even | Any evenly enclosed subgraph may be erased. | " |
| **R2i**. Iteration | $a(b) \rightarrow a(ab)$; $a \rightarrow aa$. | T13, C1 |
| **R2e**. Deiteration | $a(ab) \rightarrow a(b)$; $aa \rightarrow a$. | " |
| **R3i**. Insert double cut | $a \rightarrow ((a))$. (()) may enclose anything or nothing. | C3, A2 |
| **R3e**. Delete double cut | $((a)) \rightarrow a$. Erase any instance of (()). | " |

The EG are of three kinds, the first of which, *alpha* (hereinafter *alpha graphs*), is isomorphic to the pa. The alpha graphs are governed by six *conventions* (definitions, more or less), and four *rules of transformation*, akin to natural deduction rules. These conventions and rules are shown in Table 6-1, where CV*n* denotes the *n*th convention, and R*n* the *n*th rule of transformation. The third column of this Table proposes BA counterparts to the *alpha* rules and conventions.[4]

SA stands for "sheet of assertion," the blank surface on which graphs are to be written. The blank page tacitly asserts *truth.* For Roberts (1973: 32, 119), this assertion, which he names SA, is the sole axiom of the alpha graphs. SA follows from A2, which asserts that the blank page denotes a primitive value; cf 2.2. CV1 and CV2 can be seen as defining "graph" and "domain". I state CV3-CV5 using *alpha* as the object language and the pa as the metalanguage. CV3 defines conjunction; CV4, the conditional; CV5, denial. For a comparison of Peirce's graphs with other notations for logic, see Roberts (1973: 136).

The BA is an uninterpreted formal system. Meanwhile, CV1-CV5 reveal that the *alpha* graphs and the dual reading of the pa share the same semantics. Hence the *alpha* graphs, unlike BA, are not self-dual; this is the main way in which they differ from BA. Peirce failed to see that a trivial alteration of the *alpha* graphs would make them self-dual. Self-duality as a property of certain formal systems was unknown in Peirce's day.

The rules of transformation **1i-3e** operationalize "step" in the context of *alpha*; in the terminology of Roberts, a step *preserves tautologies*. Roberts (1973: §3.2) shows that **1i-3e** are CTV consequences. **2i,e** can be seen as analogs of T13 and C1; **3i,e** are C3 and A2 in new guises. **2i,e** and **3i,e** are bidirectional. If **1i** is invoked only in contexts where it is equivalent to $(a())=\perp$ (the complement of

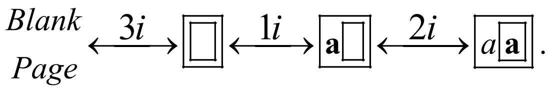
4. *LoF* includes a few diagrams in the spirit of Peirce's graphs; see chapter 12, the notes thereto, and p. 115.

C2), **1i** too becomes bidirectional. Any step invoking a bidirectional rule is analogous to a BA equational step. In EG demonstrations below, I indicate this bidirectionality via a double-headed arrow. However, a step invoking **1e** cannot be retraced; **1e** is thus unredeemedly non-bidirectional and alien to boundary logic. Fortunately, we do not require this rule. If the inserted /erased (sub)formula evaluates to ⊥, **1i,e** follow trivially from A2. More generally, we have:

**6.1.1. Theorem**. **1i,e** preserve tautologies.

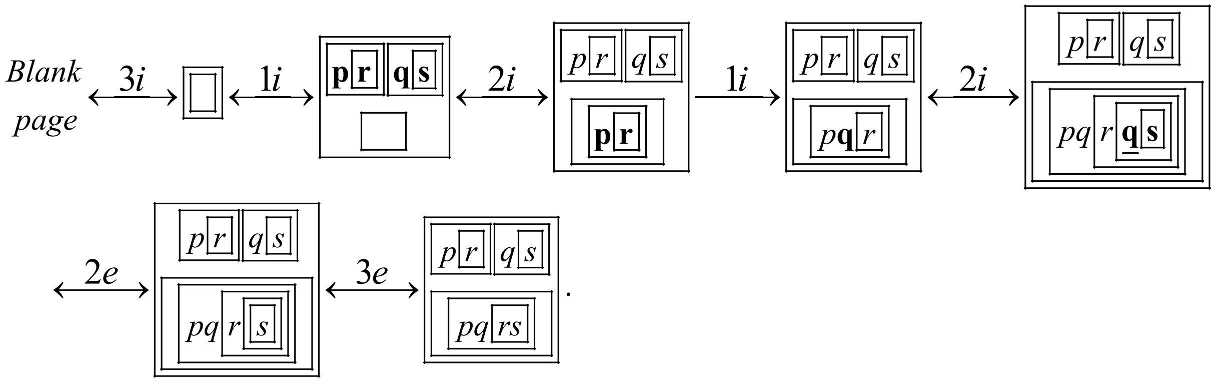
*Proof.* See §A.11.

Since 3i,e is equivalent to B2, and **2i,e** is analogous to B4, the converse of 6.1.1—that the pa is derivable from *alpha*—merely requires an *alpha* demonstration of the dual of B3, to wit:


□

Hence **1i**-**3e** form a basis for the pa. There is a sense in which B3 does the work of **1i,e**. I submit, however, that B3 is more intuitive than **1i,e**, and eliminates any need to keep track of depth parity.

I now present, by way of example, Sowa's (2002) demonstration of Leibniz's *Praeclarum Theorema*, verified by TVA in §5.1. Variables inserted by **1i** or duplicated by **2i** first appear in bold; variable instances about to be eliminated by **2e** are underlined.

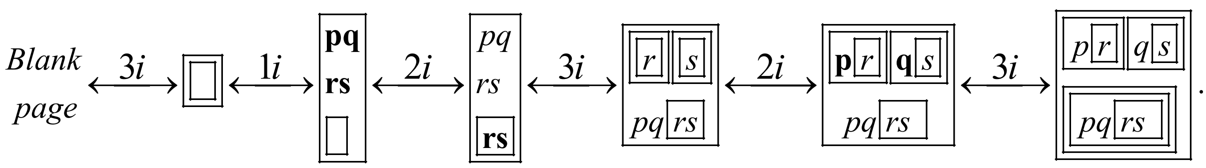

□

The first use of 1i is bidirectional as discussed above **6.1.1**, but the second is not, as this step has no pa analog.

I now verify the *Theorema* via a pa calculation. In keeping with the semantics of the *alpha* graphs, the pa translation invokes the dual reading.

*Cal.* $[(p\to r)\wedge(q\to s)]\to[(p\wedge q)\to(r\wedge s)] \Leftrightarrow ((pr')(qs')\underline{((}pq(rs)\underline{))})$ [C3] $= ((\underline{p}r')(\underline{q}s')pq(rs))$ [B4,2x] $=$ $((\underline{r'})(\underline{s'})pq(rs))$ [C3,2x] $= (rspq(rs))$ [TR] $= (pq(rs)rs)$ [B3] $= (pq())$ [C2]$= \perp$ . □

Reading this calculation backwards suggests the following *alpha* demonstration:


□

This demonstration invokes **1i** once at the outset to insert, in no particular order, one instance of each variable appearing in the *Theorema*. **1i** employed in this manner is analogous to C2 and hence is bidirectional. Since **2i,e** and **3i,e** are analogous to B4 and C3, respectively, and {B1,C2,C3,B4} is a pa basis (Table 6-2), **1e** is redundant. Since the remaining *alpha* rules are all bidirectional, *alpha* can be recast as an equational system. Sowa's demonstration, on the other hand, invokes **1i** twice, each time requiring careful thought about what to insert. Sowa states that the demonstration of the *PM* counterpart of the *Theorema*, *3.47, involves 43 steps and five axioms. Having demonstrated the *Theorema* in four ways: by TVA, *alpha* (twice), and pa calculation, I invite the reader to decide which method is the most perspicuous and easiest to learn.

## 6.2. Some Ba Postulate Sets.

"*Any finite… selection of statements* (*preferably true ones, perhaps*) *is as much a set of postulates as any other. …*'*postulate*' *is significant only relative to an act of inquiry*; *we apply the word to a set of statements… to which we have seen fit to direct our attention*." Quine (1982: 35)

Table 6-2 includes a variety of postulate sets (bases) which are one or more of: important bench-marks, relevant to an evaluation of *LoF*, little known, or have otherwise piqued my curiosity. The first eight rows of Table 6-2 consist of CTV bases to be discussed in §6.3. The remainder of the Table consists of Ba and pa bases, none of which are mentioned in Prior (1962) or Epstein (1995: 407-9); logicians, evidently, are not in the habit of consulting the Boolean algebra literature. The references in Huntington (1933) and Bernstein (1934) suggest that Boolean algebraist do not necessarily sin the other way. If a basis includes a pair of axioms asserting that a connective commutes and associates, I have replaced the pair with B1. I have added B1 to all pa bases, even though no author did so. The "length" of a basis is the number of BA symbols required to express it. For other details of how I operationalize the "length" of a basis, see the Note to Table 6-2. With two exceptions (McCune and Schröder), the Ba and pa bases seem intuitive.

*Leibniz.* In two brief memoranda, written in 1690, published in 1903, and translated into English as chapters 9 and 10 of Leibniz (1966), Leibniz set out a 'logical calculus' with primitive conjunction and denial, respectively denoted by juxtaposition and 'non-*a*'. I shall cite passages in these memoranda as (9.*m*) and (10.*n*), where *m* and *n* are paragraph numbers. I also have taken the liberty of re-ordering Leibniz's axioms (actually "undemonstrated propositions," as Euclian geometry was the sole axiomatic system before the 19th century) and restating them in pa notation.

Leibniz's system has the power of **2** because:

- His axioms are tantamount to a CTV basis called L in §6.3;
- His '=' is (unwittingly) a dyadic congruence relation.

The latter is easy to show. (9.3) is *a*=*a*; hence '=' is tacitly reflexive. (10.5) reads '*a*=*b* means that one can be substituted for the other…[*a* and *b*] are equivalent', which I take as tantamount to R1. The symmetry and transitivity of '=' can be derived from R1 and reflexivity, so that '=' is an equivalence relation (2.3.8). By virtue of (9.8) and (9.11), '=' is also a congruence relation (3.3.12).

I derive Lukasiewicz's CTV axioms as follows. Leibniz's (9.6) is C1, and (10.9) is B3 and C2. Two of Lukasiewicz's axioms are immediate: *Cal.* ($a\underline{a}$)*a* [C1] = (*a*)*a* [B3] = (); (*a*)*ab* [B3] ()*b* [C2] = (). ☐ I thus include B3, C2, and C1 in Leibniz's "basis." Lukasiewicz's remaining axiom is what I call *Syll1*, (*a*′*b*)(*b*′*c*)*a*′*c*. At this point the reader would do well to refer to §5.5, as Leibniz intended letters in his formalism to stand for terms. *Syll1* can be read as asserting the validity of the syllogism in Barbara. While *Syll1 per se* cannot be found in Parkinson (1966: chpts. 9,10), Leibniz (1966: 33, 42) freely assumed Barbara, its equivalent in categorical form. He (p. 105) purports to derive Barbara from his version of the medieval logicians' *dictum de omni et nullo* taken as an axiom. Hence I take Leibniz as granting *Syll1*. The upshot is a generous reading of Leibniz that makes him the inventor of Ba, interpreted as a logic of terms. Leibniz failed to see that alternation as well as conjunction could interpret concatenation. Thus he missed duality and De Morgan's laws.[5]

*Grassmann.* In 1872, Robert Grassmann (brother of the better known Hermann) published a curious book titled *Die Formenlehre* ("The Theory of Forms"), setting out a theory of *magnitudes*. He defined a magnitude as "anything that is or can be the subject of thought, insofar as it has one value, not more" (Grassman 1966: Be-6).[6] The primitive values of the Theory of Magnitudes are *stems*, denoted by an *e* (which may or may not have numerical subscripts) and defined as follows:

> "…a magnitude that is initially posited, and which therefore does not result from [a combination] of other magnitudes… The initial stems of the universe are the God-given properties of particles, the ether, and the spirit, of whose synthesis the entire universe consists." Grassmann (1966: Be-6)

He then applied that theory to four subjects: numbers, 'combinations,' 'externals,' and 'concepts' (i.e., logic); only this last will concern us here.

Grassmann wrote before Peano, Hilbert, Huntington, and others formulated the current understanding of an axiomatic theory. Hence 'axiom' and 'postulate' do not appear in his work. The axioms I propose below for the Theory of Concepts are BA translations of propositions Grassmann states without proof. The part of the Theory of Magnitudes Grassmann deemed applicable to his Concepts consists of two primitive binary operations, denoted by '+' and '·', governed by the following laws (Be-7: 2). I have taken the liberty of modernizing Grassmann's terminology:

a) Magnitudes are closed under '+' and '·';
b) These operations commute and associate;
c) The identity elements for '+' and '·' are 0 and 1, respectively;
d) Each operation distributes over the other.

The closure property (a) I take as tacit throughout this book; B1 captures the essence of (b). $a\perp=a$ nicely summarizes the dual pair (c); ditto for C5 and (d).

5. After writing this section, I discovered Lenzen (2004: 3,4,§4) and Hailperin (2004: 324-37). Lenzen, reviewing papers he published in German in the 1980s, concludes that his system L1, extracted from Leibniz's work, is isomorphic to sentential logic and Boolean algebra. It is not evident whether Lenzen based his conclusion in part on the original texts underlying chpts. 9,10 of Leibniz (1966). Hailperin does base his discussion on these texts, but reaches no conclusion about the strength of the implied system. The true value of Leibniz's work was not appreciated before the 1980s because Couturat (the editor of Leibniz 1903) and Parkinson (the editor and translator of Leibniz 1966), failed to appreciate the strength of Leibniz's system. Moreover, when Leibniz (1966) was published, the two 20th century logicians with the strongest interest in the history of the subject were either dead (Lukasiewicz) or soon to be (Prior). According to Lenzen, Rescher (1954) was the first to see that the power of Leibniz's formal systems had been seriously underestimated.

6. I am very grateful to Lloyd Kannenberg, the translator of Hermann Grassmann's *Ausdenungslehre*, for having taken up my invitation that he translate the *Formenlehre*, and for making his unpublished translation available to me. Page numbers refer to this unpublished translation. I discovered the *Formenlehre* thanks only to Grattan-Guiness's (2000: 157-60) discussion thereof. Grassmann (G-9) mentions Leibniz but no work on logic more recent than Hegel's. In particular, Grassmann appears to have had no knowledge of Boole's work.

Grassman (Be-7: 3) then introduced two laws peculiar to the Theory of Concepts. They warrant quotation in full:

> "1. The sum and… product of two equal stems gives the same stem again, and
> 2. The product of two different stems is zero."

The Theory of Concepts is a model of BA. Let the 'stems' be () and $\perp$, and let $a+b \Leftrightarrow ab$ and $a \cdot b \Leftrightarrow ((a)(b))$. Then the first law is equivalent to the PA equations ()()=(), $\perp\perp=\perp$, ((())(()))=(), and $((\perp)(\perp))=\perp$; Table 6-2 retains the first two. I propose to translate the second law as $()\perp=()$, true by virtue of A2. Thus it would seem that Grassmann unknowingly anticipated the PA in 1872. He defined complementation in context (Be-14: 29) via a pair of equations that are notational variants of the dual pair B3 and J1. Table 6-2 retains B3. With the postulation of J1, the Theory of Concepts becomes isomorphic to BA.

*Schröder* (1966). Lejewski (1960: 23) set out a Ba basis, S1-S9, which he distilled from 3 axioms and 6 definitions spread over 140pp of vol. 1 of the *Vorlesungen* (originally published in 1890 and discussed in Brady 2000). S1-S9 is a peculiar basis, because it is cast in ponential form, and a prolix one because Lejewski's notation includes equivalents of the $\cup$, $\cap$, $\subset$, overbar, **U**, and $\varnothing$ of set theory, and of the truth functors $\wedge$, $\rightarrow$, and $\leftrightarrow$.

Translating S1-S9 into the pa reveals that S1 is B3; S2, *Syll1*; S3, B2; S4, $a'()=()$; S6 is $(ab)c=((a'c)(b'c))$, a variant of C5; S7 is $a'(bc)(a'b')(a'c')=()$; B3 implies S8; S9 follows trivially from B2 and B3. I drop S5 because it is the contradual of S6, replace S1, S3, S8, and S9 with B2 and B3. The result is the 6 equation Schröder-Lejewski basis in Table 6-2. Comparing Schröder's basis with that of Sheffer (1913) reveals that the two bases have $(ab)c=((a'c)(b'c))$ in common. But Sheffer's basis requires only two other postulates, his versions of C3 and B2. Hence Schröder's basis is amply redundant; also see §6, fn. 10.

*Johnson* (*1892*). In a three-part article in *Mind* (then the leading philosophical journal published in English), the British logician W E Johnson set out a system whose syntax—juxtaposed letters with and without overbars—translates trivially into BA: if $\alpha$ is a formulae, $(\alpha)\Leftrightarrow\overline{\alpha}$ . Johnson interpreted juxtaposition as conjunction and the overbar as complementation. His axioms were C3, C1, axioms equivalent to B1, and the contradual of C6, $(ab)(ab')=a'$, which he called the Law of Dichotomy. I will refer to it as C6. §A.4 includes a demonstration of J1, C3, B4, and C1 from C6 and B1, thus proving that Johnson's axioms form a pa basis, and that C3 and C1 are redundant axioms. Johnson's verbose exposition falls short of subsequent taut expositions of Ba and CTV. He did claim Peirce as an important influence. Prior (1962) repeatedly cites Johnson (1892) but nowhere mentions his system. To my knowledge, the only discussion of Johnson's system is Meredith and Prior (1968).

*Implication algebra* (Abbott 1969). Consider the following trivial syntax: If $a,b,c$ are formulae, then $ab$, $(ab)c$, and $a(bc)$ are formulae. Given the semantics $ab \Leftrightarrow a\rightarrow b$ and $(ab)c \Leftrightarrow (a\rightarrow b)\rightarrow c$, this syntax suffices for CTV statements whose sole connective is $\rightarrow$. The axioms PIA1, $(ab)b=(ba)a$, and PIA2, $a(bc)=b(ac)$, result in *positive implication algebra* (PIA), in which all intuitionistically valid tautological equivalences whose sole connective is the conditional are demonstrable. Parentheses are required because the conditional does not associate. Introduce the symbol $\perp$ with intended reading *false*. $a\perp$ then defines intuitionist negation, $\neg a$ (Prior 12.3); no axioms are required. The path to classical logic begins by adding $(ab)a=a$ [C4] to PIA, yielding *implication algebra* (IA; Abbott 1969: §7-4; Wolfram 2002: 803). IA stands to the implicational calculus as BA stands to the

**Table 6-2.**
**Selected CTV/Ba Axioms, pa Initials, Reexpressed in pa Notation.**

| *Year* | *Author* | | *Axioms/Initials. Syll1: $(a'b)(b'c)a'c$. Syll2: $(ab')(bc)ac$.* | *Diff.* | *Length* |
|---|---|---|---|---|---|
| 1885 | Peirce[7] | CTV | B3, C2, $((a'b)a)a$, *Syll1*, $(a'b'c)b'a'c$ | 3 | 55 |
| 1917 | Nicod | CTV | B3, *Syll2* | 4 | 20 |
| 1924 | Lukasiewicz-Bernays | CTV | $a'ab$, $(aa)a$, *Syll2*, $(ab)ba$ | 3 | 36 |
| 1929 | Lukasiewicz | CTV | $a'ab$, $(aa)a$, *Syll1* | 4 | 32 |
| 1942 | Rosser[8] | CTV | $a'ab$, $(aa)a$, *Syll2* | 4 | 28 |
| 1948 | Lukasiewicz-Wajsberg | CTV | C2, $((a'b)r)(r'a)s'a$ | 5 | 25 |
| [9] | | CTV | PC1, PC2, B3 | 3 | 36 |
| | | | | | |
| 1690 | Leibniz | na | B3, C1, C2, *Syll1* | 2 | 32 |
| 1872 | Robert Grassmann | na | B1, B3, C5, ()()=(), ⊥⊥=⊥, ()⊥=() | 1 | 45 |
| 1890 | Schröder-Lejewski | Ba | B2, B3, *Syll1*, $a'()=()$, $a'(bc)(a'b')(a'c')$, $(ab)c=((a'c)(b'c))$ | 2 | 79 |
| 1892 | Johnson | na | B1, C1, C3, contradual of C6 | 2 | 32 |
| 1904 | Huntington | Ba | B2, J1, C5, $ab=ba$ | 2 | 35 |
| 1913 | Sheffer[10] | Ba | B2, C3, $(ab)c=((a'c)(b'c))$ | 3 | 36 |
| 1933 | Huntington | Ba | $ab.c = a.bc$, $ab=ba$, C6 | 3 | 30 |
| 1933 | Robbins-McCune | Ba | $ab.c = a.bc$, $ab=ba$, dual of C6 | 6 | 26 |
| 1968 | Meredith[11] | Ba | $(a'b)cb=bca$, C4 | 4 | 20 |
| 1969 | Abbott | Ba | C2, C4, $(ab)c=(ba)c$, $ab=ba$ | 2 | 31 |
| 1969 | *LoF* | pa | J1, B1, C5 | 1 | 33 |
| 1986 | Bricken | pa | B1, B4, C2,C3 | 1 | 28 |
| 1989 | Koppelberg[12] | Ba | B1, B3, $a(a'b')=a$, C5 | 2 | 42 |
| 2000 | Veroff et al | Ba | $((ab)(a(bc)))=a$, $ab=ba$ | 4 | 20 |
| 2002 | McCune et al | Ba | $(((bc)a)(b(a'(ad)))) = a$ (DN1) | 6 | 21 |

7. This is Prior's 3.11, his reading of the system in Peirce (W5: 162-90, 1885).

8. Eves 1990: 256, L′; Prior 6.3. $ab \Leftrightarrow a \wedge b$ because Rosser's primitive connective is conjunction.

9. This basis has never been articulated to my knowledge. It is closely related to PC1-3, discussed in §5.4 and §A.9, fn. 3. Church (1956) and Mendelson (1964) invoked axioms trivially equivalent to B3, namely $((a'\perp)\perp)a$ and $(a'b)(ab)b$, respectively.

10. Expressed using the Sheffer stroke, C3 is $((aa)(aa))=a$, B2 is $a(b'b)=a$ (and has the effect of J1), and C5 is $(ab)c = ((a'c)(b'c))$. *LoF* asserts (p. 107) that $(ab)c = ((c'a)(b'a))$ and the dual of $a'(b'b) = a$ form a Ba basis, but gives neither proof nor citation. In effect, Spencer-Brown alleged that replacing $a(b'b)=a$ with its dual enables a proof of C3. Bernstein (1934: 880) proved that $(ab)c=((a'c)(b'c))$ and an axiom equivalent to the Robbins axiom form a Ba basis.

11. Meredith and Prior (1968: 221), cited in McCune et al (2002). Meredith (1969: 269) also proposed $a(b(ac)=((cb)b)a$, $(a'(ba))=a$, (1) and (2) of his "third abridgement of Sheffer."

12. This is the only postulate set that embodies the definition (3.3.10) of Ba as a complemented distributive lattice; it is also the set in Mann (2003). 3.3.11 conjectures that these postulates are independent. The postulates **B1-B5** in Halmos and Givant (1998: 42) are very similar. Note that their $a(b'b)=a$ combines B2 and B3.

| | | | | | |
|---|---|---|---|---|---|
| 2002 | Bricken | pa | B1, B4, C2 | 1 | 21 |
| 2007 | Meguire | pa | B1-B4 | 1 | 27 |

**Note.** *Length* = Number of BA symbols required to state the axioms in BA notation.
A primed variable counts as three symbols, '()' as 1. I treat a CTV axiom as an equation ending in '=()', increasing its length by 2. I also:
(i) Eliminate axioms requiring that the cardinality of the base set be at least 2;
(ii) B1 trivially implies TR, via the Erasure Principle; in turn, TR and B1 trivially imply Associativity (§A.1.1). Hence B1 replaces Ba axioms for commutativity and associativity, and is added to every pa basis. No basis in this Table, as originally published, included B1;
(iii) Shorten axioms of the form $(\phi)=()$ to $\phi=\perp$, $(\phi)=\perp$ to $\phi=()$, and $(\phi)=(\gamma)$ to $\phi=\gamma$;
(iv) If two axioms form a dual pair, I eliminate the longer of the two;
(v) Translate the Sheffer stroke by $(ab)$ and not by $a'b'$.
*Diff.*: A subjective assessment of the ingenuity (1 least, 6 most) required to derive J1-C6, L1, TR, and associativity from the given basis.

CTV. $a\perp$ now defines classical negation. Adding the axiom $\perp a=bb$ [C2] to IA yields the CTV (Prior 3.12) and renders C3, $(a\perp)\perp=a$, and B3, $aa=bb$, demonstrable. Hence IA∪C2 is equivalent to Ba, interpretable as classical logic.

*Huntington* (*1904*). *LoF* rightly cited this paper, the wellspring of self-aware Boolean axiomatics. My summary of Huntington's axioms follows Stoll (1974: §4.1) and Eves (1990: 216, 257). Huntington defined Boolean algebra as a set $B$ with at least two members and closed under two binary and one unary operations. The binary operations are dual to each other, so that his remaining eight axioms are grouped into four dual pairs. The binary operations commute ($\mathcal{B}1$), have distinct identity elements ($\mathcal{B}2$), distribute over each other ($\mathcal{B}3$), and have inverses defined in terms of the unary operation and the identity elements ($\mathcal{B}4$). Given $\mathcal{B}1$-$\mathcal{B}4$, associativity is a theorem (Eves 1990: 217-19). Since the pa has both a primal and dual reading, a pa initial contains the same information as a dual pair of Huntington axioms.

Interpreting Huntington's two binary operations as $ab$ and $(a'b')$, BA satisfies Huntington's basis as follows:[13]

$\mathcal{B}1$. TR. Dually, $((a)(b))=((b)(a))$, also by true by TR;
$\mathcal{B}2$. B2. Dually, $((a)\underline{(())})$ [B2] $= \underline{((a))}$ [C3] $= a$;
$\mathcal{B}3$. C5. Dually, $((bc)a')$ [C3,2x] $= (((\mathbf{b'})(\mathbf{c'}))a')$ [C5] $= \underline{((}(\mathbf{a'}b')(\mathbf{a'}c')\underline{))}$ [C3] $= (a'b')(a'c')$;
$\mathcal{B}4$. B3 and J1.

Huntington (1933) derived his 1904 Ba basis from B1, C1, and C6, a nontrivial exercise. Huntington (1933a) showed that B1, C6 ⊢ C1. Kauffman (1990), using the pa, considerably simplified Huntington's proofs; see §A.4 for details. That B1 and each of {J1,J2}, {B3,B4}, and C6 form a pa basis requires demonstrating J1,J2 ⊢ C6 (*LoF*), C6 ⊢ B3,B4 (§A.4), and B3,B4 ⊢ J1,J2 (§A.1).

McCune (1997), using computer assisted methods, verified Robbins's 1933 conjecture that C6 could be replaced by its dual. Dahn's (1998) concise reworking of his proof, using boundary notation, is in §A.17.

13. Also see Wolfram (2002: 773). Bernstein (1916) combined B2 and B4 into $a(b'b)=a$ (J1, in effect) and its dual. Montague and Jan Tarski (1954) later showed that each of $ab=ba$ and $(a'b')=(b'a')$ renders the other redundant.

*McCune et al* (2002), employing computer-assisted search and proof, confirmed that the single axiom DN1, stated in terms of join and complement, is a Ba basis. The pa translation of DN1 is shown in Table 6-2. McCune et al also found another one axiom basis (Sh1) for Ba stated using just the Sheffer stroke; its pa translation is $(((bc)a)(b((ba)b)))=a$.[14] Sh1 has the same length as DNI but features three instead of four variables. Their Theorem 3 says that no single axiom for Ba, having the Sheffer stroke as sole connective, can be shorter than Sh1.

Other bases that are as short as DN1 and Sh1 are those of Robbins-McCune, Meredith, and Bricken's (2002) basis with C2 replacing his $(()a)=\perp$. The basis of Veroff et al (McCune et al 2002: 2), whose pa translation is $((ab)(a(bc)))=a$[15]and $(ab)=(ba)$ is even shorter (by one symbol) than McCune's. But the four parentheses in the latter axiom are needed only because Veroff's sole operation is the Sheffer stroke, and so can be eliminated by (iii) in the Note to Table 6-2. The resulting basis is the shortest Ba/pa basis in Table 6-2.

Wolfram (2002) drew up a list of 582 Ba consequences, with no consequence featuring more than two variables or six instances of the Sheffer stroke. He then counted how many steps were required for the Waldmeister theorem proving computer program to prove each consequences on this list. He did this for each of the eight bases described on his p. 808. The graphical summary of his results on p. 1175 reveals that bases differed importantly in the number of proof steps required. Sheffer's 1913 and Veroff's basis were more or less tied for the smallest average number of proof steps per consequence. The Robbins-McCune basis did poorly; its dual, Huntington's (1933a) basis, {C6,B1}, did not compete. This computer "horse race" suggests a more objective way of ranking the bases in Table 6-2 by difficulty.

The single axiom bases DN1 and Sh1 fared worst of all. Worse yet, the demonstrations of very elementary properties from such bases can be extremely elaborate. For example, to demonstrate $ab=ba$ [$ab.c=a.bc$] from DN1, McCune et al (2002: Th. 1) require 87 [119] steps. Wolfram's (2002: 810-11) derivation of Sheffer's (1913) Ba axioms from Sh1 requires 343 steps, 81 lemmas, and equations with as many as 128 operators. McCune et al (2002: Th. 2) require 158 steps to achieve the same result. Even demonstrating that the Sheffer stroke commutes requires 42 lemmas (Wolfram) or 93 steps (McCune)!

## 6.3. Other Historical Systems Related to the pa.

A pa basis can serve as a CTV basis and vice versa. CTV axioms take the form '$\alpha \rightarrow \beta$' or can be re-expressed as such. pa initials are of the form '$\alpha=\beta$', and are easier to work with, especially for those whose mathematical habits are those of elementary algebra. The distinction is not essential, however, because any axiom of the form $\alpha \rightarrow \beta$ is equivalent to the equational form $(\alpha)\beta=()$. B1 is absent from the CTV bases in Table 6-2, because no CTV basis contains a pair of postulates asserting that one of $\wedge$ or $\vee$ both commutes and associates. Instead, these bases were designed so that this can be demonstrated.

Nicod (1917: 34) proposed a two axiom basis for the CTV, formulated using only the Sheffer stroke, read as NAND. The longer of these axioms is more easily understood when re-expressed using the conditional as well as the stroke (*PM*, p. xviii). Invoking the dual reading, so that $a|b \Leftrightarrow$

14. DN1: *Dem.* $(((bc)a)(b(a'(ad))))$ [B4] $= (((bc)a)(b(a'\underline{(\mathbf{a'}ad)})))$ [J1] $= (((bc)a)(b\underline{(a')}))$ [C3] $= (((bc)\underline{a})(b\underline{a}))$ [C5] $= (\underline{((bc))}(b))a$ [C3] $= (bcb')a$ [OI,2x] $= (b'bc)a$ [J1] $= a$.
Sh1: Let $a|b \Leftrightarrow (ab)$. *Dem.* $(((bc)a)(b((\underline{b}a)\underline{b})))$ [B4,2x] $= (((bc)a)(b\underline{((a))}))$ [C3] $= (((bc)\underline{a})(b\underline{a}))$ [C5] $= (\underline{((bc))}b')a$ [C3; OI,2x] $= \underline{(b'ba)}a$ [J1] $= a$. □

15. Erasing $c$ from this axiom yields the Robbins axiom.

(*ab*) and $a \rightarrow b \Leftrightarrow (a(b))$, and treating any outermost parentheses as redundant, Nicod's axioms are $a'a$ and $(ab')(bc)ac$, the shortest (20) CTV basis in Table 6-2. The substitutions $a'/a$ and $b'/b$ reveal that the latter axiom is an instance of *Syll1*. Because it is shorter by four symbols, I give it a distinct name, *Syll2*. Nicod then condensed his two axioms into one (*PM*, p. xix). From this single axiom and a variant of *modus ponens*, he derived the axioms of *PM*, thereby proving that his single axiom was a CTV basis. Lukasiewicz later simplified Nicod's single axiom into a well-known single axiom (Prior 6.4; Quine 1982: 87), not shown in Table 6-2, whose pa translation has length 23.

In 1929, Lukasiewicz (Prior 1.4a; Quine 1982: 85) proposed a basis I call L, having a straightforward pa interpretation. $(aa)a$ is one half of C1. $a'ab$ is B3 and is the other half of C1 when $b/a$. *Syll1* asserts the transitivity of the conditional and the validity of Barbara. *Syll1* with $a$=() yields another hoary chestnut, *modus ponens*. Nicod's two axiom basis, just discussed, is, in effect, L with $(aa)a$ omitted. (Note also the similarity of Nicod's basis to Rosser's.) Evidently, a slight modification of *Syll1* renders $(aa)a$ redundant; I have not seen this fact mentioned in print.

Lukasiewicz and Bernays, working independently, earlier proposed a basis including *Syll2* and $(ab)ba$ in place of *Syll1*, yielding a revised version (Prior 6.11) of the truth-functional axioms of *PM*.[16] The literature is silent about the similarity between Rosser and Lukasiewicz-Bernays, as well as about how *PM*'s $(ab)ba$ is redundant in L and Rosser. In 1948, Lukasiewicz proved that $((a'b)r)(r'a)s'a$ is the shortest possible single axiom from which all formulae involving '→' alone can be demonstrated (Prior 2.15d). Wajsberg (Prior 3.3) showed in 1937 that taking F as primitive, and adding $\mathsf{F} \rightarrow a$ (in effect C2), $\sim a =_{\mathbf{df}} a \rightarrow \mathsf{F}$, and $\mathsf{T} =_{\mathbf{df}} \sim \mathsf{F}$ to any axiom system adequate for '→' alone, results in a CTV axiom system.

The radically "arithmetical" foundation BA proposes for Ba and the truth functors has a curious precedent. Van Horn (1917), writing in ignorance of Sheffer (1913), purported to derive *PM* 's CTV axioms from a single axiom giving the semantics of the Sheffer stroke (here notated by ↑), restated as follows: $[|a|=|b|] \rightarrow [|a \uparrow b| \neq |a|]$, and $[|a| \neq |b|] \rightarrow [|a \uparrow b| = \mathsf{T}]$. This axiom follows trivially from A1, A2, keeping in mind that $[a \uparrow b \Leftrightarrow (a)(b)] \leftrightarrow [() \Leftrightarrow \mathsf{T}]$ and $[a \uparrow b \Leftrightarrow (ab)] \leftrightarrow [(()) \Leftrightarrow \mathsf{T}]$. The axiom also follows from 4.1.2 above. Nicod (1917: 40) praised Van Horn's paper, but claimed that its derivations of the *PM* axioms were flawed, because Van Horn freely invoked what is here called T16, without being aware that he was invoking a metatheorem needing proof. Even though Van Horn's and Nicod's papers were published side by side, the only citation of Van Horn I have encountered is Grattan-Guiness (2000: 434). While no one, to my knowledge, has revisited Van Horn's paper, the pa reveals that its intuition was sound.[17]

Byrne's (1946) Ba notation, based on juxtaposition as sole connective and the unary prime, translates into the pa as follows: (1) remove parentheses not immediately followed by a prime; (2) if a

16. For a (nontrivial) demonstration of C5 from these axioms, see Halmos & Givant (1998: 37-8).

17. The way *LoF* grounds Ba in a bit of Boolean arithmetic has precedents. Shannon's (1938) basis for **2** is (a) his arithmetical postulates (1)-(3), isomorphic to Table 2-1a; (b) his assumption (4) that *B* has two members; and (c) a loose definition of complementation. Table 2-1b is isomorphic to his "theorems" (7a) and (7b). Prior (1962: 4-13), following Polish practice, grounds his exposition of sentential logic in the Boolean arithmetic of 0 and 1. Cole (1968) derives Ba from analogues of Table 2-1, R1, and T16. Malmstadt et al (1973: 281, §3-4.1) sketch a derivation of **2** from Boolean addition (dually, multiplication) and complementation, taken as axiomatic. Rudeanu (1974: Example 1.1) asserts but does not show that his axioms for Ba (namely OI, absorption, Boolean versions of J1, C3, C5, and the duals of all the preceding, as per his Definition 1.1) can, in the case of **2**, "be easily established by direct verification" from the operation tables for Boolean addition, multiplication, and negation. (Shannon 1938 makes a similar assertion about his (1a)-(8).) Most of Rudeanu's axioms are redundant, in that J1, C5, and OI alone form a pa basis, and the pa is a Ba by 3.3.10.

pair of parentheses is primed, remove the prime; (3) replace all subformulae of the form $\alpha\alpha'$ by (). Hence Byrne's notation differs from the pa only in the manner in which parentheses are employed. Curiously, six of the eight consequences Byrne proved are C1-C6 in *LoF*. Byrne proves that his algebra is Boolean by deriving Huntington's (1933a) basis, C6 and order irrelevance. Virtues of Byrne's system include his leaving juxtaposition uninterpreted, and his dispensing with Boolean 0 and 1. Byrne's paper is well-known, but to my knowledge his notation has yet to be imitated. Likewise, the boundary mathematics literature, such as it is, does not cite Byrne (1946).[18]

The system AB of Anderson and Belnap (1959) features one unary operation, denial, denoted by an overbar, and one connective, disjunction, denoted by $\vee$. Erasing all instances of $\vee$, and enclosing in parentheses that which lies under an overbar, results in the equivalent pa formula. AB is semantically identical to the primal reading of the pa.[19] The sole axiom of AB is $b(a)ac$ = (), true by B3 and C2. The rules of inference of AB are (1) from $bac$, infer $b((a))c$, true by C3, and (2) from $(a)c$ and $(b)c$, infer $(ab)c$. *Cal.* $(a'\underline{c})(b'\underline{c})(ab)c$ [B4,2x] = $\underline{(}a'\underline{)(}b'\underline{)}(ab)c$ [C3,2x] = $\underline{ab(ab)}c$ [B3] = (). □

The creators of AB went on to found relevance logic, a fact consistent with AB's disavowal of *modus ponens* and silence re substitution. Hunter (1971: §37.6) sets out an effective proof procedure for AB based on refutation trees, but the resulting proofs are a good deal more complicated than ones based on B3 and B4. For example, Hunter requires 16 lines to verify that $(p\to q\to r)\to(p\to q)\to(p\to r)$. The corresponding pa calculation is trivial: *Cal.* $(\underline{p}'q'\underline{r})(\underline{p}'q)p'r$ [B4,3x] = $\underline{(}q'\underline{)(}q\underline{)}p'r$ [B3; C2] = (). □

Translated into BA, Schütte's (1977: 17) basis for the CTV is J1 and the A2 instance $(\perp)$ = (). His rules I$\wedge$, I$\vee$, and I$\to$ are all instances of B3. I cannot determine whether an equivalent to B4 resides somewhere in Schütte's system.

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18. Byrne sets out four Ba axioms: $B$ has at least 2 members, OI, $\forall xyz[xy{=}x \to y'x{=}z'z]$, and $\forall xy\exists z[y'x{=}z'z \to xy{=}x]$. The latter two axioms, often misquoted, are essentially the Consistency Principle, 3.3.13. Stoll (1963: §6.3) derives Huntington's (1904) axioms from Byrne's. I do not grant Byrne's axioms the pride of place granted them in Meguire (2003), because I now appreciate that existential quantifiers in algebraic axioms are a *faute de beauté*, as such axioms are incompatible with the universal algebra concept of *variety*.

19. To Hunter (1971: §37) I owe my discovery of the system AB. In a striking feat of bravado, Anderson and Belnap (1959) lay out AB, prove it sound, complete, and decidable, and prove its axioms independent, all in less than 300 words. AB thus enjoys the dubious distinction of probably being the tersest version of sentential logic ever devised. Hunter (§§37.1-5) expands the proofs of these metatheorems to all of three whole pages. The proof of T17 in §A.9 makes possible an equally terse statement of the pa.

# Chapter 7.
# Why the Indifference?

“[The pa] *is a very beautiful version of the propositional calculus, and I cannot understand why it has not become a standard method in logic text-books… Spencer-Brown’s theory has gained great popularity among various people, but logicians have taken little interest in it.*”

Grattan-Guiness (1982: §5.1).

The approach to Ba and the sentential connectives described in this book remains unadopted, despite LoF having been published more than 40 years ago, and notwithstanding the merits I and others have claimed for it. I begin exploring the reasons behind this fate by setting out what this curious book reveals about its origins. Spencer-Brown worked out many of the ideas in *LoF* while teaching an introductory course in logic (pp. xii; the page references below without citation are to *LoF*); *LoF* may have begun as lecture notes for this course. He derived his version of BA (namely, one using ‘ $\urcorner$ ’ instead of ‘()’, and with no analogue to ‘⊥’) by working backwards from **2** and CTV (p. 112). He (p. xii) attributed some key insights to his having designed electronic circuits during the 1960s, but did not cite Shannon’s (1938) celebrated result that the algebra of switching circuits is a model for the CTV (and thus also for the pa).

*LoF* does not sufficiently disclose the extent to which it built on conventional Boolean algebra. It rightly cited some classics, such as Huntington (1904, 1933) and Sheffer (1913), as well as Boole and Peirce, but did not cite any text on Boolean algebra extant at the time of writing (e.g., the 1958 ed. of Hohn 1966; Whitesitt 1961; Arnold 1962; Goodstein 1963). *LoF* did not cite any of the following logic texts, all standard at the time of writing: Hilbert and Ackermann (1950), the 1950 ed. of Quine (1982), Rosenbloom (1950), Quine (1951), Rosser (1953), Church (1956), Suppes (1957), Carnap (1958), Nidditch (1962), and Kneebone (1963). The only text on modern formal logic cited in *LoF* is Prior (1962), invoked only once to make a minor point about syllogisms. Worst of all, *LoF* is silent about quantification and hence oblivious to the central role of first order logic.

Spencer-Brown argues (chpt. 11) that certain infinite pa formulae with a finite recursive representation have an “imaginary” truth value, arising in a manner analogous to the way complex numbers arise from the roots of certain polynomial equations with real coefficients, also having a recursive interpretation.[1] He alleges that such truth values have momentous implications for mathematics, philosophy, and engineering. For example, he argued that they extinguished the paradoxical character of self-reference[2]:

> “All we have to show is that the self-referential paradoxes, discarded with the Theory of Types, are no worse than similar self-referential paradoxes, which are considered quite acceptable, in the ordinary theory of equations. The most famous such paradox is… ‘This statement is false.’” (p. ix)

and:

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1. If a Boolean equation has a pa representation that is not recursive, *LoF* (p. 57) says that is “of the first degree.” Recursive equations are of “degree higher than one.” If neither () nor (()) solve an equation of higher degree, then *LoF* (pp. viii-x, 58) argues that it has an “imaginary” solution. Relating imaginary Boolean values to extant work on recursive arithmetic and functions (e.g., Mendelson 1997: chpts. 3,5; Kneebone 1963: chpt. 10) and to Bochvar’s (1981) paradox logic are possible directions for future research.

2. On self-reference, logically and philosophically contemplated, see Bartlett (1992).

> "[Recursive Boolean] equations have hitherto been excluded from the subject matter of ordinary logic by the Russell-Whitehead theory of types" (p. xviii)[3]

The Theory of Types is meaningful only if the ground logic is of order greater than zero, and such is never the case in *LoF*. In fact, *LoF* is innocent of polyadic predicates, as well as of all but a few trivial bits of naïve set theory (e.g., the Boolean algebra of classes makes a very brief and casual appearance in Appendix II). Spencer-Brown asserts that his imaginary truth values render the well-known limitative theorems of Gödel and Church (Stoll 1963: §§9.9, 9.10; Mendelson 1997: Ths. 3.37-3.54) "…less destructive than was hitherto supposed" (p. xvii), but gives no details. These limitative theorems all presuppose first order logic, which lies completely beyond the capabilities of pa and Ba. §A.16 discusses why the most important limitative theorem is far simpler to prove, and hence harder to evade, than was commonly understood when *LoF* was written. Meanwhile, *LoF* is silent about nonclassical and infinitary logics and the theory of recursive functions, conventional topics which render chpt. 11 less radical than would seem at first blush.

While *LoF* does not claim that BA suffices to ground all of mathematics, others stride boldly where angels fear to tread: "…the propositional calculus…develops naturally from [A1 and A2]. Thus the act of severance leads inexorably to logic and through *PM* to the whole of mathematics." (Croskin 1978: 187) This statement would be true if BA implemented the quantification and second order logic *PM* requires, and if *PM* had succeeded in its aims. (A classic critique of *PM* is Quine 1995: 3-36, first published in 1941.)

*LoF* also indulges in philosophical speculation (pp. v, vi, xix-xxii, 85, 89-96, 101-106) and invokes dubious etymologies (pp. 93, 101, 105, 106, 109, 126). Spencer-Brown claims (p. ii) to have studied under Wittgenstein[4] (whom he cites four times) and R D Laing, but is silent on how he learned mathematics and logic. Elsewhere, he claims to have worked with Lord Cherwell in the 1950s and the mathematician J C P Miller in the 1960s.

Most damaging to Spencer-Brown's reputation is how the subsequent evolution of mathematics has falsified a number of *LoF*'s predictions about the future course of mathematics. Writing in 1967, Spencer-Brown claimed that:

> "...if we confine our reasoning to an interpretation of Boolean equations of the first degree only, we should *expect* to find theorems which will always defy decision, and the fact that we do seem to find such theorems in common arithmetic may serve, here, as a practical confirmation of this obvious prediction. To confirm it theoretically, we need only to prove (1) that such theorems *cannot* be decided by reasoning of the first degree, and (2) that they *can* be decided by reasoning of a higher degree. (2) would of course be proved by providing such a proof of one of the theorems.

3. Before touching on Russell's paradox and the like, Spencer-Brown should have read Prior (1962: §III.3.3), a book he cites. *LoF* also cites the 1958 edition of Fraenkel, Bar-Hillel, and Levy (1973: chpt. III) in support of its contention that there have been prior attempts to "… rehabilitate, on a logical rather than on a mathematical basis, something of what was discarded with the Theory of Types…" (*LoF*, p. xix, fn. 8). Fraenkel et al never claim that type theory was ever the standard resolution to Russell's paradox. On type theory and how it addresse the set-theoretic paradoxes, also see Hatcher (1982: chpt. 4).

4. Wittgenstein haunts *LoF* as well as much mid-20$^{th}$ century British philosophy. I leave to others the pleasure of tracing the specific influence of Wittgenstein's *oeuvre*, the *Tractatus* in particular, on *LoF*. A similar pleasure undoubtedly awaits the Peirce or Whitehead expert willing to give *LoF* a close reading.

> "I may say that I believe that at least one such theorem will shortly be decided by the methods outlined in [*LoF*]. In other words, I believe that I have reduced their decision to a technical problem which is well within the capacity of an ordinary mathematician who is prepared, and who has the patronage or other means, to undertake the labour." (pp. 99-100; emphasis in original)

More specifically:

> "…I found evidence, in unpublished work undertaken in 1962-65, suggesting that the four-colour map theorem [*sic*] and Goldbach's conjecture are undecidable with a proof structure confined to Boolean equations of the first degree, but decidable if we are prepared to avail ourselves of equations of higher degree." (p. xix)[5]

Regarding Fermat's Last Theorem (FLT), Spencer-Brown wrote:

> "…it is my guess that Fermat (who was apparently too excellent a mathematician to make a false claim to a proof) used [imaginary truth values] in the proof of his great theorem, hence the 'truly remarkable' nature of his proof, as well as its length." (p. 99).

Spencer-Brown was asserting that certain mathematical conjectures, all very well known at the time he wrote, were unprovable using standard mathematics grounded on classical bivalent logic, but could be proved using mathematics grounded in the 3-valued logic (i.e., one incorporating an imaginary truth value) he introduced in chpt. 11 of *LoF*. In the nearly 40 years that have elapsed since *LoF* first appeared, nothing of this sort has eventuated. Instead, seven years after *LoF*'s first publication, Haken and Appel announced their proof of the Four Color Map Theorem, one based on conventional discrete mathematics requiring a large amount of machine computation (for a definitive treatment see Haken and Appel 1989). Wiles (1995) finally proved FLT using mathematics that, however intricate and difficult, are probably ultimately grounded in first order logic and the ZFC axioms. All this may help explain why Spencer-Brown's work has been ignored, and why it need not be swallowed whole.[6]

5. Spencer-Brown repeated his claim of a proof of the Four Colour Map theorem in a letter to the editor of *Nature*, dated 17.12.76.

6. A good introductory course in formal logic for nonspecialists is Hodges (1977), which employs refutation trees and linguistic examples.

# Chapter 8.

# Conclusion.

"*Logical laws are the most central and crucial statements of our conceptual scheme, and for this reason the most protected from revision by the forces of conservatism*; *but… they are the laws an apt revision of which might offer the most sweeping simplification of our whole system of knowledge*." Quine (1982: 3).

Let there be a blank surface upon which marks may be written. The mark can be the ‘⏋’ of *LoF* (proposed by Peirce in 1886), a simple closed curve (proposed by Peirce from 1896 onwards), or Croskin's (1978) ‘()’, adopted here. The symbol () is the sole primitive constant and is both operator and operand. Whatever the mark is taken to be, it is essential that it have a distinguishable "interior" and "exterior," as the mark serves as the boundary between the interior and exterior. The mark and the blank page are the boundary primitive values. Interpreting the blank page as one of ‘true’ or ‘false’ gives rise to boundary logic.

The only means we have, at this stage, to distinguish anything is to write another mark on the state we wish to distinguish. To mark the exterior, we write ()(); to mark the interior, (()). The Law of Calling says that ()() is indistinguishable from simple (). The Law of Crossing says that (()) cannot be distinguished from the blank page. Hence the exterior and interior of a mark are distinguished simply by how each interacts with another mark: the exterior of a mark is idempotent; the interior, nilpotent. Let Calling and Crossing be the sole axioms. *LoF* shows, albeit in a rather cryptic fashion, that the primary arithmetic (PA) emerges from these notions plus an equivalence relation among formulae, logical equivalence. Thus the PA can be seen as Boolean arithmetic notated so as to lay bare its tree structure. For another recapitulation of the PA, see 2.3.4.

Inserting letters anywhere in a PA formula yields a pa (primary algebra) formula. Combine the PA and pa to obtain boundary algebra (BA). Letting the blank page interpret Boolean 0, and () interpret Boolean 1 (or vice versa), yields **2**. The set $B$={(),(())} corresponds to the base set of **2**, and a letter (a.k.a. variable) can assume any value of $B$. Interpreting BA for classical logic results in boundary logic, which is equational rather than ponential. An equational logic privileges tautological equivalence, not tautology, and invokes the substitution of equals for equals instead of *modus ponens*.

An initial is a tautological equivalence verified by a decision procedure, used to derive other equivalences. Any Boolean algebra basis, or set of CTV axioms, translates into a set of pa initials. The preferred initials of this book, $abc=bca$, $a'a=()$, $(())a=a$, and $(ba)a=b'a$, and the well-known consequences $aa=a$, $a()=()$, $(a')=a$, and $(a'b')r=((ar)(br))$ facilitate a proof method I call *calculation*, which is similar to, but easier than, that of Peirce's *alpha* existential graphs. pa calculations are much easier than the proofs taught in standard texts, including natural deduction proofs. The pa facilitates clausal reasoning, and trivializes the derivations of the inference rules of conventional logic. The pa and the CTV share a common metatheory.

The pa is, at minimum, a simple yet powerful notation for the truth functors and Boolean algebra, revealing the unity and simplicity underlying the seeming diversity of truth functors. Moreover, because the CTV and **2** are models of the pa, BA highlights the seldom mentioned axiomatic role of Boolean arithmetic for these systems. Boundary methods should prove fruitful for any formal system having a lattice structure, e.g., nonclassical logics, mereology, and relation algebra (Givant 2006).[1] The boundary analogue to refutation trees, and the connection between the BA and

1. This book has touched on how fruitful the boundary point of view can be for lattices (§3.3), groupoids (§3.4), and quantification (§5.6). Meguire (2004) shows that boundary syntax suffices to express normal

topology, should also be explored. BA suggests that mathematical logic, set theory, theoretical computer science, and probability[2] all share a common source: the mental act of making a distinction, for which the marker is the boundary sign (). I invite others to explore whether BA may be seen as a nominalist grounding for **2** and sentential logic, i.e., one free of the notion of set.

Ninety-five years after Boole's first book and 30 years after *PM*, Berkeley (1942) noted that formal logic and Boolean algebra had been little applied.[3] Outside of computer science, electrical engineering, and formal philosophy, this appears to be the case down to the present day (Hehner 2004). Incorporating BA into Hehner's (2004, 2007) Unified Algebra, an integrated notation for Ba and numerical mathematics, warrants exploration. Seen as a demotic version of the hieratic languages **2** and the CTV, BA could facilitate the wider application of logic and Boolean methods. BA could even be taught in secondary schools, as a gentle introduction to logic and formal languages, and as the foundation of information technology.

In an essay on the teaching of basic practical mathematics, the computer scientist Eric Hehner wrote as follows:

> "Logic has been well studied and is now well understood, but is not well used. Programmers learn that logic is a foundation of programming, but they don't often use it to program.Mathematicians study logic, but don't often use it in their proofs. Logic is a tool, like a knife. People have looked at it from every angle; they've described how it works at great length; now it's time to pick it up and use it. To use logic well, one must learn it early, and practice it a lot. …there is a simple basic algebra that can be taught early and used widely." Hehner (2004)

We have seen that this "simple basic algebra" hinges on a mere two departures from the trivial arithmetic of 0 and 1: 1+1=1 and -1=0. I heartily concur with Hehner's wish, and invite readers to employ the contents of this book to help make it come true.

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modal logic, ZFC set theory, category theory, and ringoids. On nonclassical propositional logics, e.g., intuitionistic, modal, relevant, and substructural, see Restall (2000) and Epstein (1995).

2. Discrete probability can be given a Boolean foundation by taking mathematical expectation as primitive, then defining probability as the expectation of a Boolean random variable. See Lad (1996: §2.2).

3. Some of the reasons Berkeley gave for why this might be the case do not apply to BA. Berkeley was employed by an insurance firm. He neither cited Shannon (1938) nor mentioned the possibility of electronic computation, which was being invented while he wrote his paper.

**Bibliographic Postscript**.

BA lies at the intersection of four disciplines: mathematics, philosophy, computer science, and electrical engineering. I included a reference below either because I found it useful when writing this book, or because it was extant and relevant at the time *LoF* was written. The references are grouped by broad topic and listed in order of increasing perceived difficulty.

There are two perspectives on *Boolean algebra*:

- *Mathematics*: Stoll* (1963: §6), Halmos and Givant* (1998: §§19-39), Abbott (1969: §§6,7), Cori & Lascar* (2000: §2), Koppelberg (1989), Givant & Halmos (2009), Burris et al (1981: §§II.1, IV.1-4). Algebras larger than **2** are typically assumed, and developed in either a *set theoretic* or *algebraic* manner. Starred references discuss *Tarski-Lindenbaum algebra,* the close connection between Ba and CTV.
- *Engineering/Computer Science*: Whitesitt (1961: §§1-3), Hohn (1966: §1, §§5.1-5), Rudeanu (1974: esp. §1; many references).

*Lattice theory*. Arnold (1962: §§3,4), Donnellan (1968), Curry (1963: §4; very relevant for logic), Davey and Priestley (2002), Burris et al (1981: §§I, II.1).

*Calculus of Truth Values*. *Mathematics*: Arnold (1962: §1), Hohn (1966: §§3.1-12), Goodstein (1963: §4), Kneebone (1963: §§2,6), Halmos and Givant (1998: §§8-18), Epstein (1995: §§II.J-M), Stoll (1974: §§2.1-5,3.5), Nidditch (1962), Machover (1996: §7), Mendelson (1997: §1), Hunter (1971: §§15-36), Smullyan (1968: Part I), Cori & Lascar (2000: §1), Curry (1963: §§5,6), Schütte (1977: §I). *Philosophy*: Girle (2002: Part One), Quine (1982: Part I), Suppes (1957: §§1,2), Bostock (1997: §2), Prior (1962: 1-71, 301-6, 318-19), Hodges (2001: §§1-7), Carnap (1958: §§2-8, 12a, 22), Zeman (1973: 1-76), Segerberg (1982).

*Calculus of Quantified Variables*. *Mathematics*: Stoll* (1974: §2.6-9, §3.6), Machover* (1996: §8), Quine* (1951: §2), Hunter (1971: §§38-59), Pollock (1990: §2.1), Smullyan (1968: Part II), Mendelson* (1997: §2), Schütte (1977: §II), Cori & Lascar (2000: §§3,4). Starred references include axiomatic set theory. *Philosophy*: Girle (2002: §§12-14), Quine (1982: Parts II, III), Bostock (1997: §§3,5), Hodges (2001: §§8-18), Carnap (1958: §§1, 9-14, 21-25). Bostock, Hodges, and Machover are best for current terminology.

For gentle introductions to logic as part of elementary mathematics, see Wolf (1998: Unit 1); to metamathematics and axiomatic thinking, see Stoll (1974: §3); to the philosophy of mathematics, see Lucas (1999).

On the history of logic and related mathematics, see Curry (1963), Grattan-Guiness (2000), Gabbay and Wood (2004), and Kneebone (1963). On Peirce's role in the early history of Ba, see Brady (2000: 1-142). The first systematic treatment of Ba is vol. 1 of Schröder (1966), written in 1890. Ba came of age as serious mathematics in the 1930s, primarily thanks to Marshall Stone and Tarski.

There is a fair secondary literature on *LoF*; see the bibliography at http://www.lawsofform.org/bib/index.html, in which the names Bricken, Kauffman, and Varela stand out, as do a number of articles in the *International Journal of General Systems*. Another URL bearing on *LoF* is http://www.enolagaia.com/GSB.html . Both sites reveal that *LoF*'s love of paradox and enigma has attracted a nonmathematical following.

**Appendix: The Null Individual and Its Controverted Ontology.**

This Appendix reviews a controversy in the foundations of *mereology*, a body of first order theories about the part-whole relation described in Simons (1987: chpts. 1,2) and Casati and Varzi (1999: chpt. 3). Mereology begins with a domain of individuals, and a primitive dyadic predicate P*xy*, read as '*x* is part of *y*.' P is assumed transitive and can be proved a partial order. Let the fusion *b* of any number of invividuals *a* be such that *a*P*b* comes out true for all *a*. An axiom asserts that the *fusion* of the members of any nonull set [of those individuals satisfying any monadic predicate] exists.

Nearly all mereological systems deny the existence of a *null individual*, one that is part of every individual; the main exception is a system advocated by R. M. Martin. For present purposes, the null individual can be deemed the mereological analogue of the null set. In this Appendix, I argue that the denotation I propose for ⊥ is controversial in a manner analogous to the controversies aroused by Martin's null individual.

To my knowledge, the null individual, under the name *null entitity*, made its first public appearance in the following passage from Martin (1943: 3): "In order to develop an unrestricted Boolean algebra… it is desirable to admit the existence of a null entity… We shall retain then the interpretation of this system as a calculus of individuals and also admit the null entity." Carnap (1956: 36), citing Martin (1943), postulated the existence of a *null thing* as one of seven possible things named by a nonunique description. He wrote: "…a natural solution offers itself if we construct the system in such a way that the spatiotemporal part-whole relation is one of its concepts. …it is possible, although not customary in the ordinary language, to count among the things also the null thing, which corresponds to the null class of space-time points. …it is characterized as that thing which is part of every thing."

Geach (1972: 200), writing in 1949 in response to the 1947 first edition of Carnap (1956), wrote: "There is a well-known convention in mathematics whereby 'the least' or 'the only' number fulfilling a condition is deemed to be zero if there is in fact *no* number thus uniquely described. This has technical advantages… Carnap proposes an allegedly similar convention for language about physical objects [, the null thing]. Further, [Carnap] describes the null thing as corresponding 'to the null class of spacetime points'—or, in plain English, as existing nowhen and nowhere!"

The following long quotation is taken from the opening paragraphs of Martin's vigorous defense of the null individual, first published in 1965 and reprinted as Martin (1979). I trust that writing ⊥ where Martin wrote 'null individual' does not do violence to his meaning.

"Is there such a thing as ⊥? Well, as an actual or concrete entitity, certainly not. There is no such actual entity, there never has been, and there never will be. If this were the whole story, one could end therewith. As a convenient technical fiction and useful notational device, however, introducing ⊥ into [first order logic] is not without interest. ⊥ can be given important roles to perform and it can be made to perform them well, so well in fact as to lend strong support to regarding its theory as a suitable appendage to logic.

"One speaks of *the* ⊥ in the sense of there being one and only one ⊥. Could there be two or more? Possibly, but there is no need for such, and anyhow it is desirable to keep traffic with the ghostly at a minimum.

"Attitudes differ as to the feasibility of introducting ⊥... Lejewski… explicitly admits a 'nonreferential name… meant to be a name that does not designate anything.' Such a name is to be read 'object which does not exists.'

"That the notion of ⊥ is no better or worse than that of the null set seems likely. Refusal to postulate one should perhaps go hand in hand with refusal to postulate the other. The null set… is a useful mathematical notion that has been with us with impunity for some time. Set theory… would be impoverished without it and technical inconveniences would result. These are perhaps not insurmountable, but little would be gained if one were to reject it. And mathematics abounds with other convenient technical fictions that by parity of reasoning would have to be forsworn, many of these depending definitionally on the null class. The more reasonable course then seems to be to admit not only the null set but also such additional 'fictions' as are feasible if strong technical reasons can be given on their behalf." (Martin 1979: 82-83)

Martin went on to cite Carnap (1956) with approval.

Bunt (1985: 56-7), nowhere mentioning Martin or Carnap, wrote of the *empty ensemble* as follows: "*emptiness*… is defined as the property of having no other parts than itself…From the transitivity of the part-whole relation it follows that all parts of an empty ensemble are empty. …it can be proved that there exists an empty ensemble, and that an empty ensemble is part of every ensemble [individual]." [emphasis in original]

Simons (1987: 13), citing Geach, summarily dismisses the null individual as follows: "Most mereological theories have no truck with the fiction of a null individual which is part of all individuals… The chief culprit in propounding this absurdity is R M Martin."

Finally, Lewis (1991: 11) writes: "If we accepted the null individual, no doubt we would identify the null set with it, and so conclude that the null set is part of every class. But it is well nigh unintelligible how anything could behave as the null individual is said to behave. It is a very queer thing indeed, and we have no good reason to believe in it. Such streamlining as it offers in formulating mereology [e.g., closure under intersection] can well be done without. Therefore, reject the null individual; look elsewhere for the null set."

Casati and Varzi (1999: 45) also distance themselves from Martin, whom they relegate to a footnote, but in a more cautious way: "…few authors have gone so far as to postulate the existence of a 'null individual' that is part of everything. Without such… (which one could hardly countenance except for algebraic reasons), the existence of an [intersection] is not always guaranteed. Likewise… complements may not be defined, e.g., relative to the universe."

In a mathematically sophisticated paper on the foundations of topology and geometry, Roeper (1997), citing no authority, unapologetically takes mereology as a model of Ba. In any event, doubting the ontological innocence of ⊥ can be forgiven, in light of how contentious the ontological implications of the null individual have been.

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# Appendix: Demonstrations, Proofs, etc.

Throughout this Appendix, '☐' signals the end of a demonstration/calculation/proof.

### A.1. The Core Demonstrations and Calculations.

"*The kind of axiom splitting that got us to this point is the least attractive and the least rewarding part of any subject. We present it… mainly because… its omission would have given a distorted picture of what much of the subject is like*." Halmos and Givant (1998: 40).

The BA basis is *abc* = *bca*, B1; (())*a*=*a*, B2; (*a*)*a*=(), B3; and (*ba*)*a* = (*b*)*a*, B4, which *LoF* calls C2.

### A.1.1. Juxtaposition Commutes, Associates.

There are two ways of showing that concatenation commutes and associates. The first is valid for any lattice, the second for any bounded lattice.

*Any lattice*.
Following Byrne (1946), I now show that concatenation commutes and associates. This proof does not require complementation and so works for any lattice.

**C1**. **aa=a**. *Cal*. *LR*: (*a$\underline{a}$*)*a* [B4] = (*a*)*a* [B3] = (). *RL*: (*a*)*aa* [B4] = (*a**a***)*aa* [B3] = (). ☐

The remaining demonstrations in this section invoke, for the nonce, the decimal point as a "temporary" notation denoting grouping. Once associativity is demonstrated, grouping ceases to matter and this notation can be discarded. In the following demonstation, B1 justifies each step in unless otherwise indicated. Underlining indicates pairs of letters to be treated as a unit.

Concatenation:

- *Commutes* (TR). *Dem*. **ab** [C1] = *ab.$\underline{ab}$* = *b.ab.a* = *ab.a.b* [apply B1 to *ab.a*] = *ba.a.b* = *aa.b.b* = *bb.aa* [C1,2x] = **ba**. ☐
- Ab = ab.ab = b.aba = b.b.aa = b.ab
- ab.a.b = b.aa.b = aa.b.b = b.b.aa = b.b.a = b.ab = a.b.b
- *Associates* (Ass.). *Dem*. If B1 is assumed to associate from the left, we have **ab.c** = *bc.a* [TR] = **a.bc**. If B1 associates from the right, we have **a.bc** = *b.ca* = *c.ab* [TR] = **ab.c** ☐

Henceforth, subformulas can be reordered at will, either silently of by invoking TR. Note that *ab.c* = *bc.a* = *ca.b*. Also, *ab.c* [TR] = *ba.c* = *ac.b* = *cb.a*. Hence all possible permutations of the string *abc* can be generated using B1 and TR.

*Bounded lattice.*
The second demonstration of TR and Associativity is trivial, but requires B2 and B3 and so is valid only for a bounded lattice. B1 is now *ab.c*=*ac.b* (Dilworth 1938: 263, Postulate 1). Concatenation:

*Commutes*: *Dem*. An instance of B1 is (*a'a*)*b.c* = (*a'a*)*c.b*. Then two applications of B3 and B2 yield *bc*=*cb*. ☐
*Associates*: *Dem*. *ab.c* [TR] = *ba.c* [B1] = *bc.a* [TR] = *a.bc.* ☐

Both forms of B1 follow from TR and Associativity:

*Dem*. *ab.c* [Ass.] = *a.bc* [TR] = *a.cb* [Ass.] = *ac.b*. Also, *ab.c* [Ass] = *a.bc* [TR] = *bc.a*. ☐

Note that $ab.c = ac.b$ [TR] $= b.ac$ [TR] $= b.ca$ [TR] $= ca.b = cb.a$ [TR] $= bc.a$. Again, repeated application of TR and B1 generates all six possible permutations of the string *abc*. Also, each form of B1 can be derived from the other form.

### A.1.2. More Consequences.

Demonstrate or calculate C2-C8 as follows.

**C2**. *Dem.* $\mathbf{()a}$ [B3; R1] $= \mathbf{(a)a}a$ [B4] $= (a\mathbf{a})aa$ [B3] $= \mathbf{()}$. □

*Remark*. An immediate consequence of C2 and B3 is that $(a)a{=}()a$ is valid. Bricken (2002) invokes B4 to justify $(a)a{=}()a$ without discussion.

**C3. ((a))=a**. *Cal. LR*: $(((a)))a$ [B4,2x] $= ((\underline{a'}\mathbf{a})\mathbf{a})a$ [B3] $= (\underline{(())}a)a$ [B2] $= (a)a$ [B3] $= ()$.
*RL*: $(a)((a))$ [TR] $= (a')a'$ [B3] $= ()$. □

*Remark.* Recall the group theory notation of §3.4. Group theory proves C3 by noting that $(a')a' = ()$ and $a'a = ()$ both hold, so that $a''$ and $a$ are both inverses of $a'$. Since group theory inverses are provably unique, $a''{=}a$. This proof carries over to the pa if we can prove that pa complements are unique, which the following lemma does.

**Lemma**. Let $b$ and $c$ be hypothetical complements of any $a{\in}B$, so that $ba = () = ca$.

*Dem*. $b$ [B2] $= b(())$ [Hyp.] $= b(ba)$ [B4] $= b(a)$. $a$ [B2] $= a(())$ [Hyp.] $= a(ba)$ [B4] $= a(b)$.

$[(a)b{\wedge}(b)a]{\to}\ a{=}b$. This demonstration holds if $c$ replaces $b$, implying that $a{=}c$. By the Euclidian property (2.3.8) of equality, $(a{=}b){\wedge}(a{=}c) \to b{=}c$. □

**C4**. *Dem.* $\mathbf{((a)b)a}$ [B4] $= ((ab)b\mathbf{a})a$ [TR] $= (\underline{(ab)ab})a$ [B3] $= \underline{(())}a$ [B2] $= \mathbf{a}$. □

*Remark*. Substituting $b'$ for $b$ in C4 yields the primal of the absorption law. The dual is:

*Dem*: $\mathbf{((ab)a')}$ [C3] $= (((a')b)a')$ [C4] $= (a')$ [C3] $= a$. □

The close connection between C4 and absorption should now be evident. The primal and dual together imply L1, $[ab]a{=}a{=}[[ab]a]$, the absorption law, because $(a'b')a \Leftrightarrow [ab]a$ and $((ab)a') \Leftrightarrow [[ab]a]$ are the case. BA is a lattice by virtue of TR, Associativity, C1, and absorption.

**C5**. *Dem.* $\mathbf{a((b)(c))}$ [B4,4x] $= a((\mathbf{a}b)(\mathbf{a}c))$ [C3; $\chi/((ab)(ac))$] $= \mathbf{(a')}\chi$ [B4] $= (a'\boldsymbol{\chi})\chi$ [$((ab)(ac))/\ \chi$; B4,3x] $= (a'(\mathbf{a'}(\underline{\mathbf{a'}ab})(\underline{\mathbf{a'}ac})))\chi$ [B3,2x] $= (a'(a'(()b)(()c)))\chi$ [C2,2x] $= (a'(a'\underline{(())(())}))\chi =$ [B2,2x] $(\underline{a'(a')})\chi$ [B3] $= \underline{(())}\chi$ [B2; $((ab)(ac))/\ \chi$] $= \mathbf{((ab)(ac))}$. □

*Remark*. Note that the demonstration requires only C2 and C3 in addition to B1-B4.

The following calculation may be easier to follow:

*Cal. RL*: $\underline{(((ab)(ac)))}a(b'c')$ [C3] $= (\underline{a}b)(\underline{a}c)a(b'c')$ [B4,2x] $= b'c'a(b'c')$ [OI] $= \underline{(b'c')b'c'}a$ [B3] $= ()a$ [C2] $= ()$.

*LR*: $(a(b'c'))((ab)(ac))$ [B4,3x] $= (a(\underline{\mathbf{a}}(\mathbf{a}b)(\mathbf{a}c)))((ab)(ac))$ [B4] $= (a\underline{((ab)(ac))})((ab)(ac))$ [B4] $= a'((ab)(ac))$ [B4,3x] $= a'(\mathbf{a'}(\underline{\mathbf{a'}ab})(\underline{\mathbf{a'}ac})$ [B3,2x] $= a'(a'(()\underline{b})(()\underline{c}))$ [C2,2x] $= a'(a'\underline{(())(())})$ [B2,2x; OI] $= (a')a'$ [B3] $= ()$. □

**C6**. *Dem.* $\mathbf{(a'b')(a'b)}$ [C3] $= \mathbf{(((}\underline{a'}b')(\underline{a'}b)\mathbf{))}$ [C5] $= (a'(\underline{(b')b'}))$ [B3] $= (\underline{(())}a')$ [B2] $= \underline{(a')}$ [C3] $= \mathbf{a}$. □

**C7**. *Dem*. $\mathbf{((a'b)c)}$ [C3] = $((a'\mathbf{(b')})c)$ [C5] = $\underline{((}(a\mathbf{c})(b'\mathbf{c})\underline{))}$ [C3] = $\mathbf{(ac)(b'c)}$. □

C8 is simpler than its *LoF* counterpart, and its demonstration is new:

**C8**. *Dem*. $\mathbf{((b'r)(a'r'))}$ [C5] = $\underline{((}(\boldsymbol{(b'r)}a)(\boldsymbol{(b'}\underline{\boldsymbol{r}}\boldsymbol{)}r)\underline{))}$ [C3; B4] = $((b'r)a)(\underline{(b')}r)$ [C3; TR] = $(a(b'r))(br)$ [B4] = $(a(b'r)\mathbf{(br)})(br)$ [TR; C3,2x] = $(a(\underline{(r')b')((r')b}))(br)$ [C5, $r'/\mathrm{A}$, $b/\mathrm{B}$] = $\mathbf{(ar')(br)}$. □

It is usually easier to calculate a consequence than to demonstrate it. C8 is an exception; the *RL* part of the following calculation is surprisingly difficult.

*Cal*. *LR*: $\underline{((}(b'r)(a'r')\underline{))}(ar')(br)$ [C3; TR] = $(rb')(rb)(r'a')(r'a)$ [C3,2x] = $(\boldsymbol{(r')}b')(\boldsymbol{(r')}b)(r'a')(r'a)$ [C6,2x] = $r'r$ [B3] = ().

*RL*: $((ar')(br))((b'r)(a'r'))$ [C7,2x] = $(r(b\underline{r}))(a'(br))(b(a'r'))(r'(a'\underline{r'}))$ [B4,2x] = $(r(b))(a'(br))(b(a'r'))(r'\underline{(a')})$ [C3; TR] = $(rb')(a'(rb))(b(r'a'))(r'a)$ [B4,2x] = $(rb')(a'\underline{\boldsymbol{(rb')}(rb)})(b\underline{(r'a')\boldsymbol{(r'a)}})(r'a)$ [C6,2x] = $(rb')(a'r')(br)(r'a)$ [TR, 3x; LR] = (). □

The *RL* part of the above calculation is much more easily verified by TVA:

$r$=(): $((a)\underline{(b())})(\underline{(b'())}(a'))$ [C2; B2, 2x] = $\underline{((a))((a'))}$ [C3,2x] = $aa'$ [B3] = ().
$r$=⊥: $(\underline{(a())}(b))((b')\underline{(a'())})$ [C2; B2,2x] = $\underline{((b))((b'))}$ [C3,2x] = $bb'$ [B3] = (). □

C1-C7 are *LoF* consequences, by different names. B2 and B3 are demonstrable in the system of *LoF*, to wit:

*Dem*. $\mathbf{(\underline{()})a}$ [J1] = $(a'a)a$ [C4] = $\mathbf{a}$. *Dem*. $\mathbf{(a)a}$ [C1] = $a'a\boldsymbol{a}$ [C3] = $(\underline{(a'a)})a$ [J1] = $()a$ [C2] = $\mathbf{()}$. □

*Huntington's Approach to Associativity.*
The above suggests that Huntington's (1904) demonstration of associativity from the basis $(a'a)$=⊥, B2, C5, and TR, as reproduced in Eves (1990: 217-19), was needlessly elaborate. From C3 (immediately below), it follows trivially that the dual of juxtaposition associates: *Dem*. $(\underline{((a'b'))}c')$ [C3] = $(a'b'c')$ [C3] = $(a'\mathbf{((}b'c'\mathbf{))})$. □ That juxtaposition itself associates then follows from 4.1.4. Hence that juxtaposition associates requires only what the proof of 4.1.4 and the demonstration of C3 require. *Proof*: C3, R1 ⊢ 4.1.4. B3, C3, B4 ⊢ R1. B3, B4, TR ⊢ C3. Hence B3, B4, TR ⊢ 4.1.4. □

### A.1.3. From Any Contradiction, Anything Can Be Proved.

*Proof*. By B4, $b(ba)$=$b(a)$ for any $a$,$b$. Now let $b$ be any pa formula whatsoever, and let $a$ be a formula such that both $a$=() and $a$=⊥ are demonstrable by hypothesis. The lhs of B4 evaluates as $b(ba)$ [let $a$=()] = $b(\underline{b}())$ [C2] = $b(())$ [J1] = $b$. The rhs of B4 evaluates as $b(a)$ [let $a$=⊥] = $b$(⊥) [B4] = $b(b$⊥) [J1] = $b(b)$ [B3] = (). Hence $b$=(), the desired absurd result. □

## A.2. Proof of 2.3.10.

**2.3.10**. **Theorem**. $R$ is an equivalence relation iff $R$ is *reflexive* and *Euclidian*.

*Proof*. I assume that the uniform replacement of letters ranging over the field of a relation is allowed (the analogous BA property is R2). Since formulae of the form $xRy$ have truth values, they can be treated as BA atomic formulae. The BA version of *Euclidian* is $(aRc)(bRc)aRb$ = (). Then:

$(aRc)(bRc)aRb$ [$a/c$] = $(aRa)(bRa)aRb$ [*reflexive*; J1] = $(bRa)aRb \Leftrightarrow bRa \rightarrow aRb$.

$(aRc)(bRc)aRb$ [$a/b$; $b/a$; $c/b$] = $(bRb)(aRb)bRa$ [*reflexive*; J1] = $(aRb)bRa \Leftrightarrow aRb \rightarrow bRa$. Hence $aRb \leftrightarrow bRa$, i.e., $R$ is *symmetric*.

$(aRc)(bRc)aRb$ [$c/b$; $b/c$] = $(aRb)(cRb)aRc$ [*symmetric*] = $(aRb)(bRc)aRc \Leftrightarrow (aRb \wedge bRc) \rightarrow aRc$. Hence $R$ is *transitive*. □

*Remark*. Using ponential methods, Lukasiewicz (1967: 97-98) derives the three properties of equivalence relations from a variant of *Euclidian*, $(cRb)(cRa)aRb$, 1 on p. 97. Reflexivity and symmetry are *6 and *7 on p. 97; transitivity is 5 on p. 98.

### A.3. Demonstrations Needed in §3.3.

The following demonstration of C3 holds for any complemented lattice.

**C3.** $((a))=a$. *Dem*. $((a))$ [L3a; L7b] = $((())(a)(a))$ [L8; TR] = $((a)(a)((b)b))$ [L9] = $[aa[(b)b]]$ [L8] = $[aa[()]]$ [TR] = $[[()]aa]$ [L7b] = $[aa]$ [L3b] = $\mathbf{a}$. □

**3.3.13. Theorem** (Consistency Principle). $a \leq b$, $a \cup b = b$, and $a \cap b = a$ are equivalent Ba statements, and these in turn have the pa equivalents $a'b=()$, $ab=b$ and $(a'b')=a$.

*Dem*: $\mathbf{a \cup b = b} \Leftrightarrow ab=b \Leftrightarrow (((a\underline{b})b)(b(a\underline{b})))$ [B4,2x] = $(((a)b)\underline{(b(a))})$ [TR; C1] = $\underline{((}(a)b\underline{))}$ [C3] = $\mathbf{a'b}$.

$\mathbf{a \cap b = a} \Leftrightarrow (a'b')=a \Leftrightarrow (((\underline{(a'b')})a)((\underline{a}'b')(a)))$ [C3; B4] = $((a'b'a)((\underline{b'})a'))$ [C3] = $((a'b'a)(ba'))$ [TR] = $((\underline{a'ab'})(a'b))$ [J1] = $\underline{((}a'b\underline{))}$ [C3] = $\mathbf{a'b}$.

Moreover, $a'b \Leftrightarrow a \leq b$ because $a'b$ satisfies the three criteria for a partial ordering:
*Reflexivity*: $a'a$ [B3] = (). *Antisymmetry*: $a \leq b \wedge b \leq a \Leftrightarrow ((a'b)(b'a))$ [Def. of =] $\Leftrightarrow a=b$.
*Transitivity*: $(a \leq b \wedge b \leq c) \rightarrow a \leq c \Leftrightarrow \underline{((}(a'b)(b'c)\underline{))}a'c$ [C3] = $(\underline{a}'b)(b'\underline{c})a'c$ [B4,2x] = $\underline{(b)(b')}a'c$ [B3] = $()a'c$ [C2] = (). □

**3.3.17. Theorem**. The cardinality of the base set $B$ in BA is necessarily 2.

*Cal*. $x=() \vee x=\perp \Leftrightarrow (x'\underline{(())})(x())(x'(\perp))(x\underline{\perp})$ [J1,2x] = $(x')(x())(x'(\perp))(x)$ [TR] = $(x())(x'(\perp))\underline{(x')(x)}$ [B3] = $(x())(x'(\perp))()$ [C2] = (). □

### A.4. Demonstrating B2-B4 from C6 and Order Irrelevance.

Huntington (1933), which *LoF* cited on p. 88, showed that C1, C6, TR, and Ass. form a basis for Ba. *LoF* is unaware that Huntington (1933a) proved C1 redundant. I now verify that $H$={C6, TR, Ass.} is a Ba basis by demonstrating $H \vdash$ B2-B4. The demonstration of Lemmas 2-3 and B2 are adapted from Huntington (1933a: 4.15); the remaining demonstrations are adapted from Kauffman (1990).[1] A.4 is more concise than Huntington's and Mann's (2003) verification that $H$ is a basis, and also implies that Johnson's (Table 6-2) axioms C1 and C3 are redundant.

**Lemma 1**. *Dem*. $\mathbf{a'a}$ [C6,2x] = $((a')(b'))\underline{((a')b')(a'(b'))}(a'b')$ [TR] = $((a')(b'))(a'(b'))((a')b')(a'b')$ [TR,4x] = $\underline{((b')(a'))((b')a')(b'(a'))(b'a')}$ [C6,2x] = $\mathbf{b'b}$. □

*Remark*. This lemma means that $a'a$ has the same value for all $a$. I go with $a'a=()$, which yields B3 and J1. $a'a=\perp$ is equally valid, and gives rise to the dual interpretation.

C3. *Dem*. $\mathbf{(a')}$ [C6] = $\mathbf{(((a'))a')(((a'))(a'))}$ [Lem. 1] = $(((a'))a')(\mathbf{(a')a'})$ [TR; C6] = $\mathbf{a}$. □

1. See http://www.lawsofform.org/logic.html. This URL also includes the demonstrations in Bricken (1986).

**Def.** $A$ **=**: $a'a$**.** Lem. 1 now implies that $A=A'A$, and that C6 can take the form $a = (a'a')A'$. $A$ and $A'$ are, of course, algebraic synomyms for () and (()).

**Lemma 2.** *Dem.* $\mathbf{A'}$ [C6] $= ((A')(A'))((A')A')$ [C3,2x] $= (AA)\underline{((A')A')}$ [Lem. 1] $= \mathbf{(AA)A'}$. □

**Lemma 3**. *Dem.* $\mathbf{A'}$ [Lem. 2] $= (\underline{AA})A'$ [Lem. 1] $= (\underline{A'}AA)A'$ [Lem. 2] $= ((AA)\underline{A'AA})A'$ [Lem. 1] $= ((\underline{AA})\underline{AA})A'$ {Lem. 1] $= \mathbf{A'A'}$. □

**B2.** *Dem.* $\underline{(())}\mathbf{a}$ [B3; Def.] $= A'\underline{a}$ [C6] $= A'\mathbf{(a'a')A'}$ [TR] $= (a'a')A'\underline{A'}$ [Lem. 3] $= (a'a')A'$ [C6] $= \mathbf{a}$**.** □

**C1**. *Dem.* $\mathbf{aa}$ [C3] $= \mathbf{((}aa\mathbf{))}$ [B2; B3] $= ((\mathbf{a'a})(aa))$ [C3,2x] $= ((a'\mathbf{(}a'\mathbf{)})(a\mathbf{(}a'\mathbf{)}))$ [C6] $= \underline{(a')}$ [C3] $= \mathbf{a}$. □

**B4**. *Dem.* $\underline{\mathbf{a'}}\mathbf{b}$ [C6,2x] $= ((a')b)((a')b')(b'a)(b'a')$ [C3,2x] $= (ab)\underline{(ab')}(b'a)(b'a')$ [TR; C1] $= (ab)\underline{(b'a)(b'a')}$ [C6] $= \mathbf{(ab)b}$. □

### A.5. α=β ⇔ α↔β ⇔ ((α′β)(β′α)) Because '↔' Is an Equivalence/Congruence Relation.

Here I employ '⇔' in place of '=' because, for the sake of argument, I am temporarily setting aside the fact that '=' is an equivalence relation.

*Symmetric*: $[\alpha=\beta]\Leftrightarrow[\beta=\alpha] \Leftrightarrow [\alpha\leftrightarrow\beta]\leftrightarrow[\beta\leftrightarrow\alpha]$.
*LR*: *Cal.* $\underline{(((\alpha'\beta)(\beta'\alpha)))}((\beta'\alpha)(\alpha'\beta))$ [C3] $\Leftrightarrow (\alpha'\beta)(\beta'\alpha)((\beta'\alpha)(\alpha'\beta))$ [B4,2x] $\Leftrightarrow$ ().
*RL*: Trivial, and also evaluates to (). □

*Transitive*: Let $\chi$ stand for $(\alpha'\beta)(\beta'\alpha)(\beta'\delta)(\delta'\beta)$. "$\alpha=\beta$ and $\beta=\delta$ implies $\alpha=\delta$" translates as $(\alpha'\beta)(\beta'\alpha)(\beta'\delta)(\delta'\beta)((\alpha'\delta)(\delta'\alpha)) \Leftrightarrow \chi((\alpha'\delta)(\delta'\alpha))$ [C5] $\Leftrightarrow ((\chi\alpha'\delta)(\chi\delta'\alpha))$.

*Cal.* $\chi\alpha'\delta \Leftrightarrow (\underline{\alpha'}\beta)(\beta'\alpha)(\beta'\underline{\delta})(\delta'\beta)\alpha'\delta$ [B4,2x] $\Leftrightarrow (\beta)(\beta'\alpha)\underline{(\beta')}(\delta'\beta)\alpha'\delta$ [C3] $\Leftrightarrow \underline{\beta'}(\beta'\alpha)\underline{\beta}(\delta'\beta)\alpha'\delta$ [TR; B3] $\Leftrightarrow$ (). $\chi\delta'\alpha \Leftrightarrow$ () follows, *mutatis mutandis*. □

*Reflexive*: *Cal.* $[\alpha=\beta] \Leftrightarrow (\underline{(\alpha'\alpha)(\alpha'\alpha)})$ [J1,2x] $\Leftrightarrow$ (). □

By virtue of satisfying conditions $C_1$ and $C_2$ below, '=' is also a *congruence relation* (cf. 3.3.12 in the text; also see Stoll 1963: 259-61). I now demonstrate this fact, $\forall a,b,c\in B$, again translating '=' as '↔':

**$C_1$**. $a=b \rightarrow ac=bc$. *Cal.* $(a'b)(b'a)(((a\underline{c})bc)((b\underline{c})ac))$ [B4,2x] $\Leftrightarrow (a'b)(b'a)(((a)b\underline{c})((b)a\underline{c}))$ [C5] $\Leftrightarrow (a'b)(b'a)((a'b)(b'a))\mathbf{c}$ [TR] $\Leftrightarrow \underline{((a'b)(b'a))(a'b)(b'a)}c$ [B3] $\Leftrightarrow$ (). □

**$C_2$**. $a=b \rightarrow a'=b'$. *Cal.* $(a'b)(b'a)(((\underline{a'})b')((\underline{b'})a'))$ [C3,2x] $\Leftrightarrow (a'b)(b'a)((ab')(ba'))$ [TR] $\Leftrightarrow ((a'b)(b'a))(a'b)(b'a)$ [B3] $\Leftrightarrow$ (). □

**$C_3$**. $a'=b' \rightarrow a=b$. *Cal.* $((\underline{a'})b')((\underline{b'})a')((a'b)(b'a))$ [C3,2x] $\Leftrightarrow (ab')(ba')((ab')(ba'))$ [TR] $\Leftrightarrow ((a'b)(b'a))(a'b)(b'a)$ [B3] $\Leftrightarrow$ (). □

*Remark.* Stoll does not mention $C_3$, which enables replacing $C_2$ with $a'=b' \leftrightarrow a=b$, a corollary of which being that either of B3 or J1 may do duty for the other. Note how the calculations reveal that $C_2$ and $C_3$ reduce to the same thing, one differing only slightly from $C_1$.

### A.6. More re B4.

*Proof.* A quick TVA proof. Given $(ba)a = (b)a$, let:

- $b=\bot$**:** Simply erase $b$. The lhs becomes $(a)a$ [B3] = (), and the rhs becomes $()a$ [C2] = ().
- $b=()$: The lhs becomes $(()a)a$ [C2] $= (())a$, and the rhs, $(())a$.

Invoke T7 twice to complete the proof. □

Equivalents of B4 include Johnson's (1892: 342) Law of Exclusion, the third consequence proved in his system, Th. 7 in Byrne (1946), (48) in Rosser (1953: 113), T8-6.j4 in Carnap (1958), T12 in Stoll (1963: 257), and exercise 3.12.a in Hohn (1966). The Implication axiom (§A.12) in Gries and Schneider (1994) is $a \to b = (a \vee b) \leftrightarrow b$. B4 is an important property of residuated lattices, in which $ab \Leftrightarrow a \cup b$, $(a'b') \Leftrightarrow a \cap b$, and $ab'$ is read as the *residual* of $a$ and $b$; see Dilworth (1938) and references therein.

(31) in Suppes (1957: 204) is B4, given that $a{\sim}b$ (in the sense of Suppes) $\Leftrightarrow$ $(a'b)$. B4 is a trivial corollary of $(b \to a) \leftrightarrow [(b \vee a) \leftrightarrow a]$, *4.72 in *PM*, T73 in Kalish et al (1980: §II.11), and (38) in Cori and Lascar (2000: 32). B4 can be obtained from $[(a \to b) \wedge (c \to b)] \leftrightarrow [(a \vee c) \to b]$ (*4.77 in *PM*, (18) in Stoll (1974: 85), and (57) in Cori and Lascar (2000: 33)), via $b/c$ and noting that $[(a \to b) \wedge (b \to b)] \leftrightarrow (a \to b)$.

One half of B4, viewed as a biconditional, is B4′, $(a \to b) \to [(a \vee b) \to b]$. B4′ is (22) in Grassmann (1966: Be-13), *2.621 in *PM* (the other half is *2.67), Zeman (1973: 2.20), and Leblanc and Wisdom (1976: 99, Example 21). Zeman shows that B4′ is merely a substitution instance of a tautology equivalent to *modus ponens*. The converse of B4′ is likewise a substitution instance of his 2.12. Zeman derives B4′ from that part of the implicational calculus intuitionists accept. B4′ is also an axiom in a system Hilbert set out in 1922 (system 1.3) and in four other CTV axiom systems set out in Epstein (1995: 408-9). $c/a$ turns Reichenbach's (1947: 39) (8h)[2], $((b \vee c) \to a) \to (b \to a)$, into the converse of B4′. Even though B4′ is a substitution instance of Mendelson's (1997: 35) axiom A3, his proof of the converse (Th. 1.11.g) requires 43 lines and the Deduction Theorem! B4′ can even be viewed as an analogue to the special case $\phi = \Delta$ of the "left" version of Bostock's (1997: §2.5) structural rule THIN.

A tautology related to B4′ is B4″, $(b \to a) \to [(b \vee c) \to (a \vee c)]$, axiom *1.6 of *PM* (Prior 6.11). Hence B4″ is included in the *PM* axiom system as modified by Lukasiewicz and Bernays (Table 6-2; Prior 6.11 net of (4)) and commonly used since (e.g., Carnap 1958: 86, P1-P4; Kneebone 1963: 43; Mendelson 1997: 45, system L1; Halmos and Givant 1998: 22, T1-T4). To obtain B4′, substitute $a$ for $c$ in B4″ and note that this axiom system trivially implies $(a \vee a) \leftrightarrow a$, C1 in the pa. Yet another tautology that is B4′ in disguise is B4‴, $(a \to c) \to [(b \to c) \to ((a \vee b) \to c)]$, one of the axioms in Kleene (1952) and (6) in §A.13. Substitute $b$ for $c$ in B4‴, note that $((b \to b) \to$ vanishes, and B4 results.

As best as I can determine, however, only one demonstration in the sources I cite invoke any of B4, B4′, or the converse of B4′: Zeman's demonstration of his 2.21. I conclude that extant expositions of the CTV are unnecessarily complicated.

B4 has the following intuitionistic interpretation. Substituting $b$ for $c$ in (6) in §A.13 yields $[(a \to b) \wedge (b \to b)] \to [(a \vee b) \to b]$. Now $b \to b$ evaluates to $\mathsf{T}$, and $(a \to b) \wedge \mathsf{T}$ evaluates to $a \to b$. Hence $(a \to b) \to [(a \vee b) \to b]$, one half of B4, is an intuitionist tautology. Meanwhile, the other half of B4, $[(a \vee b) \to b] \to (a \to b)$, cannot be demonstrated using axioms (1) to (8a) in §A.13.

### A.7. B2, B3, C1-C5, and J1 Are Standard Identities in Logic and Boolean Algebra.

Lipschutz (1964) is an elementary text on set theory and sentential logic published a few years before *LoF*. Table A-1 shows the correspondence between Lipschutz's (p. 195) "Laws of the Algebra of Propositions" and BA. 8a and 8b excepted, each law is actually a dual pair; 8a is self-dual. Corresponding to each law is a law for the algebra of sets (p. 104). The table also shows which of

2. *Cal*. $((b \vee c) \to a) \to (b \to a) \Leftrightarrow ((bc)\underline{a})b'a$ [B4] $= \underline{(}(bc)\underline{)}b'a$ [C3; OI] $= b'bac$ [B3] $= ()ac$ [C2] $= ()$. □

these laws Huntington (1904) deemed postulates (P) and consequences (C). Only two basic BA notions are missing from Table A-1: A1, a trivial consequence of 5 and 6, *and B4*.

**Table A-1: Correspondences between Lipschutz and Ba.**

| | | *Algebra of Propositions* | BA | *Huntington* (1904) |
|---|---|---|---|---|
| 1 | | Idempotent | C1 | C |
| 2 | | Associative | B1 | C |
| 3 | | Commutative | B1 | P |
| 4 | | Distributive | C5 | P |
| 5 | | Identity | A2 | C |
| 6 | | " | C2 | C |
| | | " | B2 | P |
| 7b | | Complement | J1 | P |
| 8a | | " | C3 | C |
| 8b | | " | A2 | C |
| 9 | | De Morgan's | Transcriptional triviality | C |

## A.8. Proof of Theorem 4.1.4 on Duality.

The proof is by *induction on formula length*, a standard technique well-explained in Bostock (1997: §2.8). The proof below follows Bostock closely in all respects but notation.

**Definition:** Given a formula α, its *length* $l(\alpha)$ = number of variables in α less 1, plus the number of left parentheses and primes in α. If $l(\beta) < l(\alpha)$, then β is *shorter* than α.

**Notation:** Given the pa formula $\alpha = \alpha\langle x_1, \ldots, x_n \rangle$, its contradual is $\overline{\alpha} =_{\mathbf{df}} \alpha\langle x'_1, \ldots, x'_n \rangle$.

**Lemma**: $\alpha^D = (\overline{\alpha})$.

*Proof*. The hypothesis of strong induction is:

For all formulae β such that $l(\beta) < l(\alpha)$, $\beta^D = (\overline{\beta})$.

There are three cases to consider.

α *is atomic*: $\alpha^D = \alpha$ [C3] = **((α))** [Def. of overbar] = $(\overline{\alpha})$.

α *is enclosed*: α = (β).

| | |
|---|---|
| $\alpha^D = (\beta)^D$ | Substitute (β) for α [R1] |
| $= (\beta^D)$ | Def. of duality |
| $= ((\overline{\beta}))$ | Inductive hypothesis |
| $= (\overline{(\beta)})$ | Def. of overbar |
| $= (\overline{\alpha})$ | Substitute α for (β) [R1]. |

α *is a concatenate*: α = βχ for some formula χ.

| | |
|---|---|
| $\alpha^D = [\beta\chi]^D$ | Substitute βχ for α [R1] |
| $= ((\beta^D)(\chi^D))$ | Def. of duality |
| $= (((\overline{\beta}))((\overline{\chi})))$ | Inductive hypothesis |
| $= (\overline{\beta}\overline{\chi})$ | C3, 2x |
| $= (\overline{\beta\chi})$ | Def. of overbar |
| $= (\overline{\alpha})$ | Substitute α for βχ [R1]. |

□

**Theorem 4.1.4**: $\alpha=\varphi \leftrightarrow \alpha^D=\varphi^D$.

*Proof.* If $\alpha=\varphi$, then $\overline{\alpha}=\overline{\varphi}$ [R2], so that $(\overline{\alpha})=(\overline{\varphi})$. Hence $\alpha^D=\varphi^D$ by the lemma above. □

The lemma / 4.1.4 is Bostock's (1997: §2.10) First / Third Duality Theorem and Quine's (1982: §12) second / 5th law of duality.

### A.9. The *LoF* metatheorems.

**3.1.9. (T10). C5** extends to any finite number *n* of divisions (2.1.7) of the subspace of depth 1.

*Proof* (*LoF*, pp. 38-39). T10 with $n$=0 is simply C2. T10 with $n$=1 yields $((a))r$ [C3] = $ar$ [C3] = $((ar))$. The following demonstration verifies the case $n$=3.

*Dem.* $r(a_1'\ a_2'\ a_3')$ [C3,2x] = $r(a_1'\,\mathbf{((}a_2'\,\mathbf{((}a_3'\mathbf{))))})$ [C5] = $((ra_1)(r(a_2'((a_3')))))$ [C5] =
$((ra_1)\underline{((}(ra_2)(r\underline{(}a_3'\underline{)))}))$ [C3,2x] = $((a_1r)(a_2r)(a_3r))$.

The case $n$=3 generalizes to any finite $n$, if both instances of '2x' in the preceding demonstration are replaced by '$(n$-1)x'. □

**4.4.2 (T14).** Let $d_\alpha^*$>2 for some formula α. Then α can be transformed, by taking steps, into an equivalent formula β such that $d_\beta^*$=2.

*Proof.* (This proof is new except for its repeated use of C7, which I took from *LoF*.) α can be seen as an *ordered tree*, one or more of whose *branches* terminate at some maximal depth $d_\alpha^*$. Let $d_\alpha^*$>2, and let β, χ, and ϕ be subformulae appearing at depths $d_\alpha^*$, $d_\alpha^*-1$, and $d_\alpha^*-2$, respectively, of any longest branch of α. Let γ denote all of α not accounted for by β, χ, and ϕ, so that α=(((β)χ)ϕ)γ. By C7, (((β)χ)ϕ)γ=(βϕ)(χ′ϕ)γ. The maximum depth of (βϕ)(χ′ϕ)γ is 1 less than that of (((β)χ)ϕ)γ. This depth-reducing procedure based on C7 can be repeated, each time suitably redefining β, χ, ϕ, and γ. When one branch of α is exhausted, switch to the longest remaining branch. Continue until no branch of α has depth>2. □

*Remark*: Each application of C7 to the *terminus* of any branch of α reduces that branch's depth by 1. This fact enables another, perhaps simpler, algorithm. Beginning at the terminus of any branch, apply C7 repeatedly until a *node* is encountered, then switch to the terminus of any other branch. Continue until no further applications of C7 are possible, at which time no branch will have depth>2. For more on ordered trees, see Smullyan (1968: §§I.0-1). Ordered trees are models of bounded semilattices.

**4.4.3. (T15).** Let the pa formula α⟨*v*⟩ contain more than two instances of the variable *v*. Then α can be transformed, by taking steps, into an equivalent formula β⟨*v*⟩, such that β⟨*v*⟩ contains at most two instances of *v*.

*Proof* (Adapted from *LoF*). By C3 and T14, there exist a subformula $f\rangle v\langle$, and the sequences of subformulae, $a_i$, $p_i$, and $x_j$, such that

| | | |
|---|---|---|
| α⟨$v$⟩ | $=((va_i)p_i)\ldots f(vx_i)\ldots$ | |
| | $=\underline{(}((v')(a_i'))\underline{p_i)}\ldots f(vx_i)\ldots$ | Apply C3 twice for each value of $i$. |
| | $=\overline{(}v'\mathbf{p_i})(a_i'\mathbf{p_i})\ldots f(vx_i)\ldots$ | Apply C5 and C3 once for each value of $i$. |
| | $=(v'p_i)\ldots(a_i'p_i)\ldots f(vx_i)\ldots$ | By OI, the disjuncts $(v'p_i)$ can grouped together, to the |

left of the $(a_i'p_i)$.

$= (v'p_i)...g(vx_j)...$ Let $g = (a_i'p_i)...f$.

$= (((\underline{v}'p_i)...))(((\underline{v}x_j)...))g$ C3, 2x; OI $g$.

$= ((p_i'...)v')((x_j'...)v)g$ T10, 2x. □

**4.4.4. (T16).** Let the variable $v$ appear in at least one of the formulae α and β. Let $v \in B$ be a possible value of $v$, and $\alpha\langle v\rangle$ be $\alpha\langle v\rangle$ with $v$ set to $v$. Then if $\forall \upsilon[\alpha\langle\upsilon\rangle = \beta\langle\upsilon\rangle]$, then α=β.

*Proof* (Adapted from *LoF*, pp. 47-49). Let $v$ *vary* between ⊥ and (). Now consider the following two mutually exclusive and exhaustive cases:

1. Variation in $v$ either alters or does not alter *both* $\alpha\langle v\rangle$ and $\beta\langle v\rangle$. If $\alpha\langle v\rangle=\beta\langle v\rangle$ after a change in $v$, then $\alpha\langle v\rangle=\beta\langle v\rangle$ must have been the case before the change. Hence α=β in this case.
2. Variation in $v$ causes the value of one of $\alpha\langle v\rangle$ or $\beta\langle v\rangle$ to change, but *not both*. If both were to change, there would exist a $v$ such that $\alpha\langle v\rangle\neq\beta\langle v\rangle$, contrary to hypothesis. Hence if varying the value of $v$ causes one of $\alpha\langle v\rangle$ or $\beta\langle v\rangle$ to change, the other must change as well, so that the reasoning of case 1 applies.

Hence $\alpha\langle v\rangle$ and $\beta\langle v\rangle$ are equivalent in all cases of $v$, from which α=β follows. □

*Remark*. If only one of $\alpha\langle v\rangle$ and $\beta\langle v\rangle$ is the case, then variation in $v$ cannot affect the value of the formula in which $v$ appears.

**4.4.5. (T17).** The pa is complete.

*Proof.* (adapted from *LoF*, pp. 50-52). Let $\alpha_k$, $\beta_k$ be formulae each containing some variable $v$. Let $k$-1 be the number of distinct variables other than $v$ appearing in either $\alpha_k$ or $\beta_k$. The proof proceeds by strong induction on $k$, with the inductive hypothesis being $\forall k<n$ $[\alpha_k=\beta_k]$ for some integer $n>0$. T15 assures us that there exist formulae $\alpha_n^*$ and $\beta_n^*$, with $v$ appearing at most twice in each, such that $\alpha_n^* = \alpha_n$ and $\beta_n^* = \beta_n$. Moreover, T14 assures us that the depths of $\alpha_n^*$ and $\beta_n^*$ need not exceed 2. Since $\alpha_n^*$ and $\beta_n^*$ are the normal forms of $\alpha_n$ and $\beta_n$, they can be written as:

(E1) $\alpha_n^* = (v'a_1)(va_2)a_3$ (E2) $\beta_n^* = (v'b_1)(vb_2)b_3$,

where $a_i$ and $b_i$, $i$=1,2,3, are suitable sub-formulae whose depth do not exceed 1.

Now evaluate $\alpha_{n-1}^*$ and $\beta_{n-1}^*$ at each of $v$=() and $v$=⊥:

$v$=(): $\alpha_{n-1}^* = ((())a_1)(()a_2)a_3 = (a_1)a_3$ ; $\beta_{n-1}^* = ((())b_1)(()b_2)b_3 = (b_1)b_3$.

$v$=⊥: $\alpha_{n-1}^* = ((\perp)a_1)(\perp a_2)a_3 = (()a_1)(a_2)a_3 = (a_2)a_3$ ; $\beta_{n-1}^* = ((\perp)b_1)(\perp b_2)b_3 = (b_2)b_3$.

The inductive hypothesis asserts that $\alpha_{n-1}^*=\beta_{n-1}^*$ can be proved. Hence:

(E3) $\alpha_{n-1}^* = \beta_{n-1}^* \to (a_1)a_3 = (b_1)b_3$, and

(E4) $\alpha_{n-1}^* = \beta_{n-1}^* \to (a_2)a_3 = (b_2)b_3$,

are provable. I now demonstate that $\alpha_n^*=\beta_n^*$. (N.B. C8 is $((A'R')(B'R)) = (AR')(BR)$ ).

$(v'a_1)(va_2)a_3$ E1, transcription of $\alpha_n^*$.

$((v'(a_1))(v(a_2)))a_3$ C8, $a_1$/A, $a_2$/B, $v$/R

$((v'(a_1)\mathbf{a_3})(v(a_2)\mathbf{a_3}))$ C5

$((v'(b_1)\underline{b_3})(v(b_2)\underline{b_3}))$ Substitute E3, E4

$$((v'(b_1))(v(b_2)))\mathbf{b_3} \qquad \text{C5}$$
$$(v'b_1)(vb_2)b_3 \qquad \text{C8}, b_1/\text{A}, b_2/\text{B}, v/\text{R}$$
$$\beta_n^* \qquad \text{E2.}$$

It remains to be shown is that there exists a value of $n$ for which the inductive hypothesis, $\alpha_k=\beta_k$ $\forall k<n$, holds. Suppose that $n$ to be 1, in which case only $k$=0 need be considered. Now if $k$=0, then $\alpha_0$ and $\beta_0$ are PA formulae. $\alpha_0=\beta_0$ can be proved in the pa if A1 and A2 are pa consequences. This is indeed the case: A1 is C1 with ()/A, and A2 is C3 with ⊥/A. Also, ⊥⊥ [C1, ⊥/A] = ⊥; (⊥) [A2] = ((())) [C3, ()/A] = (); ⊥()= ()⊥ [C2, ⊥/A] =(). □

**4.4.5. Alternate Proof** (adapted from Kneebone 1963: 48).

Completeness means that *all tautologies are demonstrable from the* pa *initials*. By T14, any formula $\alpha$ has an NF representation $\beta=(a_i^*...)_j....$ Keep in mind that for some $j$, $i$ may equal 1. Every step in the derivation of $\beta$ from $\alpha$ is justified by invoking one of C7, C3, or B4, which are all initials or consequences derivable from the initials. Hence if $\beta$ is derivable from the initials, then so is $\alpha$. Also if $\beta$ is a tautology, $\alpha$ is as well, because $\beta=\alpha$ by T14. To demonstrate that $\beta$ is a tautology, it suffices to consider two cases:

*Case 1.* $\forall j\,(a_i^*...)_j=\perp$**.** If every disjunct contains some variable in both primed and unprimed form, then each disjunct (and hence $\beta$ as well) simplifies to ⊥ by J1**.**
*Case 2.* $\exists j\,(a_i^*...)_j=()$. This is demonstrable if there exists a variable $x$ such that one disjunct is simply ($x$) and another is (($x$)). Then $\beta$=() by B3.

B3, B4, J1, C3, and C7 are either initials, or can be derived from the initials. Hence if $\alpha$ is a tautology, the initials suffice to verify that fact. □

**4.4.7.** (**T18**). The initials making up certain bases are independent.

Each proof below points out an aspect of each initial not shared by the other initials making up the given basis. This aspect suffices to prove that each initial cannot be derived from the others. Hence the initials are independent of each other. This approach is that of *LoF*.

J1-2: *Proof* (adapted from *LoF*). In J2, $r$ changes from one instance at depth 0 to two instances at depth 2. In J1, nothing changes depth. J2 does not create or eliminate any variable. J1 creates/eliminates $a$. □

B1-B4: *Proof*. Only B1 has no boundaries. Only B3 creates/erases all instances of a variable. Only B2 creates/erases boundaries. Only B4 creates/erases some (but not all) instances of a variable. □

To prove independent the axioms for an abelian group, omit each of the axioms in turn and obtain the model on the right:

- Commutativity: group
- Associativity: commutative loop.
- Identity: commutative inverse semigroup
- Inverse: commutative monoid.

{C6, B1}: *Proof*. C6 creates/eliminates a variable and five boundaries. B1 alters no variable instances and has no boundaries. □

### A.10. Two Important Metatheorems on the Expressive Power of BA.

**5.4.1. Theorem**. BA ⊢ CTV and CTV ⊢ BA. BA and the CTV have the same expressive power.

*Proof*. The proof is in two parts. The first derives in BA the rule *modus ponens* (hereinafter “MP”) and the CTV basis Prior 1.4c , PC1–PC3 below and hereinafter PC1–3. Hence BA ⊢ CTV. I then derive B1, B3, and B4 (simple form) in the CTV, and note that R1 and R2 are CTV metatheorems —see the references in §3.1. Hence CTV ⊢ BA.

BA ⊢ CTV. The basis PC1–3 and the rule modus ponens are all easy BA consequences:

PC1: *Cal*. $\varphi\to\xi\to\varphi \;\Leftrightarrow\; (\varphi)(\xi)\varphi$ [TR] $= (\xi)(\varphi)\varphi$ [B3] $= (\xi)()$ [C2] $= ()$. ☐

PC2: *Cal*. $[\varphi\to\xi\to\nu]\to[(\varphi\to\xi)\to(\varphi\to\nu)] \;\Leftrightarrow\; ((\varphi)(\xi)\nu)(\underline{(\varphi)}\xi)(\varphi)\nu$ [TR; B4] $= ((\xi)(\varphi)\nu)(\xi)(\varphi)\nu$ [B3] $= ()$. ☐

PC3: *Cal*. $[\sim\varphi\to\sim\xi]\to\xi\to\varphi \;\Leftrightarrow\; (\underline{((\varphi))}(\xi))(\xi)\varphi$ [C1; TR] $= ((\xi)\varphi)(\xi)\varphi$[B3] $= ()$. ☐

MP: *Cal*. $(\alpha)(\alpha'\beta)\beta$ [TR] $= (\alpha'\beta)\alpha'\beta$ [B3] $= ()$. ☐

CTV ⊢ BA. The proof relies on the CUT rule (Table 5-3), which is derivable in the CTV. PC1–3 + MP ⊢ *PM* holds simply because PC1–3 is a CTV basis and MP is *1.11. To show that *PM* ⊢ BA, translate BA concatenation as alternation and ‘=’ as the biconditional. Then note that B3 is *2.08 in *PM*, and that the two halves of B4, viewed as a biconditional, are *2.621 and *2.67. Recall that the only purpose of B1 is to prove that alternation commutes and associates, facts which the *PM* axioms *1.4 and *1.5 assure. Now invoke CUT, letting $\Gamma$= PC1–3+MP, $\varphi$=*PM*, $\Theta$= BA, and $\Delta,\Phi = \varnothing$. Then the following clause is valid: ([PC1–3+MP ⊢ *PM*] ∧ [*PM* ⊢ BA]) → [PC1–3+MP ⊢ BA]. ☐

*Remark*. PC1–3 neither are, nor claim to be, “obvious” and “elementary.” The calculations for PC1 and PC2 reveal that they are B4 in disguise. The axioms PC1 and PC2 are popular because they facilitate the proof of the Deduction Theorem (Machover 1996: 7.7.3), stated in Table 5-3 above. PC3 and its variants (see §6, fn. 9) govern negation. J1 fills that role in *LoF*; simpler axioms of this nature are B3, C2, or C3. Note that every calculation in the first half of the proof invokes B3 in the final step.[3]

*Theorem*. The pa and Ba are isomorphic.

*Proof.*

The proof is in two parts. First, I show that there exists a variant of Ba whose postulates are all pa consequences. Second, I show that B1–B4 are Ba identities.

The Ba variant employed in the first part of the proof is that of Huntington (1933), who defines Ba as a ⟨+,-,1⟩ algebra of type ⟨2,1,0⟩. This variant readily translates the pa as follows: concatenation ⇔ infix '+', ‘(*a*)’ ⇔ '-*a*', and ()⇔ 1. Huntington's Ba postulates are:

- '+' commutes (TR) and associates (Ass.);
- An identity that translates as C6.

Thus his Ba postulates are all pa consequences.

Turning to the second part of the proof, see the description of Ba in Eves (1997: 8.2), an accessible re-exposition of Huntington’s (1904) landmark paper on Boolean axiomatics. Then Eves's:

- *B1* (Th. 5) implies that concatenation commutes (associates);
- *B2* implies the primal and dual interpretations of B2;

3. PC1–3 commands pride of place in Church (1956: 119) and Bostock (1997: 387). PC1, PC2, and the converse of PC3 are three of the six axioms in Frege’s *Begriffschrifft* (Prior 1.1). In 1930, Lukasiewicz derived Frege’s three other axioms from PC1-3.

- *B4* implies the primal and dual interpretations of B3;
- *B3* implies the primal and dual interpretations of C5.

I now need to demonstrate B4 as a Ba consequence:

*Dem*. **(ab)a** [C3,2x] = $((a')(b'))a$ [C5] = $((\underline{a'\mathbf{a}})(b'\mathbf{a}))$ [B3] = $((\underline{()})(b'a))$ [B2] = $\underline{((b'a))}$ [C3] = **b′a**. □

This demonstration of B4 invoked C3. To complete the proof of the theorem, I need to derive C3 from B2, B3, and C5, as follows:

**C3.** *Dem*. **(a′)** [B2] = **(())**$(a')$ [B3] = $((\mathbf{a'})\mathbf{a'})(a')$ [C5] = $((\underline{(\mathbf{a'})a'})((\mathbf{a'})a))$ [B3] = $((\underline{()})((a')a))$ [B2,2x] = $(((a')a)(\underline{()}))$ [B3] = $(((a')a)(\mathbf{a'a}))$ [C5] = $(\underline{(a')})\underline{(a')}))\mathbf{a}$ [B3] = $(())a$ [B2] = **a**. □

Hence B1−B4 are either postulates or consequences of Huntington's (1904) formalisation of Ba. □

*Remark*. This is not to say that the pa is just a notation for **2**. B2 means that the blank page is part of pa syntax and is a lattice bound. $(a)a$ [C1] = $\underline{(a)a}\mathbf{a}$ [B3] = $()a$ [C2] = () means that complementation can have an empty scope, in which case it denotes a primitive value and the lattice bound other than (()). I know of no hint of either fact in any extant treatment of Ba.

## Systems Going Beyond *LoF*.

### A.11. Theorem 6.1.1: 1i,e preserve tautologies.

*Proof.* Let $\gamma$ be a subformula of the formula $\alpha$. It follows from T14 that there exists a formula $\beta=\alpha$, also containing $\gamma$, such that the depth of $\gamma$ in $\beta$ does not exceed 2. Alternatively, if $\gamma$ is to be inserted in $\alpha$, the result is equivalent to inserting $\gamma$ in some $\beta$ whose depth also does not exceed 2. Hence only three cases need be considered: $\gamma$ has depth 0, 1, or 2. As the EG map very naturally into the dual reading of the pa, the objective is to reduce $\beta$ to $\perp$.

Erase $\gamma$ at depth 0. *Cal.* $a\gamma b' \to ab' \Leftrightarrow (a\gamma b'(ab'))$ [TR] = $((ab')ab\gamma')$ [J1] = $\perp$.
Insert $\gamma$ at depth 1. *Cal.* $(ab')\to(a\gamma b') \Leftrightarrow ((ab')\underline{((a\gamma b'))})$ [C3; B1] = $((ab')ab'\gamma)$ [J1] = $\perp$.
Erase $\gamma$ at depth 2. *Cal.* $(a(b\gamma))\to(ab') \Leftrightarrow ((a(b\gamma))((ab')))$ [C3] = $((\underline{a}(b\gamma))ab')$ [B4] = $(\underline{((b\gamma))}ab')$ [C3] = $(b\gamma ab')$ [B1,2x] = $(b'b\gamma a)$ [J1] = $\perp$ .

The cases where depth = 0 or 2 justify *Erase Even*. The case depth=1 justifies *Insert Odd.*
I now show that the three remaining possibilities do not reduce to tautologies:

Insert $\gamma$ at depth 0. *Dem.* $ab'\to a\gamma b' \Leftrightarrow (ab'(a\gamma b'))$ [B1; B4,2x] = $(ab'\gamma')$.
Erase $\gamma$ at depth 1. *Dem.* $(a\gamma b')\to(ab') \Leftrightarrow ((a\gamma b')\underline{((ab'))})$ [C3] = $((\underline{a\gamma b'})ab')$ [B4,2x] = $(ab'\gamma')$.
Insert $\gamma$ at depth 2. *Dem.* $(ab')\to(a(b\gamma)) \Leftrightarrow ((ab')\underline{((a(b\gamma)))})$ [C3] = $((\underline{a}b')a(b\gamma))$ [B4] = $(\underline{(b')}a(b\gamma))$ [C3] = $(ba(b\gamma))$ [B4; TR] = $(ab\gamma')$. □

*Remark*. The six demonstrations making up 6.1.1 require only C3 (**3i,e**), B4 (**2i,e)**, B1 (tacit in the EG), and J1. I infer that J1 in effect plays the same role in BA that **1i and 1e** play in EG.

### A.12. The Axioms of Gries and Schneider Are pa Identities.

*Definition of* $\leftrightarrow$: $p\leftrightarrow q \Leftrightarrow (p'q')(pq)$.

$\leftrightarrow$ *Associates*: $((p\leftrightarrow q)\leftrightarrow r) = (p\leftrightarrow(q\leftrightarrow r))$.

I will demonstrate this identity in a novel way, by showing that both sides have the same normal form. I begin by finding the normal form of the lhs using C7: $((A'B)C) = (AC)(B'C)$.

*Dem*. $(((p'q')(pq))r')((p'q')(pq)r)$ [C7] = $(p'q'r')(pqr')((p'q')(pq)r)$ [C7] = $(p'q'r')(pqr')(p(\underline{p}q)r)(q(p\underline{q})r)$ [B4,2x] = $(p'q'r')(pqr')(pq'r)(qp'r)$.

Notice that the pa translation of the rhs can be obtained by substituting $q$ for $p$, $r$ for $q$, and $p$ for $r$ into the pa translation of the lhs. Hence the normal form of the rhs is $(q'r'p')(qrp')(qr'p)(rq'p)$ [OI] = $(p'q'r')(qp'r)(pqr')(pq'r)$ [OI] = $(p'q'r')(pqr')(pq'r)(qp'r)$. Hence the normal forms of both sides are identical. □

$\leftrightarrow$ *Commutes*: $p\leftrightarrow q = q\leftrightarrow p$.
*Dem*. $(p'q')(pq)$[TR,2x] = $(q'p')(qp)$. □

*Identity of* $\leftrightarrow$: **true** $= q\leftrightarrow q$.
*Dem*. $(q'q')(qq)$ [C1,2x] = $(q')q'$ [B3] = $()$. □

*Definition of* **false**: **false** = ~**true** $\Leftrightarrow$ $(())$.
Trivial if $()$ $\Leftrightarrow$ **true**. Follows trivially from A2 if $(())$ $\Leftrightarrow$ **true**.

*Definition* of $p|q \Leftrightarrow p\vee q \Leftrightarrow pq$.

| *Commutes*, *Associates*: $p|q = q|p$, $(p|q)|r = p|(q|r)$.
Both follow from B1.

| *Idempotent*: $p|p = p$.
This is just C1.

| *distributes over* $\leftrightarrow$: $p|(q\leftrightarrow r) = p|q \leftrightarrow p|r$.
*Dem.* Transform rhs into lhs: $((pq)(pr))(pqpr)$ [C5] = $((q)(r))p(pqr)$ [B4; TR] = $p((q)(r))(qr)$. □

*Golden rule*: $(p\ \&\ q) \leftrightarrow p = q\leftrightarrow(p|q)$.

*Dem*. Simplify lhs: $(((p'q'))p')((p'q')p)$ [C3] = $(p'q'p')((p'q')p)$ [TR; C1] = $(p'q')((p'q')p)$ [B4] = $(p'q')p'$ [B4] = $(q')p'$[C3] $= qp'$.
Simplify rhs: $(q'(pq))(qpq)$ [TR; C1] = $(q'(pq))(pq)$ [B4] = $(q')(pq)$ [C3] = $q(pq)$ [B4] = $qp'$. □

*Axioms governing not, ~*:

*Implication*: $\sim p|q = (p|q)\leftrightarrow q$.
This is simply the second half of the preceding.

*Consequence*: $p|\sim q = \sim q|p$.
Follows trivially by TR.

*Excluded Middle*: $p|\sim p$.
This is just B3.

*Distributivity of not*, ~: $(\sim(p\leftrightarrow q) \leftrightarrow \sim p) = q$.

*Dem*. $((((p'q')(pq)))\underline{(p')})(((p'q')(pq))p')$ [C3] = $((p'q')(\underline{p}q)p)(((\underline{p'}q')(pq))p')$ [B4, 2x] = $((p'\underline{q'})q'p)((\underline{(q')}(pq))p')$ [B4; C3] = $(\underline{(p')}q'p)((q(p\underline{q}))p')$ [C3; B4] = $(pq'\underline{p})((q\underline{p'})p')$ [TR; C1; B4] = $(q'p)(q'p')$ [TR; C6] = $q$. □

### A.13. Kleene's CTV axioms.

The following axioms for the CTV are from Kleene (1952: 82). I now derive Kleene's axioms as pa consequences.

**1a**. Same as PC1 in §A.10.

**1b**. *Cal*. $(a\rightarrow b)(a\rightarrow(b\rightarrow c))\rightarrow(a\rightarrow c) \Leftrightarrow (a'b)(a'b'c)a'c$ [TR; B4] = $(a'b)(b')a'c$ [TR] = $(a'b)a'(b')c$ [B4] = $(b)a'(b')c$ [TR] = $a'(b')b'c$ [B3] = $a'()c$ [C2,2x] = $()$. □

**2.** (*Modus ponens*). *Cal*. $(a\wedge(a\rightarrow b))\rightarrow b \Leftrightarrow ((a'(a'b)))b$ [TR] = $(((a'b)a'))b$ [B4,2x] = $(((a'b)a'b)b)b$ [B3] = $((())b)b$ [B2] = $(b)b$ [B3] = $()$. □

*Remark*. This is MP in §A.10.

**3**. *Cal*. $a\rightarrow(b\rightarrow(a\wedge b)) \Leftrightarrow a'b'(a'b')$ [B3] = $()$. □

**4a**. *Cal*. $(a\wedge b)\rightarrow a \Leftrightarrow ((a'b'))a$ [B4,2x] = $((a'ab')a)a$ [B3]= $((()b')a)a$ [C2] = $((())a)a$ [B2] = $(a)a$ [B3] = $()$. □

*Remark*. The same reasoning holds for $(a\wedge b)\rightarrow b$. This is Kleene's 4b.

**5b**. *Cal*. $b\rightarrow(a\vee b) \Leftrightarrow b'ab$ [TR; B3] = $()a$ [C2] = $()$. □

*Remark*. The same reasoning holds for $a\rightarrow(a\vee b)$, except that TR is not needed. This is Kleene's 5a.

**6**. *Cal*. $(a\rightarrow c)\rightarrow((b\rightarrow c)\rightarrow((a\vee b)\rightarrow c)) \Leftrightarrow (a'c)(b'c)(ab)c$ [B4,2x] = $(a')(b')(ab)c$ [C3,2x] = $ab(ab)c$ [B3] = $()c$ [C2] = $()$. □

**7**. *Cal*. $(a\rightarrow b)\rightarrow((a\rightarrow\neg b)\rightarrow\neg a) \Leftrightarrow (a'b)(a'b')a'$ [B4,2x] = $b'(b')a'$ [B3] = $()a'$ [C2] = $()$. □

**8**. *Cal*. $(\neg\neg a)\rightarrow a \Leftrightarrow ((a'))a$ [B4,2x] = $((a'a)a)a$ [B3] = $((())a)a$ [B2] = $(a)a$ [B3] = $()$. □

*Remark*. B3 and the LR half of C4 are acceptable alternatives.

(6) requires two instances of C3. All other steps in the above calculations are justified by B2-B4, order irrelevance, and C2.

(8) assures that the CTV basis (1a) – (8) treats negation in the manner classical logic requires. The resulting logic is classical because $a\vee{\sim}a$, the law of excluded middle, is demonstrable from these axioms. Boundary logic is a classical logic simply because $(a\vee{\sim}a)\leftrightarrow$T, interprets B3. (1a) – (7) also hold in *intuitionistic logic*, where $\wedge$,$\vee$, and $\rightarrow$ are all taken as primitive, and no subset of $\{\wedge,\vee,\rightarrow,{\sim}\}$ is EA – whence the large number of axioms. Intuitionistic logic replaces (8) with (8a) F$\rightarrow a$, which results in an axiom set from which none of B3, C3, and C4 is demonstrable. Devising a boundary syntax and proof theory for intuitionist logic would be an interesting exercise.

### A.14. Abelian Groups.

Let GA1 be $ab.c=ac.b$; GA2, $()a=a$; and GA3, $(a)a = ()$. Deduce that group product commutes and associates from GA1, as follows:

TR: *Dem*. An instance of G1 is $(a)ab.c{=}(a)ac.b$ [GA3,2x] $\Rightarrow$ $()b.c{=}()c.b$ [GA2,2x] $\Rightarrow$ $bc{=}cb$. □
Ass.: *Dem*. $ab.c$ [TR] = $ba.c$ [GA1] = $bc.a$ [TR] = $a.bc$. □

McCune's (1993) single axiom for an abelian group, recast in the boundary notation of this book, and assuming that juxtaposition associates from the left, is *G*0: $abc(ac){=}b$. *G*0 reveals the fundamental calculation rule for abelian groups: if the subformulae $\alpha$ and $(\alpha)$ appear in the same depth, erase them both. G0 highlights a crucial difference between abelian groups and the pa, illustrated by the following example. In abelian group theory, the expression $abc(ac)$ simplifies as $abc(ac)$ [3.4.1] = $abc(c)(a)$ [OI] = $b(a)a(c)c$ [GA3,2x] = $b()()$ [GA2,2x] = $b$. If the same expression is taken as a pa formula, it simplifies as $abc(ac)$ [B4] = $abc(a)$ = [GA1] = $bc(a)a$ [B3] = $bc()$ [C2] = $()$. In nonabelian group theory, $abc(ac)$ [3.4.1] = $abc(c)(a)$ [GA3] = $ab()(a)$ [GA2] = $ab(a)$.

McCune's (1993) computer-generated proof that G0 suffices to axiomatize commutative groups requires 101 steps to prove TR, 28 further steps to prove GA2 and GA3, and 108 yet further steps to prove Associativity. The ponderousness of computer exercises involving G0 resembles that of McCune's single axioms for Ba, DN1 and Sh1, discussed in §6.2.

I do not derive GA1-GA3 from G0 here, but conclude with the following derivation of Associativity given G0, TR, GA2, and GA3.

*Dem*. $a.bc$ [G0] = $a.ab.c(ac).c$ [TR,2x] = $a.ab.(ca)c.c$ [TR] = $a.ab.c.(ca)c$ [TR] = $(ca)ca.ab.c$ [GA3] = $().ab.c$ [GA2] = $ab.c$. □

I conclude that McCune's computer generated proof is unnecessarily involved.

### A.15. How Rosser Grounded Ba in Point Set Topology.

Unless otherwise specified, all axioms, definitions, and theorems in this section are from Rosser (1969). Let $X$ be a set of abstract objects called *points* (a primitive notion), with typical elements $x$ and $y$ (Def. 2.1). A *basis* is a set $B$ whose members are nonempty subsets of X, called *basis sets*, satisfying the axioms RO1 and RO2 below (Axioms 2.1, 2.2).[4] Let $B_n$, $n\in\mathbf{N}$, denote arbitrary basis sets of $X$, and let $x$ range over the elements of $X$.

RO1: The basis *covers* $X$, i.e., every point in $X$ is contained in at least one basis set.
$\forall x\exists B_0[x\in(B_0\subseteq X)]$.

RO2: If $x$ is an element of the basis sets $B_1$ and $B_2$, then there exists a basis set $B_3$ whose elements include $x$ and that is a subset of the intersection of $B_1$ and $B_2$.
$\forall x\exists B_3[(x\in B_1 \wedge x\in B_2) \rightarrow (x\in B_3 \wedge (B_3 \subseteq B_1\cap B_2))]$.

A set $X$ for which RO1 and RO2 hold is a *topological space*,[5] whose *open sets* are those sets that can be formed as the union of two or more basis sets of X (Def. 2.2). The open sets of $X$ include $X$ itself (Th. 2.3) and the empty set (Th. 2.4). Let the *closure* of $P$, denoted Cl($P$), be the set of all $x$ such that if $x$ is a member of a basis set, that basis set must also include a member of $P$ (Def. 2.3).

4. Let $x$, $y$, and $w$ range over $B$, a set of sets. Then Metamath theorem **isbasis2g** expresses the predictate "$B$ is a basis" as $\forall x,y\forall z\in(x\cap y)\exists w[z\in w \wedge w\subseteq(x\cap y)]$. Remarkably, the only axiom of standard (ZFC) set theory required to prove **isbasis2g** is Extensionality.
5. Kolmogorov and Fomin (1975: 81), Th. 2. The corresponding Metamath theorem is **istopg**, http://us.metamath.org/mpegif/istopg.html. Let $y$ and $z$ range over some basis $B$ of $X$. Then **istopg** defines the predicate "$X$ is a topology" as $\forall y[y\subseteq X \rightarrow (\bigcup y)\in X] \wedge \forall y,z[(y\cap z)\in X]$. Again, the only ZFC axiom required to prove **istopg** is Extensionality.

Let Co($P$) denote the *complement* of $P$ relative to $X$ (Def. 2.4). Letting Cc($P$) $=_{df}$ Co(Cl($P$)), $P$ is *regular* iff $P$= Cc(Cc($P$)). Regular sets are also open sets (Def. 2.8).

Let $P$ and $Q$ be any two regular sets of $X$. Let (()) $\Leftrightarrow$ empty set, () $\Leftrightarrow$ $X$, ($P$) $\Leftrightarrow$ Cc($P$), and $PQ$ $\Leftrightarrow$ Cc(Cc($P \cup Q$)) (Def. 2.9). Then Th. 2.18 shows that the resulting algebra is a model of BA: A1 is an instance of (2.18), A2 is (2.11), OI is (2.20) and (2.21), B3 is (2.16), and C5 is (2.22).

I conjecture that Rosser's approach can be connected to that of this book, and invite someone versed in point set topology to pursue this.

### A.16. *LoF* Sheds No Light on the Incompleteness of Peano/Robinson Arithmetic.

A succint definition of *Robinson arithmetic* is:

- The universe of discourse is the natural numbers (naturals);
- There exists a function, called *successor*, whose domain is the naturals and whose range is every natural but 0;
- Addition and multiplication are binary operations over the naturals, defined in the usual recursive fashion.

Augmenting Robinson arithmetic by the axiom schema of induction yields Peano arithmetic.

Spencer-Brown claimed that the "imaginary" truth values of *LoF*'s chpt. 11 reduce the scope and importance of the classic limitative theorems. (*LoF* mentions the theorems of Gödel—Peano arithmetic is incompletable—and of Church: first order logic with uninterpreted predicates is undecidable.) In the 1950s and 60s, those who believed that those theorems required much or all of the formal machinery (e.g, first order logic, Peano arithmetic, Gödel numbering, recursive function theory, number theory) employed at that time to state and prove those theorems could be forgiven. Subsequent work by Smullyan (1991: chpts 1,2; 1994: chpts. 4,9; 2001) and Boolos (1998) has shown how to dispense with most of this machinery.

Smullyan's (1994: chpt. 4) proofs require little more than an elementary formal language capable of the self-reference required for diagonalization, and some way of coding strings as natural numbers. Smullyan has also rightly drawn attention to Tarski's theorem (a finitely axiomatized formal system strong enough for Robinson arithmetic cannot define its own truth predicate), because its philosophical implications rival those of Gödel's theorem, yet it is much easier to prove. Smullyan (1994) reviews how first order Peano arithmetic is but one of many incompletable formal systems for which Tarski's theorem holds. Smullyan also shows that the self-referential paradoxes that *LoF* (p. ix) tried to explain away cannot be easily dismissed.

Boolos (1998: 383-88) states and proves the following metatheorem:

**Theorem.** "There is no algorithm $M$ whose output contains all true sentences of arithmetic and no false ones."

*Proof sketch*. The requisite Gödel sentence builds on the following variant of Berry's paradox: "the least number not definable by a formula, having $n$ symbols, of the language of [first order] arithmetic."

Let [$n$] abbreviate $n$ (a natural number) successive applications of the successor function, starting from 0. Boolos then defines several related predicates, starting with $Cxz$, which comes out true iff an arithmetic formula containing $z$ symbols "names" (see below) the number $x$. The construction of $C$ is only sketched. This sketch assumes that every formula in the language of arithmetic has a Gödel number; this is the only mention of Gödel numbering in the entire proof. The "language of arithmetic” is first order logic, augmented by the successor function, and the defined binary relation '<' and trinary relation '×'. Boolos tacitly assumes that the language of arithmetic includes enough formal machinery for these symbols to have their usual meanings. The other defined predicates are:

$Bxy \leftrightarrow \exists z(z<y \wedge Cxz)$,
$Axy \leftrightarrow \neg Bxy \wedge \forall a(a<x \rightarrow Bay)$,
$Fx \leftrightarrow \exists y((y=[10]\times[k]) \wedge Axy)$. $k$ = the number of symbols appearing in $Axy$.

$Fx$ "names" $n$ if the output of $M$ includes the sentence $\forall x(Fx \leftrightarrow (x=[n]))$. Thus Berry's paradox is formalized. The balance of the proof, requiring but 12 lines of text, shows that while this sentence is true in a semantic sense, no algorithm $M$ can prove it true. Thus arithmetic truth necessarily outruns proof, the essence of Gödel’s famous result. □

Boolos’s proof requires less than two pages, because he assumes rather than demonstrates that his syntax can be arithmetized. His proof requires but two existential quantifiers, shown above in the definitions of the predicates $Bxy$ and $Fx$, and is:

- Silent about the axioms and theorems of Robinson and Peano arithmetics;
- Innocent of infinity in any form, of any notions from proof theory, and of any facts about the connectives or quantifiers;
- Is intuitionistically valid.

Boolos even claimed that his proof dispenses with diagonalization.

### A.17. Robbins Algebras Are Boolean: The Proof Restated Using Boundary Notation.

A *Robbins algebra* is a $\langle B,\cup,'\rangle$ algebra of type $\langle 2,1\rangle$, with $\cup$ assumed to commute and associate, and with the *Robbins equation* R (explained below) as the sole additional postulate. My notation for Robbins algebra replaces $\cup$ with concatenation, and the overbar with enclosure by parentheses or, in the case of a single letter, a postfix prime. (Hence $a' =: (a)$.)

In 1933, Robbins conjectured that the algebras now named after him were in fact Boolean. This conjecture was finally proved by McCune (1997: §2) using computer methods. Dahn (1998) reworked McCune’s proof to bring it closer to the “tree of lemmas” style of mainstream mathematics. I rework Dahn’s proof below, using the boundary notation of the preceding paragraph, in order to show how boundary notation can simplify nontrivial contemporary mathematics.

Dahn works hard the Boolean function $\delta(a,b) =: \overline{\bar{a} \cup b}$, whose sentential logic equivalent is $a \nrightarrow b$. (N.B: $\{\nrightarrow,\neg\}$ is expressively adequate; see Table 4-.) Dahn’s variant of R can be elegantly reexpressed using δ as $\delta(ab,\delta(a,b)) = b$. In pa notation, $\delta(a,b)$ is written as $(a'b)$, so that Dahn’s variant of R is $((ab)(a'b)) = b$. The proof below invokes the alphabetic variant of R, $((ab)(ab')) = a$, at least once. For a derivation of the Huntington equation, $(a'b')(a'b) = a$, in Robbins algebra, see Mann (2003: §5).

**Theorem**. Robbins algebras are Boolean algebras.

*Proof.* We begin with some notation, U0-U2, then the preliminary facts **δ1-δ3**:

**U0**: $a_0 =: (a'a)$;

**U1**: $a_n =: a_{n-1}a = a_0 \overbrace{a...a}^{n}$;

$a_n =: aa_{n-1} = \overbrace{a...a}^{n} a_0$;

**δ1**: $a'=b'$ [algebraic substitution] $\rightarrow (a'c)=(b'c)$;

**δ2**: $((aa)a_0)$ [U0] $= ((aa)(a'a))$ [R] $= a$;

**δ3**: $(a_2' a)$ [U1; δ2] $= ((aaa_0)((aa)a_0))$ [R] $= a_0$.

**U2**: $a_m a_n = a_k a_l$ if $m+n = k+l$.

*Remark*. U0-U2 establish that numerical subscripts work like the superscripts denoting powers in numerical algebra. This notation of Dahn's is required because we cannot assume that Robbins algebras are idempotent until they are proved equivalent to Boolean algebra. U0 and U1 implicitly define $(a'a)$ as the Boolean 0. Note that **δ1** can be written $a'=b' \rightarrow \delta(a,c)= \delta(b,c)$. Also recall that if equality is a congruence relation, then $a'=b'$ and $ac=bc$ both follow from $a=b$. **δ2** and **δ3** are the familiar Boolean identities $a''= a$, $aa=a$, and $a_0 b=b$ in other guises.

Let L*n* denote "Lemma *n*." Dahn then proves the 11 lemmas L1-L7 below. Each lemma equates the first and last formula in its proof. I have replaced Dahn's α with χ because α is easily confused with *a*. "TR" signals the reordering of concatenated subformulae.

**L1**. $(a_3' a_0)$ [δ3] = $(a_3'(a_2' a))$ [U1] = $((a_2 a)(a_2' a))$ [R] = $a$.

**Def**. $\chi =: (a_3' a_1 (aa))$.

**L2a**. $a$ [R] = $((a_3' a_0 a)((a_3' a_0)a))$ [U1] = $((a_3' a_1)((a_3' a_0)a))$ [L1] = $((a_3' a_1)(aa))$.

**L2b**. $(aa)$ [R] = $((a_3' a_1(aa))((a_3' a_1)(aa)))$ [L2a] = $((a_3' a_1(aa))a)$ [Def. χ] = $(\chi a)$.

**L2c**. [L2b] $(\chi a) = (aa)$ [δ1] $\rightarrow ((\chi a)a_0) = ((aa)a_0)$ [δ2] = $a$.

**L2d**. $a_3'$ [R] = $(\chi((a_1(aa))a_3'))$ [TR] = $(\chi(a_3'((aa)a_1)))$ [U1] = $(\chi((aaa_1)((aa)a_1)))$ [R] = $(\chi a_1)$.

**L2**. [L2d] $a_3' = (\chi a_1)$ [δ1] $\rightarrow (a_3' a) = ((\chi a_1)a)$ [U1] = $((\chi a a_0)a)$ [L2c] = $((\chi a a_0)((\chi a)a_0))$ [R] = $a_0$.

**L3**. $((a_1 a_3)a)$ [R] = $((a_1 a_3)((a_3 a)(a_3' a)))$ [L2] = $((a_1 a_3)((a_3 a)a_0))$ [U2; U1] = $((a_4 a_0)(a_4' a_0))$ [R] $=a_0$.

**L4**. $((a_1 a_2)a)$ [U2] = $((a_3 a_0)a)$ [L1] = $((a_3 a_0)(a_3' a_0))$ [R] = $a_0$.

**L5**. $((a_1 a_3)a_0)$ [L4] = $((a_1 a_3)((a_1 a_2)a))$ [U1] = $((a_1 a_2 a)((a_1 a_2)a))$ [R] = $a$.

**Def**. $\beta =: ((a_1 a_3)a a_3')$.

**L6**. $(\beta a)$ [L1] = $(\beta(a_3' a_0))$ [L3] = $(\beta(a_3'((a_1 a_3)a)))$ [TR] = $(\beta(((a_1 a_3)a)a_3'))$ [R] = $a_3'$.

**L7**. $(\beta a)$ [L5] = $(\beta((a_1 a_3)a_0))$ [L2] = $(\beta((a_1 a_3)(a_3' a)))$ [TR] = $(\beta((a_1 a_3)(a a_3')))$ [R] = $(a_1 a_3)$.

According to Mann (2003: 7), Winker (1992) proved that if any of the following can be derived in Robbins algebra:
- B2 or C3;
- There exists $x \in B$ such that $xx=x$ (an instance of C1) holds;
- There exist $x,y \in B$ such that one of $xy=y$ or $(xy)=y'$ holds,

then all Robbins algebras are Boolean. L6 and L7 tells us that $x=a_1$ and $y=a_3$ satisfy $(xy)=y'$. Since L6 and L7 hold for any Robbins algebra, all Robbins algebras are Boolean. □

*Remark*. Surprisingly, Dahn's proof relied on the most complex of Winker's five possible sufficient conditions for Robbins algebras to be Boolean. Mann (2003: §§5,6) restates, using a uniform conventional notation, the proofs of Winker (1992) and McCune (1998) by showing that Robbins algebra satisfies $\exists x \exists y[xy=y]$.

## *A Précis of Mathematical Logic.*

*First Order Logic* (**FOL**) unites two calculi: that of *truth values* (CTV) and that of *quantified variables* (CQV). For a masterly précis of FOL and its extension to axiomatic set theory, see Fraenkel, Bar-Hillel, and Levy (1973: §V.2). For more leisurely expositions, consult the references cited under "Quantifier Logic" in the Bibliographic Postscript. For a treatment more sophisticated than the one below, see http://plato.stanford.edu/entries/logic-classical/ .

A *string* consists of a single symbol, or of concatenated symbols. *Symbol* is undefined except in semiotic theory. $B$={T,F} is the set of possible *truth* or *primitive values*.

***CTV***. A *statement* (*sentence*, *proposition*) is a string that can be assigned a truth value. Statements include *formulae*, i.e., strings that satisfy a formation rule. S*ubformula* and *atomic formula* are defined in 2.1.3. A *statement letter* (*sentential variable*) stands for any member of some set of statements. (Truth) *functor, connective*, and *operator* are defined in 3.1.2; these relate to mappings from $B^n$ onto $B$, $n \in N$. The constants T and F are 0-*ary* functors by assumption. Common functors include the prefix ~ "not", and the infix connectives ∧ "and", ∨ "or", → "if", ↔ "iff", and| "NAND".

An *atomic valuation* (i) assigns an element of $B$ to every atomic formula, and (ii) completely describes the mapping $f_k$: $B^n \to B$ for every $n$-ary operator $k$. A statement consisting of $m \in N$ statement variables (*arguments*) and constants, linked by connectives, is a *truth function* from $B^m$ onto $B$. The *truth value* of a statement is the image of its truth function under some atomic valuation. If the image is T [F], the statement is *valid* [invalid]. If a statement is valid under [all/some] atomic valuations, the statement is [*tautologous*/*satisfiable*]. If a statement is not valid under any atomic valuation, its denial is a tautology. If all atomic valuations satisfying α also satisfy β, and vice versa, α and β are *tautologically equivalent*, denoted α↔β.

***Proof*** (adapted from Halmos and Givant 1998: §13). An *axiom* is a statement asserted true without proof. The *rule of detachment* is: If α and α→β are both tautologies, then β is also a tautology. Let $i,j,k,n \in N$. A *formal proof* (*demonstration* in *LoF*-speak, or simply *proof*) is an ordered sequence of $n$ statements with typical statement $\alpha_k$ $1 \le k \le n < \infty$. A *step* transforms $\alpha_k$ into $\alpha_{k+1}$. For each $\alpha_k$, $i,j<k$, $\alpha_k$ is either (i) a substitution instance of some definition, axiom, or already proved consequence, resulting from the application, often tacit, of R1 and R2 (§3.1), or (ii) the result of applying detachment to some pair $\alpha_i$ and $\alpha_j$. (i) alone suffices for equational logics (e.g., boundary logic), for which detachment is just a special case of *propositional consequence* (§5.3). If there exists a proof whose last statement is $\alpha_n$, $\alpha_n$ is *provable* and a *theorem*.

CTV is *sound* (all provable formulae are valid), *complete* (all valid formulae are provable), and *decidable* (there exist algorithms, e.g. TVA, for determining whether any finite formula is valid). The *primitive basis* (DeLong 1971: 91) of a formal system consists of its primitive symbols, defined constants, rules of formula formation, axioms and rules of inference, and a truth definition. A *model* is an interpretation of a formal system under which its formulae all come out true; see Suppes (1957: §4.2) or Mendelson (1997: §2.2).

***CQV***. A *variable* stands for any member of a nonempty collection (*domain* [*of interpretation*]) of physical or abstract *individuals*, each having a *name*. A *term*, denoted by a lower case letter, is a name, variable, or a function thereof. A term letter may be uniformly replaced by another term letter. *Predicate letters* are upper case. Associated with each predicate and function letter is a nonnegative integer $n$, called its *arity*. By convention, a predicate [function] letter with an arity of 0 is a statement variable [*constant*]. An *atomic formula* (aka predicate) consists of a predicate letter followed by $n$ terms. An atomic formula such that $n$=1 [$n$>1] is *monadic* [*polyadic*]. ∀ [∃] is the *universal* [*existential*] *quantifier*. A quantifier operates on the variable that immediately follows it; ∀$x$

[$\exists x$] translates as "for all [for some] $x$." A CQV formula consists of quantified variables, and atomic formulae linked by truth functors.

Let $\alpha,\beta$ be arbitrary CQV formulae. $\exists x\alpha =_{\mathbf{df}} \sim\forall x[\sim\alpha]$, so that there is in fact only one quantifier. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be vectors of variables, of unspecified dimension. Let $Q_i$ be one of $\forall$ or $\exists$, and let $Q(\boldsymbol{x})$ be a string of the form $Q_1x_1Q_2x_2$... known as a *prefix*. Let a *matrix* $M(\boldsymbol{x},\boldsymbol{y})$ consist of atomic formulae and truth functors but no quantifiers, with each $x_i$ and $y_i$ appearing at least once. $Q_i$ *binds* $x_i$, and $x_i$ is a *bound* variable; the $y_i$ are *free*. An atomic formula has a truth value only if its variables are all bound. If $\boldsymbol{y}$ has dimension 0 ($\geq 0$), then $\alpha$ is *closed* (*open*). CTV formulae are the special case when both $\boldsymbol{x}$ and $\boldsymbol{y}$ have dimension 0. The *prenex* form of $\alpha$ is then $Q(\boldsymbol{x})M(\boldsymbol{x},\boldsymbol{y})$, The *scope* of $Qx_i$ is $M(\boldsymbol{x})$ by default, or overridden by parentheses. If $\boldsymbol{x}$ has dimension 1, and if $\alpha$ is closed and does not lie within the scope of another quantifier, then $\alpha$ is an *elementary quantification*. Writing $\forall y_1\forall y_2$... to the left of an open formula results in its *universal closure*. $\alpha$ is *valid* (is a "law of logic") if it evaluates to T for all nonempty domains.

Let '$\alpha\rangle x\langle$' denote that any instances of $x$ in $\alpha$ are bound. CQV requires three axioms in addition to any basis sufficient for the CTV: $\forall x\alpha\langle x\rangle \rightarrow \alpha\langle a/x\rangle$ (called UI), $\alpha\rangle x\langle \rightarrow \forall x\alpha\rangle x\langle$, and $\forall x[\alpha\rightarrow\beta]\rightarrow (\forall x\alpha\rightarrow\forall x\beta)$ (Bostock 1997: 236). Fitch devised these axioms and Quine (1951) popularized them; they enable dispensing with the rule of generalization. CQV *with identity* includes a primitive dyadic predicate, denoted by infix '=', assumed *reflexive* ($x= x$) and governed by the axiom schema $(x=y)\rightarrow(F\langle x\rangle\leftrightarrow F\langle y//x\rangle)$, where $F$ is any CQV formula.

CQV is provably sound (Hunter 1971: §42), complete (§46), and undecidable. Some necessary conditions for a CQV formula to be undecidable include a domain with infinitely many individuals, a matrix that is not a substitution instance of a monadic formula, quantified variables nesting at least three deep, and a prefix with at least one $\exists$ preceding a $\forall$.[1]

A *first order theory* is FOL augmented with at least one primitive intepreted predicate, and some some additional (*proper*) axioms involving the primitive predicates. Much, perhaps all, of mathematics can be formulated as first order theories involving at least one polyadic predicate and a suitable domain of abstract individuals. The claim that all of mathematics can be formalized by axiomatic set theory (AST) is equivalent to the claim that mathematics is a first order theory having but one binary predicate. AST is undecidable because Robinson arithmetic (§A.16) is undecidable, and the axioms of Robinson arithmetic are AST theorems (see fn. 2 on p. vi). The undecidability of AST in turn implies that AST requires quantified variables nesting *at least* three deep. Tarski and Givant (1987) show that AST does not require formulae with quantified variables nested more than three deep.

**Table of Cross-References between *LoF* and this book.**

| *LoF* | *Here* | | *LoF* | *Here* | | *LoF* | *Here* | | *LoF* | *Here* |
|---|---|---|---|---|---|---|---|---|---|---|
| A1 | Table 2-1 | | C2 | B4 | | T1 | 2.3.2 | | T10 | 3.1.9 |
| A2 | " | | C3 | C2 | | T2 | 2.3.3 | | T13 | 3.1.10 |
| R1 | 3.1.7 | | C4 | C4 | | T3 | 2.3.4 | | T14 | 4.4.2 |
| R2 | 3.1.8 | | C5 | C1 | | T4 | 2.3.5 | | T15 | 4.4.3 |
| J1 | 3.1.6, §A.1 | | C6 | C6 | | T5 | 2.3.6 | | T16 | 4.4.4 |
| J2 | C5 | | C7 | C7 | | T6 | 2.3.7 | | T17 | 4.4.5 |
| C1 | C3 | | C9 | C8 | | T7 | 2.3.9 | | T18 | 4.4.7 |

1. Grädel et al (1997) review many results on the decidability and computational complexity of FOL.

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