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MATHEMATICS AND CIVILIZATION

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KARL MANNHEIM AND OSWALD SPENGLER DEFEND diametrically opposed positions on the possibility of a sociology of mathematics. Mannheim argues that mathematics is exempt from sociological explanation; mathematics is not an ideology, and mathematical truths are not culturally relative. This view has been reinforced by Pythagoreans or Platonists who believe that mathematical truths are eternal objects that exist independently of the flux of historical experience. Most historians, philosophers, and sociologists of science have adopted a Mannheimian view mathematics.

Spengler, on the other hand, holds that mathematics is culturally relative; each culture has its own conception of number. No other student of the social foundations of mathematics, has ventured to defend this extreme claim. Spengler's notion of the "soul" of a civilization cannot provide the basis for an adequate sociological analysis. However, we endorse Spengler's goal of explaining mathematics in terms of the particular social and historical forms in which it is produced.

Spengler's argument is summarized in two statements: (1) *"There is not, and cannot be, number as such."* There are several; number-worlds as there are several cultures; and (2) "There is no mathematic but only mathematics." Spengler's objective in his analysis of "number" is:

> to exemplify the way in which a soul *seeks* to actualize itself in the picture of its outer world - to show, that is, in how far culture in the *"become"* state can express or portray an idea of human existence - I have chosen number, the primary element on which all mathematics rests. I have done so because mathematics, accessible in its full depth only to the very few, holds a quite peculiar position amongst the creations of the mind.[1]

The "peculiar position" of mathematics rests on the fact that it is at once a "science" (like logic, only "fuller", "more comprehensive"), a "true art," and a "metaphysic."

Spengler draws two analogies in sketching the nature and origin of number. As "the symbol of causal necessity," number, like God "contains the ultimate meaning of the world-as-nature." And like myth, number originated in the "naming process" through which humans sought "power over the world." *Nature,* the numerable, is contrasted with *history,* the aggregate of all things that have no relationship to number.

Spengler argues against treating earlier mathematical

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*This article is a Word version of the original article with some minor corrections. Footnote 4 in the original article on p. 298 has been incorporated into the text.

[1]Quotations in this section are from Oswald Spengler, *The Decline of the West* (New York, 1926), pp. 56-70.

events as stages in the development of "mathematics". This is consistent with his thesis on the incommensurability of Cultures and with his cyclical view of historical change. His general schema of Classical and Western styles and stages in "Culture," "number," and "mind" is essentially an analysis of world views. It an attempt to articulate the nature and significance of the insight that numbers and their meanings are components of world views. This is reflected in Spengler's attempt to correlate mathematical and other sociocultural "styles": for example, " Gothic cathedrals and Doric temples are *mathematics in stone."*

Spengler is aware of the problem of the limits of a " "naturalistic" approach to number and pessimistic about a solution:

> There are doubtless certain characters of very wide ranging Validity which are (seemingly at any rate) independent of the Culture and century to which the cognizing individual may belong, but along with these there is a quite particular necessity of form. which underlies all his thought as axiomatic and to which he is subject by virtue of belonging to his own Culture and no other. Here, then, we have two very different kinds of *a priori* thought-content, and the definition of a frontier between them, or even the demonstration that such exists is a problem that lies beyond all possibilities of knowing and will never be solved.

Finally, we want to draw attention to two ideas Spengler discusses that merit consideration as "working hypotheses." The first is his claim that "the greatest mathematical thinkers, the creative artists of the realm of numbers, have been brought to the decisive mathematical discoveries of their several Cultures by a deep religious intuition." This follows from his central thesis that "the number-thought and the world idea of a Culture are related." Thus, number thought is not merely a matter of knowledge and experience, it is a "view of the universe." This reinforces our notion that Spengler sees number as infused with and imbedded in world views.

The second claim Spengler makes is that a "high mathematical endowment" may exist without any "mathematical science"; he cites, for example, the discovery of the boomerang, which can only be attributed, he argues, to people "having a sure feeling for numbers of a class that we should refer to the higher geometry."

Sociologists of mathematics have been bold enough about challenging the Platonic conception of number, but they have hesitated to follow Spengler. His ideas must seem mad to scholars and laypersons, and specialists and non-specialists alike, to whom the truth of number relations appears to be self-evident. And yet, the "necessary truth" of numbers has been challenged by mathematical insiders and outsiders. One of the outsiders is Dostoevsky:

> …twice-two-makes-four is not life, gentlemen. It is the beginning of death.

> Twice-two-makes-four is, in my humble opinion, nothing but a piece of impudence ... a farcical, dressed up fellow who stands across your path with arms akimbo and spits at you. Mind you, I quite agree that twice-two-makes four is a most excellent thing; but if we are to give everything its due, then twice-two-makes-five is sometimes a most charming little thing too.[2]

Dostoevsky's remarks are not merely a matter of literary privilege. Mathematicians and historians of mathematics have also challenged the conventional wisdom on number. Morris Kline, for example, has pointed out that (1) "Ordinary arithmetic does not apply to *all* physical situations"; and (2) "we can only know this through experience with these situations." There is, in brief, a *rationale* for pursuing the Spenglerian program for a sociology of mathematics based on the views of at least some mathematicians, historians of mathematics, and observers of numbers such as Dostoevsky. Our work is an exploration of the potentials and limits of Spengler's sociology of mathematics.

Our objective in this essay is to present briefly and with a minimum of technical detail some of the tentative results of our examination of the comparative history of mathematics from a sociological perspective. In the larger study on which this essay is based we are (1) examining the development of different forms of mathematics at different times and places; (2) identifying the noted mathematicians, the social positions they held and how they were related to one another; and (3) looking at the social conditions within and outside of mathematical communities as they go through phases of progress, stagnation, and decline. The degree of "community" among mathematcians, the level of specialization, the extent of institutionalization and the relative autonomy of the social activity of mathematics. it should be stressed, are variable across time and space.

In general, we find both a long-term "logic of development" in the history of mathematics, and also a number of variations among the types of mathematics produced in different cultures. The latter "horizontal" variations are *prima .facie* evidence for the Spengler thesis. But what about the long-term trends? These too are socially determined, and in two different senses.

First, and in a weaker sense, the "longitudinal" development of mathematics does not occur without interruptions, nor does it unfold in a single cultural context. We are, therefore, interested in these questions: (1) what factors cause mathematicians to move along a certain sequence at some times and not others? (This is tantamount to asking: what is truth in mathematics at any particular point in time?) (2) Why does a sequence stop for two hundred, or a thousand years and then start up again? (3) Why do particular mathematicians at particular times and

[2] F. Dostoevsky, "Notes From the Underground," pp. 107-240 in *The Best Short Stories of Dostoevsky* (New York, n.d.).

places make the major advances and not others. We believe it is possible to identify social factors that affect the variations, interruptions, progressions, and retrogressions characteristic of the longitudinal development of mathematics.

There is a stronger sense in which the longitudinal development of mathematics towards more "advanced" forms is socially determined. The longitudinal development of mathematics reveals the social aspects of mathematical work. Much of mathematics is created in response to stimuli from within the mathematical community, especially as mathematicians go about playing competitive games with one another. The development of higher and higher levels of abstraction, for example, reflects the increasing self-consciousness of mathematicians about their own operations. This in turn reflects higher levels of specialization and institutional autonomy among mathematicians. Our argument suggests that the Spengler thesis is true in a very strong form: “number”, and all that It stands for metaphorically, is a *socially created activity.*

We begin our essay by reviewing variations in the development of mathematics across civilizations. We then briefly discuss the influence of practical interests, competition, and social organization on the development of mathematics. The following discussion is not intended to offer direct and unequivocal sup- for. the Spenglerian program. Our goal is more modest: to identify some of the main social patterns in the history of mathematics that substantiate Spengler's *general* hypothesis about the cultural nature of number.

II Mathematics and Civilization

The world history of mathematics has not unfolded in a unilinear, unidirectional manner. The Greeks, for example, took a step backward from the Babylonian achievements in notation; different types of mathematical systems have developed in different parts of the world; and rival forms of mathematics have sometimes developed *within* societies. Hindu mathematics, especially in the period before the influx of Greek astronomy (ca. 400) placed unique emphasis upon large numbers. Geometry, arithmetic, number theory, and algebra were ignored in favor of the use of numbers in "sociological" schemes. The *Upanishads* (ca. -700 to -500) are full of numerical description: of the 72,000 arteries; the 36,360, or 36,000 syllables; the 33, 303, or 3306 gods; the 5, 6, or 12 basic elements out of which the world is composed. The wisdom of the Buddha is illustrated by the gigantic numbers he can count out (on the order of 8 times 23 series of 10^7), and his magnificence is shown by the huge number of Bodhisattvas and other celestial beings who gather to set the scenes for his various sutras. The Hindu cosmology includes a cyclical view of time that enumerates great blocks of years called *yugas.* There are four *yugas* ranging from 432,000 to 1,728,000 years, all of which together make up one thousandth of a *kalpa* or 4,320,000,000 years. .

This emphasis upon immense, cosmological numbers, gives A distinctively Hindu view of the near-infinite stretches of being that surround the empirical world. It seems almost inevitable that the Hindus should have invented zero or *sunya* (emptiness) in Sanskrit. The concept *sunya,* developed about 100, was the central concept in Madhyamika Buddhist mysticism, and preceded the invention of the mathematical zero about 600. Classical Indian world views are permeated with mathematics, but of a special kind. It is a mathematics for transcending experience but not in the direction of rationalistic abstraction. Instead. numbers are used for purposes of *mystification* or *impressiveness;* they are symbols in a mathematical rhetoric to awe listener into a religious posture. In general, numbers were for *numerological* rather than mathematical purposes. The social roots of this distinctive mathematical system lie in the particularly exalted status of Indian religious specialists. The concrete as opposed to abstract nature of Hindu large numbers may also have been suggested by a social reality: the great variety of ethnic groups making up Indian society, institutionalized in the ramifications of the caste system.

Chinese mathematics, on the other hand, also has a cosmological significance, but on an entirely different scale. Its bias is ideographic. Numbers, and higher mathematical expressions, are written as concrete pictures. The system of hexagrams that make up the *I Ching,* the ancient book of divination, was continuously reinterpreted in successive Chinese cosmologies as the basic form of the changing universe. Chinese arithmetic and algebra were always worked out in positional notation. Different algebraic unknowns, for example, could be represented by counting sticks laid out in different directions from a central point. Chinese algebra, at its height around 1300, could be used to represent fairly complex equations, and included some notion of determinants (i.e., the pattern of coefficients). But it could not be developed in the direction of abstract rules. The ideographs (and the social conditions of their use) helped preserve the concreteness of mathematics.

Why did Chinese mathematics take this form? No doubt for some of the same reasons that account for the maintenance of ideographic writing among the Chinese intellectuals. Both gave a concrete aesthetic emphasis to Chinese culture. The ideographic form had technical limitations that a more abstract form - an alphabet, a more mechanical mathematical symbolism would have overcome. Ideographs are hard to learn; they require a great deal of memorization. But these limitations may in fact have been the reason why Chinese intellectuals preferred to retain them. For a difficult notation is a social advantage to a group attempting to monopolize intellectual positions. This may be contrasted with the algorithmic imperative characteristic of periods of rapid commercial expansion.

Writing and mathematics were highly esoteric skills in the ancient civilizations when they were first developed. Those who possessed these skills were almost exclusively state or religious dignitaries. Hence it should not be surprising that writing and mathematical notation were conservatively retained in forms that were very difficult to read and interpret, except by those could spend a long time in acquiring familiarity with them. Sanskrit, for example, was written without vowels and without Spaces between the words. Egyptian writing was similarly conservative. Chinese writing and mathematics are notable because archaic styles lasted much longer than anywhere else. The development of ideographs and mathematical notation in China was in the direction of greater aggregative complexity and aesthetic elaboration, not of simplification and abstraction. The Chinese literati thus managed to make their tools progressively more difficult to acquire. This is in keeping with the unusually high position that Chinese intellectuals maintained in the state institutionalized through the examination system used to select officials in many dynasties.

Historians of mathematics often comment that the lack of a "good" notation was the reason why mathematics did not progress further at some particular time and place. But this begs the question. Why *wasn't* a more appropriate symbolism invented then? Instead, we should envision a struggle between monopolizing and democratizing forces over access to writing and mathematics. Monopolistic groups were strong in highly centralized administrations, such as ancient Egypt, the Mesopotamian states, and China. Democratizing forces won the upper hand in decentralized situations, and/or under social conditions where there was a great deal of private business activity – as in ancient (especially Ionian) Greece, and periods in ancient and medieval India. The predominance of these forces, of course, does not mean there were no counter-forces. Greek mathematics also had some conservative elements, especially in the Alexandrian period when difficult rhetorical form of exposition limited the development of algebra. The specific character of mathematics in given world cultures is due to the differential incidence of such conditions.

Greek mathematics is distinguished by its emphasis on geometry, generalized puzzles, and formal logical proofs. This is the intellectual lineage of modern Western mathematics. But the history of Greek and European mathematics also shows a divergent type that rose to prominence *following* the establishment of the classical form. During the Alexandrian period, another form of arithmetic was developed that was used neither for practical calculations nor for abstract puzzle-contests. This was a type of numerology that used the real relations among numbers to reveal a mystical cosmology. The system was connected with verbal symbolism through a set of correspondences between numbers and letters of the Hebrew or Greek alphabets. Any word could be transformed into a related number that in turn would reveal mathematical relations to other words.

The social conditions involved in the creation and development of this alternative mathematics are connected with religious movements. Numerology is related to Hebrew Cabbalism, to Christian Gnosticism, and to the Neo-Pythagorean revival of the time (especially by Philo of Alexandria, ca. 50). The most prominent expositor of this new mathematics was Nichomachus (ca.100). Like Philo, he was a Hellenistic Jew (living in Syria) – in short, part of the Jewish-Greek intellectual milieu of the Levant in which the major religious movements of the time were organized.

Finally, it is worth noting that there are variants even in modern European mathematics. There are conflicts between alternative notational systems in the 1500s and 1600s; and a century-long battle between the followers of Newton and those of Leibniz over the calculus. In the nineteenth century, a major dispute arose between Riemann, Dedekind, Cantor, Klein, and Hilbert and critics such as Kronecker and Brouwer. This split has continued and widened in the twentieth century into schools of formalists, intuitionists, and now yet others in conflict over the foundations of mathematics. Without following up the matter here, we suggest that these splits can be explained by social factors both outside and within the mathematical community.

III. The Social Roots of Mathematics

The social activities of everyday life in all the ancient civilizations gave rise to arithmetic and geometry, the two major modes of mathematical work. Each of these modes is associated with specific types of social activity. The development of arithmetic is stimulated by problems in accounting, taxation, stock-piling, and commerce; and by religious, magical, and artistic concerns in astronomy, in the construction of altars and temples, in the design of musical instruments, and in divination. Geometry is the product of problems that arise in measurement, land surveying and construction and engineering in general. Arithmetic and geometrical systems appear, in conjunction with the emergence of literacy, in all the earliest civilizations – China, India, Mesopotamia, Egypt, and Greece. These mathematical systems are, to varying degrees in the different civilizations, products of independent invention and diffusion. .

The *discipline* of mathematics emerged when sets of arithmetic and geometrical problems were assembled for purposes of codification and teaching, and to facilitate mathematical studies. Assembling problems was an important step toward unifying mathematics and stimulating abstraction. An even more Important step was the effort to state general rules for solving all problems of a given type. A further step could be taken once problems were arranged so that they could be treated in more general and abstract terms. Problems that had arisen in practical settings could now be transformed into purely hypothetical puzzles, and problems could be invented without explicit reference to practical issues. The three famous puzzles proposed by Greek geometers of the -5th and -4th centuries are among the earliest examples of such puzzles: to double the volume of a cube (duplication of the cube), to construct a square with the. Same area as a given circle (quadrature of the circle), and to divide a given angle into three equal parts (trisection of the angle). Such problems may have been related to the non-mathematical riddles religious oracles commonly posed for one another. One account of the origin of the problem of duplicating the cube, for example, is that the oracle at Delos, in reply to an appeal from the Athenians concerning the plague of -430, recommended doubling the size of the altar of Apollo. The altar was a cube. The early Hindu literature already refers to problems about the size and shape of altars, and these may have been transmitted to Greece by the Pythagoreans, a secret religio-political society. The problem is also a translation into spatial geometric algebra of the Babylonian cubic equation $x^3 = v$.

The duplication, quadrature, and trisection problems were popular with the Sophists, who made a specialty out of debates of all kinds. A generation or two later, Plato introduced the constraint that the only valid solutions to these problems were those in which only an unmarked straightedge and a compass were used. This meant that special mechanical devices for geometrical forms cou1d not be used in mathematical competitions. The goal was apparently to stiffen and control the competitive process by stressing intellectual means and "purely gentlemanly" norms. This development was related to social factors in the Platonist era. Plato's Academy was organized to help an elite group of intellectuals gain political power; and it represented the opposition of an aristocracy to democratization and commercialization. It is not surprising that this elite group of intellectuals developed an ideology of extreme intellectual purity, glorifying the extreme separation of hand and brain in the slave economy of classical Greece.[3]

The three famous Greek puzzles and other problems became the basis of a mathematical game of challenge-and-response. Various forms of this game are important throughout most of the subsequent history of Western mathematics. Prior to the nineteenth and twentieth centuries, however, the challenge and response competitions were often initiated, endorsed, or rewarded by patrons, scientific academies, and governments. Prizes were sometimes offered for solutions to practical problems. Economic concerns as well as governmental prestige were often mixed in with the struggles for intellectual preeminence.

At about the same time that they initiated mathematical contests, the Greek mathematicians took two further steps that led to new mathematical forms. They stipulated that a formal, logical mode of argument must be used in solving problems; this represented a further development of earlier methods of proof. And by extending this idea they created *systems* of interrelated

[3] These developments are discussed in more detail in S. Restivo, The Social Relations of Physics, Mysticism, and Mathematics, Kluwer Academic Publishers, Dordrecht 1983: 239-252. This note corrects and updates the original note.

proofs. This culminated in the *Elements* of Euclid shortly after -300. In addition to a collection of problems, Euclid presented an explicit body of abstractions in the form of definitions, postulates, and axioms. Euclid, like Aristotle, did not use the term "axiom" but something closer to "common notion." They both self-consciously worked at codifying past human experiences. The process of "systematization-and-abstraction" is one of the two major paths to new mathematical forms. The other major path is an "empirical" one.

The empirical path to new mathematical forms involves applying existing mathematical concepts and methods to new areas of experience. Most of the early Greek geometrical puzzles, for example, concerned flat figures. But the methods of plane geometry could be easily extended to solid geometry, and then to the properties of spheres or of conic sections; the work on conic sections eventually led to work on curves of various shapes. The intermittent periods of creativity in Alexandrian mathematics (especially from -300 to -200 and 150-200) were largely devoted to these extensions. No new level of abstraction (with the exception of trigonometry, considered below) was Reached, but a number of new specialties appeared.

Looking back, now, at the development of the other branch of mathematics, arithmetic, we find some of the same processes noted for geometry. The effort to find general rules for solving numerical problems led gradually to what we call algebra. Here again we see mathematicians developing the practice of posing problems primarily to challenge other mathematicians. For example, there is the famous problem, attributed to Archimedes (-287 to -212): find the number of bulls and cows of various colors in a herd, if the number of white cows is one third plus one quarter of the total number of black cattle; the number of black bulls is one quarter plus one fifth the number of the spotted bulls in excess of the number of brown bulls, etc. Such problems, involving unknown quantities, led over a very long period to the introduction of various kinds of notations and symbolisms. These took quite different directions in ancient and medieval China and India, the Arab world, and later in medieval and Renaissance Europe. The creation of a highly abstract symbolism which could be mechanically manipulated to find solutions did not appear until the late 1500s and 1600s in Europe.

Over this period, and to different degrees in different parts of the world, algebra underwent an empirical extension. Problems were deliberately created to increase the number of unknowns, and to raise them to successively higher powers. Equations of the form $ax + b = c$ gave way to those on the order of $ax^4 + by^3 + cz^2 = g$. The complexity of these, of course, could be extended indefinitely (Vieta in the 1580s, for example, was challenged to solve an equation involving x^{45}); but the extensions also gave rise to efforts to find general rules for solving higher order equations (i.e., empirical extensions tended to promote abstract extensions).

At the same time, arithmetic was developing in other directions. What we would call elementary arithmetic (solving numerical problems in, for example, addition, subtraction, multiplication, and division) continued to stimulate efforts to find general rules for solving particular problems. There was tremendous variation from one system of numerical symbols and calculating rules to another in terms of the ease or difficulty with which they could be applied to solving practical problems. Most of the ancient forms of notation made working with large numbers, fractions, or complex operations like division or the extraction of roots difficult; the exposition of problems was usually rhetorical. That is, problems were express in words. A great deal of mathematical creativity went into the development of notational systems that could be readily manipulated. Among the most important of these innovations were the invention of decimal place notation and the zero sign in India; the standardization of positional methods for writing multiplication and division (in Europe ca. 1600); and the invention of logarithms by the Scotsman Napier in 1614, for use in astronomy, navigation, and commerce.

A different development in arithmetic led to what we now call “number theory.” This focused on the properties of numbers themselves. As early as Eratosthenes (ca. -230), efforts were made to find prime numbers and to produce a general formula for doing so. There were also various propositions about how numbers are composed of other numbers (e.g., the Pythagorean work on “triangular” and “square” numbers, an anticipation of the work that led up to the seventeenth century mathematician Fermat’s famous theorem that every prime number of the form 4n+1 is a sum of two squares). Number theory was particularly popular in the Alexandrian period in an occultist, cabalistic form. In its more standard puzzle-solving form, it has remained popular among mathematicians from the Renaissance through the twentieth century.

One more branch of mathematics, based on a combination of arithmetic and geometry, developed in the Alexandrian period. Measurements of angles and lines, and the calculation of their ratios, led to the creation of trigonometry (especially by Hipparchus, ca. -140 and Menelaus, ca.-lOO). Trigonometry spread to medieval India and the Arab world, and in Renaissance Europe provided the basis for Napier's development of logarithms.

The overall picture so far, then, shows mathematics arising from practical geometry and arithmetic. The development .of abstract mathematical puzzles and the extension of mathematics to new areas leads to the emergence of new fields. Geometry becomes systematic, and is progressively applied to plane and solid figures, to conics, and eventually to trigonometry. Arithmetic gives rise to algebra in successfully more complex forms (based on practical calculating systems), and to number theory.

The creation of new fields continued in modern Europe. They grew out of the processes of abstraction, the extension of results to new empirical areas, and the combination of existing mathematical fields into hybrid fields.

The combination of algebra with a new coordinate representation in geometry by Descartes and Fermat produced analytic geometry. Consideration of the problems of motion and the study of curves gave rise to the calculus in the 1600s. Calculus was then applied to successively more complex functions (empirical extension); and eventually (in the 1800s) it was generalized into an abstract theory concerning such things as the rules for solving equations, and the general properties of all functions (abstract extension). It should be noted that the drive towards creating new fields by abstraction and extension seems to be characteristic of highly competitive periods.

Geometry itself experienced a rapid series of branchings around 1800 and thereafter, the best known being the non-Euclidean geometries. But there was also the creation of descriptive geometry by Monge, projective geometry by Poncelet, higher analytical geometry by Plucker, modern synthetic geometry by Steiner and Von Staudt, and topology by Mobius, Klein, and Poincare. In the late nineteenth and early twentieth centuries systems unifying these different geometries were formulated by Klein, Hilbert, and Cartan.

In algebra, there was a parallel set of developments after 1800. The effort to find a general solution for the quintic and other higher-order equations led to the creation of the theory of groups by Abel, Galois, Cauchy, and others. This theory focused on an abstract pattern among the coefficients of equations, and opened up a new area of inquiry in abstract mathematics. Abstract algebras were created by Boole, Cayley, Sylvester, Hamilton, and Grassman. All of these new tools were applied to other branches of mathematics. Dedekind applied set theory to the calculus, Cantor applied it to the concept of infinity, and others applied it to topology, number theory, and geometry. These developments led to the creation of yet another even more abstract field toward the end of the nineteenth century. This was the field of "foundations," concerned with the nature of mathematical objects themselves and with the rules by which mathematics should be carried out. Foundations research has been the focus of a number of opposing schools, and has led to what are probably the most intense controversies in the history of mathematics.

The basic forms of mathematics, arithmetic and geometry, arise from practical problems in construction, taxation, administration, astronomy, and commerce. Moreover, the stimulus of practical concerns does not simply disappear once mathematics is launched. For example, the basic forms of arithmetic, including the number system, developed over a very long period, during which virtually the sole interest in improvement was to facilitate practical calculations. The same can be said for the invention of logarithms, and much of the development of trigonometry. Other advanced forms of mathematics were also stimulated by efforts to solve practical problems. The development of the calculus was linked to problems in ballistics and navigational astronomy. Descriptive geometry and Fourier's analysis answered problems in the production of new machinery in the industrial revolution. Practical concerns do not tell the whole story of mathematics, but they are one component that continuously shapes its history. This suggests a general principle: an increase in the amount, type, intensity, or scope of practical concerns in a society will stimulate mathematical activity. The relationship between economic concerns and mathematics is especially strong; commercial growth tends to be very stimulating for mathematics. Mathematical innovations will also tend to occur when there is a shift to new productive technologies (and perhaps when there are shifts to new technologies of warfare and transportation, and shifts to more intensive administrative modes of organization). This implies a link between the development of modern European mathematics and the development of capitalism. Since this is one factor among several, it does not imply that mathematics must come to an end in non-capitalist societies. It does, however, suggest that the form and content of mathematics (within the constraints noted by Spengler) as we know it today is a product of specific lines of cultural development.

The roots of mathematics in practical concerns are more apparent in some cases than in others. For example, the history of Chinese mathematics from Yü the Great Engineer's discovery of a magic square on the back of a Lo River tortoise (a myth probably created during the Warring States period around -500) to the highest achievements of the late Sung and early Yuan dynasties (for example, Chu Shih-Chieh's "Precious Mirror of the Four Elements," written in 1313, at the end of the "Golden Age" of Chinese mathematics) is primarily a history of an inductive "mathematics of survival." Chinese mathematics never ventured far from problems of everyday life such as taxation, barter, canal and dike construction, surveying, warfare, and property matters. Chinese mathematicians could not organize an autonomous mathematical community, and consequently failed to establish the level of generational continuity that is a necessary condition for long-term mathematical development. This helps to explain why the Chinese did not develop the more abstract forms of higher mathematics.

Conditions in ancient Greece were more favorable for abstract mathematics. The commercial expansion in Greece in the -600s stimulated mathematical growth. Learned merchants practiced and taught mathematical arts, and master-student relationships across generations fostered mathematical progress. Political and economic changes in Greek civilization led to the development of an increasingly elitist and self-perpetuating intellectual community, culminating in the oligarchic conditions and intellectual elitism of Plato's time.

The achievements of the "thinking Greeks" depended on a division of labor that divorced hand and brain. The "thinkers" had the "leisure" to reflect on and elaborate mathematics. The class structure of the slave-based society that developed in the post-Ionian period conditioned the development of classical mathematics. Arithmetic was left to the slaves who carried out most commercial transactions, and householders for whom simple calculations were a part of everyday life. The elite intellectual class, by contrast, courted geometry which was considered democratic and more readily adapted to the interests of the ruling classes than arithmetic. What we know as "Greek mathematics" is a product of the classical period.

The development of specialties within the division of labor, left unchecked, tends to foster virtuosity. Such specialization tends to increase the specialists' distance from the order and spectra of everyday phenomena and to increase the importance of human-created phenomena, especially symbols. The result is an increase in the level of abstraction and the development of ideologies of purity. This is essentially what occurred in classical Greece. Hand and brain slowly reunited following Plato's death; there is already evidence of an increased interest in linking mathematical and practical concerns in Aristotle. In the Alexandrian period, hand and brain were more or less united, but the ideology of purity retained some vitality. This is notably illustrated by Archimedes, whose work clearly exhibited a unity of hand and brain but whose philosophy echoed Platonist purity.

The decline of Greek commercial culture was accompanied By the decline of Greek mathematical culture. The achievements of Archimedes, which brought Greek mathematics to the threshold of the calculus, mark the high point of Greek mathematics. When mathematics was revived in the European commercial revolution (beginning haltingly as early as the twelfth century A.D.), many aspects of the Greek case were recapitulated. European mathematics moved on in the. direction of the calculus, rooted in problems of motion. It picked up, in other words, essentially where Archimedes had left off, and under the influence of the Archimedean (and more generally, Greek) corpus as it was recovered and translated. By 1676, Newton was writing about mathematical quantities "described by continual motion." The concept of function, central to practically all seventeenth and eighteenth century mathematics, was derived from studies of motion. Newton and Leibniz helped to reduce the basic problems addressed in the development of the calculus - rates of change, tangents, maxima and minima, and summations - to differentiation and anti-differentiation. Infinitesimals nurtured earlier in the debates of theologians and scholastics, entered into the process of production. Abstract intellectual ideas of a Euclidean realm of the straight, the flat, and the uniform gave way to the ideas of an increasingly energetic world of guns and machinery characterized by skews, curves, and accelerations.

The search for algorithms, time-saving rules for solving problems, is evident in the writings of the inventors of the calculus (e.g. in Leibniz's "De geometria recondita et analysi indivisibilium atque infinitorium" of 1686). As the industrial “machine" of capitalist society was fashioned, so was the “machine of the calculus." Descartes' analytic geometry, the other great contribution to the development of pre-modern European mathematics, was also characterized by an algorithmic imperative.
It was, in spite of the conflicts between Cartesians and Newtonians, from the very beginning in constant association with the development of the Newtonian-Leibnizian calculus. The historian of mathematics Boutroux has characterized Descartes' analytic geometry as an industrial process; it transformed mathematical research into “manufacturing.”

The idea that the calculus is linked to the emergence of capitalism (or at least early industrialization) is further suggested by the Japanese case. When the Japanese established a monetary economy and experienced a commercial revolution in the seventeenth century, they also worked out a "native calculus."

IV Puzzles and Proofs

Mathematicians, from the earliest times onward, and especially in the West, have posed puzzles for one another. This practice tends to make mathematics a *competitive game.* Some periods have been dominated by public challenges such as those that the Emperor Frederick's court mathematician posed to Leonardo Fibonacci (ca. 1200), those that Tartaglia and Cardano posed for one another in sixteenth century Italy, or those that gave Vieta such high acclaim at the French court in the 1570s. Such puzzle-contests have been important for several reasons. They often involved pushing mathematics into more abstract realms. Mathematicians would try to invent problems which were unknown in practical life in order to stump their opponents. And the search for general solutions to equations, such as those that Tartaglia found for cubic equations, and Vieta found for the reduction of equations from one form to another, was directly motivated by these contests.

The emphasis on *proofs* which has characterized various periods in the development of mathematics was partly due, in our view, to a heightening of the competitiveness in these contests. Greek mathematicians rationalized the concept and method of proof at a time when mathematics was popular among the elite class of philosophers and there was a lot of competition for power and attention in the intellectual arena. This was the same period during which the wandering Sophists challenged one another to debating contests and in doing so began to develop canons of logic. This is completely analogous to the development in mathematics, in terms of both cause and effect. The analogy turns into a virtual identity when we realize that many of the mathematicians of the time *were* Sophists, and that many of the formal schools that were organized in the classical period (e.g., the Academy) used prowess in mathematics as a grounds for claiming superiority over competing institutions. Stressing proofs was a way of clarifying the rules of the game and escalating the intensity of competition. In general, competitive puzzle-contests are probably responsible for much of the inventiveness characteristic of Western mathematics.

This analysis should not obscure the economic stimulus to the initial development of proofs. Thales, the philosopher-merchant, is credited with carrying the idea of a proof to a more general level than the Babylonians and Egyptians. We can conclude that at least symbolically Thales personifies the need among the Ionians of his era to develop a comprehensive and organized understanding of physical reality and successful computational methods in the context of the increasingly well-organized economy that they were products and fashioners of. Thales' proofs were probably crude extensions of Babylonian or Egyptian "rules" for checking results. In any case, the process of constructing proofs was rationalized over the next three hundred years and eventually led to Euclidean-type proofs.

Concern for proof has varied a great deal in the history of mathematics. The Chinese and Hindu mathematicians ignored proofs almost entirely; indeed, they would often present problems without solutions, or with incorrect solutions. That these practices were the result of a relatively uncompetitive situation in mathematics in these societies is suggested by several facts. The social density of mathematicians in these societies was rather low; we rarely hear of more than a few mathematicians working at the same time, whereas in Greece and Europe the numbers in creative periods are quite high. Most of the Oriental mathematicians were government officials, and thus were insulated from outside competition, while most of the ancient Greek and modern mathematicians were private individuals or teachers in competitive itinerant or formal educational systems.

In the Islamic-Arabic world, there was a flurry of mathematical activity in the period 800-1000 (and later to some extent). There was some concern for proofs (in the works of Tahblt Ibn Qurra, for example), but this was much more limited than in classical Greece. The Greek works they translated stimulated an awareness of and interest in proofs among the Islamic-Arabic mathematicians. The limited emphasis on proofs reflects the fact that their "community" was not as densely populated as the Greek mathematical community, competition was not as intense, and master-student chains and schools were not as well organized.

In modern Europe, the emphasis on proofs has grown steadily. In the 1600s, Fermat presented his theorems without proofs, and in the 1700s, Euler offered proofs that were not very rigorous. The early 1800s saw a shift towards more rigorous standards of proof; earlier solutions were rejected, not because they were incorrect, but because the reasoning behind them was not sufficiently universal and comprehensive. This went along with a massive increase in the number of people engaged in mathematics (which in turn was the result of the expansion of educational systems, especially in Germany and France, and other social changes). Both this shift towards rigor, and the earlier invention of proofs, had important effects upon the nature of mathematics. For both pressed mathematics toward new levels of abstraction: proofs had to invoke more abstract elements than particular numerical examples, and rigorous proofs stimulated the systematic consideration of the nature of mathematics in the nineteenth century.

V Abstraction and Self-consciousness

Let us return to the "main line" development of Western puzzle-solving mathematics. That development has consisted of an increasing awareness that levels of abstraction have been created by the mathematicians themselves. Mathematicians moved beyond naïve realism when they gradually began to use negative numbers instead of dropping negative roots of equations, as Hindu, Arab, and medieval European mathematicians had done). Later they came to recognize that imaginary numbers could be used despite their apparent absurdity. Gauss established. a new basis for modern algebra by creating a representational system for complex numbers. Nineteenth century higher mathematics took off from this point. Mathematicians finally realized that they were not tied to common-sense representations of the world, but that mathematical concepts and systems could be deliberately created. The new, more abstract geometries (projective, non-Euclidean) popularized the point, and stimulated the creation of new algebras and more generalized forms of analysis. The objects with which modern mathematics deals, however, *are real* in the following sense. They are not *things,* as was once naively believed; they are, rather, operations, activities that mathematicians carry out. The imaginary number i is a shorthand for a real activity, the *operation* of extracting a square root from a negative number. This operation. of course, cannot be carried out. But mathcmaticians had long been used to working backwards from solutions-not-yet-found, to the premises, by symbolizing the solution by an arbitrary designation (e.g., *x).* This symbol represented the result of an imaginary operation. The imaginary number *i,* then, could be used as the basis for other mathematical operations, even though the operation of producing it could never actually be performed. The ordinary arithmetic operations, the concepts of a function, the concept of a group - all of these are operations of different degrees of complexity. A natural whole number itself is not a thing but an operation - the operation of counting (and perhaps also other operations whose nature modern mathematicians are untangling).

Modern mathematics has proceeded by taking its operations as its units. These are crystallized into new symbols which can then be manipulated as if they were things. A process of reification has gone on in conjunction with the emergence of the notion that abstractions are self-created by mathematicians. Thus mathematics has built upon itself hierarchically by treating operations as entities upon which other operations can be performed. The Western trend in symbolism, then, is not an "accidental" feature of Western mathematical uniqueness; the symbolism was *created* precisely because the mathematical community was pushing towards this degree of self-consciousness.

Mathematics, like other modern activities, has been affected by specialization on a level unknown in earlier historical periods. As a result, the "causal power" of mathematics itself in the relationships between mathematical and other social activities has steadily increased. Mathematical ideas have increasingly become the generative basis for new mathematical ideas. The work setting and institutional context of mathematical activity has become a social foundation of a higher order than the social foundation of subsistence productive activity. Mathematics continues to be socially rooted *within* the mathematical community; it is especially important to recognize the social nature of the symbols mathematicians create for communication within their own ranks.

The development of Western higher mathematics, then, is a social development. For the objects with which mathematicians deal are *activities of mathematicians.* In building upon the operations already in existence, and making them symbolic entities upon which further operations can be performed, mathematicians are self-consciously building upon previous activities in their intellectual community. Mathematics thus *embodies* its own social history, and uses it as the base upon which its current community activities are constructed.

Western mathematics thus depends upon a particular kind of long-term organization of the intellectual community. This is an organization in which strong links are maintained across generations, and in a highly self-conscious and competitive form. The new attempts to competitively consume the old. We suggest that the important linkages of teachers and pupils typically found among European mathematicians, together with strong external competition among different mathematical "lineages," have been the social basis for this pattern. Once the pattern of competitive self-consciousness was established, subsequent rounds of competition could only escalate the degree of self-reflection and inventiveness among mathematicians. Out of this situation arose the hyper-reflexive concerns of twentieth-century foundations research.

VI Conclusion

> All thought, in its early stages, begins as action. The actions which you [King Arthur] have been wading through have been ideas, clumsy ones of course, but they had to be established as a foundation before we could begin to think in earnest. Merlyn the Magician

We are suggesting that the history of mathematics can be *ex*plained sociologically. This task requires a sociology both of the *external* conditions - economics, religious, political - of the societies within which mathematical activities are situated, and a sociology of the *internal* organization of the mathematical activities. We recognize that the notion of "internal and external factors" is an analytic device. The Spenglerian idea of mathematics as a world view is not, in the end, compatible with a strict adherence to internal-external analysis. Our conclusion is that the mathematics of any particular time embodies its own social history. This process becomes increasingly intense as and to the extent that mathematical activity becomes and remains more clearly differentiated from other social activities and more autonomous. But "autonomy" simply means that mathematicians communicate more intensively with each other than with outsiders. It does not mean that mathematicians are more removed from social social determinants or that they have unimpeded access to "objective reality." Their activities remain at all times coupled to the social activities of insiders and outsiders, and thus unfold in an environment of multiple social and historical determinants. This is the rationale for our defense of a Spenglerian approach to the sociology of mathematics.

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