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Operations Research -- Sample Homework Assignments

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Homework #1

Matrix Algebra Review: The following problems are to be found in Chapter 2 of the text, *Operations Research (3rd edition)* by W. Winston:

(1.) Exercise # 7&8, p. 31
Use the Gauss-Jordan method to determine whether each of the following linear systems has no solution, a unique solution, or an infinite number of solutions. Indicate the solutions (if any exist).

a.)
$$\begin{aligned} x_1 + x_2 \quad\quad &= 2 \\ -x_2 + 2x_3 &= 3 \\ x_2 + x_3 &= 3 \end{aligned}$$

b.)
$$\begin{aligned} x_1 + x_2 + x_3 \quad\quad &= 1 \\ x_2 + 2x_3 + x_4 &= 2 \\ x_4 &= 3 \end{aligned}$$

(2.) Exercise #5, p. 35
Determine if the following set of vectors is independent or linearly dependent:

$$V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right\}$$

(3.) Exercise # 4, p. 40: *Find A^{-1} (if it exists) for the following matrix:*

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

Linear Programming Model Formulation: Formulate a Linear Programming model for each problem below, and solve it using **LINDO** (available both on the Apollo workstations and Macintoshes of ICAEN.) Be sure to state precisely the definitions of your decision variables, and briefly explain the purpose of each type of constraint. State verbally the optimal solution. *(All exercises are from Chapter 3 of the text. For instructions on LINDO, see §4.7 and the appendix of chapter 4 of the text.)*

(4.) Exercise #2, page 73 *(Manufacturing of Heart Valves)*
"U.S. Labs manufactures mechanical heart valves from the heart valves of pigs. Different heart operations require valves of different sizes. U.S. Labs purchases pig valves from three different suppliers. The cost and size mix of the valves purchased from each supplier are given in the table below. Each month, U.S. Labs places one order with each supplier. At least **400*** large, 300 medium, and 300 small valves must be purchased each month. Because of limited availability of pig valves, at most 500 valves per month can be purchased from each supplier. Formulate an LP that can be used to minimize the cost of acquiring the needed valves."

	Cost per valve	Percent Large	Percent Medium	Percent Small

Supplier 1	\$5	40	40	20
Supplier 2	\$4	30	35	35
Supplier 3	\$3	20	20	60

*The original problem statement specifies a requirement of 500 large valves per month, which is infeasible given the specified availability from each supplier. I have modified the requirement so as to make the problem feasible.

(5.) *(Exercise 3, page 112)*
" I now have \$100. The following investments are available during the next three years:

Investment A: Every dollar invested now yields \$.10 a year from now and \$1.30 three years from now.
Investment B: Every dollar invested now yields \$0.20 a year from now and \$1.10 two years from now.
Investment C: Every dollar invested a year from now yields \$1.50 three years from now.

During each year, uninvested cash can be placed in money market funds, which yield 6% interest per year. At most \$50 may be placed in each of investments A, B, and C. Formulate an LP to maximize my cash on hand three years from now."

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Solutions:

1a.)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

That is $(x_1,x_2,x_3)=(1,1,2)$.

1b.)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right]$$

There are infinite solutions, since $(x_1,x_2,x_3,x_4)=(a+2,-1-2a,a,3)$ for all $a\in R$.

(2.) These three vectors are dependent, since

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}.$$

(3.) The inverse does not exists, since the three row vectors are dependent.
Note: (1,2,1)+(1,2,0)=(2,4,0) implies they are dependent.

(4.) The original problem statement specifies a requirement of 500 large valves per month, which is infeasible given the specified availability from each supplier. I have modified the requirement so as to make the problem feasible.

Define: X1=# of valves produced by Supplier 1,
Define: X2=# of valves produced by Supplier 2,
Define: X3=# of valves produced by Supplier 3, respectively.

The formulation and LINDO outputs are as below.
Optimal solution is: Supplier 1 produces 500 valves, Supplier 2 produces 500 valves, and Supplier 3 produces 250 valves.

```
MIN     5 X1 + 4 X2 + 3 X3
SUBJECT TO
    2)    0.4 X1 + 0.3 X2 + 0.2 X3 >=   400
    3)    0.4 X1 + 0.35 X2 + 0.2 X3 >=   300
    4)    0.2 X1 + 0.35 X2 + 0.6 X3 >=   300
    5)     X1 <=   500
    6)     X2 <=   500
    7)     X3 <=   500
 END

: go
     LP OPTIMUM FOUND  AT STEP      7

          OBJECTIVE FUNCTION VALUE

  1)        5250.00000

 VARIABLE         VALUE          REDUCED COST
       X1       500.000000         0.000000
       X2       500.000000         0.000000
       X3       249.999954         0.000000

 ROW         SLACK OR SURPLUS     DUAL PRICES
    2)          0.000000        -15.000000
    3)        125.000008          0.000000
    4)        124.999992          0.000000
    5)          0.000000          1.000000
    6)          0.000000          0.500000
    7)        250.000046          0.000000

 NO. ITERATIONS=       7
```

(5.) Define A1=$ invested in A at the beginning of year 1,
B1=$ invested in B at the beginning of year 1,
C2=$ invested in C at the beginning of year 2,
Ft=$ invested in F at the beginning of year t, t=1,2,3.

The LP formulation and LINDO outputs are shown below.

The optimal solution is:
First year-- invest $13.541664 in A, $50 in B, and 36.458336 in money market funds.
Second year--invest $50 in C.
Third year--invest $55 in money market funds.
Total returns is $50.904.

MAX 1.3 A1 + 1.5 C2 + 1.06 F3 (Total return at the end of year 3)
SUBJECT TO
2) A1 + B1 + F1 = 100 (Available $)
3) 0.1 A1 - C2 + 0.2 B1 + 1.06 F1 - F2 = 0 (The output of first year=the input of second year)
4) - F3 + 1.1 B1 + 1.06 F2 = 0 (The output of second year=the input of third year)
5) A1 <= 50 (Upper limit for each investment)

6) B1 <= 50
7) C2 <= 50
END

```
: go
    LP OPTIMUM FOUND  AT STEP     6

          OBJECTIVE FUNCTION VALUE

  1)        150.904160

 VARIABLE        VALUE          REDUCED COST
       A1        13.541664         0.000000
       C2        50.000000         0.000000
       F3        55.000000         0.000000
       B1        50.000000         0.000000
       F1        36.458336         0.000000
       F2         0.000000         0.230567

 ROW        SLACK OR SURPLUS     DUAL PRICES
    2)         0.000000           1.435417
    3)         0.000000          -1.354167
    4)         0.000000          -1.060000
    5)        36.458336           0.000000
    6)         0.000000           0.001417
    7)         0.000000           0.145833
```

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Homework #2

(1.) Find A^{-1} for the following matrix:

$$A = \begin{bmatrix} 7 & 3 & 2 \\ 7 & -2 & 5 \\ 5 & 6 & -1 \end{bmatrix}$$

and use this result to solve the equations:

$$\begin{cases} 7x_1 + 3x_2 + 2x_3 = 2 \\ 7x_1 - 2x_2 + 5x_3 = 0 \\ 5x_1 + 6x_2 - x_3 = 4 \end{cases}$$

(2.) The following problem (Exercise 3, §3.5, page 76 of Winston, 3rd edition) refers to Example 7 of that same section:

> "Suppose that the post office can force employees to work one day of overtime each week. For example, an employee whose regular shift is Monday to Friday can also be required to work on Saturday. Each employee is paid $50 a day for each of the first five days worked during a week and $62 for the overtime day (if any). Formulate an LP whose solution will enable the post office to minimize the cost of meeting its weekly work requirements."

Also, use LINDO (or other LP software) to find the optimal solution of the LP. (Is it integer-valued?)

(3.) Exercise 11, §3.8, page 92 of Winston, 3rd edition:

> Eli Daisy produces the drug Rozac from four chemicals. Today they must produce 1000 lb. of the drug. The three active ingredients in Rozac are A, B, and C. By weight, at least 8% of Rozac must

consist of A, at least 4% of B, and at least 2% of C. The cost per pound of each chemical and the amount of each active ingredient in one pound of each chemical are given in the table below:

Chemical	Cost per lb.	A	B	C
1	\$8	0.03	0.02	0.01
2	\$10	0.06	0.04	0.01
3	\$11	0.10	0.03	0.04
4	\$14	0.12	0.09	0.04

It is necessary that at least 100 pounds of chemical 2 be used. Formulate an LP whose solution would determine the cheapest way of producing today's batch of Rozac.

Also, use LINDO (or other LP software) to find the optimal solution of the LP.

(4.) Use the Simplex Method to solve the following two LPs:

a. Maximize $x_1 + x_2$
subject to
$x_1 + 5x_2 \le 5$
$2x_1 + x_2 \le 4$
$x_1 \ge 0,\ x_2 \ge 0$

b. Minimize $x_1 + 2x_2$
subject to
$x_1 + 3x_2 \ge 11$
$2x_1 + x_2 \ge 9$
$x_1 \ge 0,\ x_2 \ge 0$

In (b), use a "Phase-One" procedure to get a starting basic feasible solution.

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Solutions:

(1.)

$$[A|I]=\left[\begin{array}{ccc|ccc}7&3&2&1&0&0\\7&-2&5&0&1&0\\5&6&-1&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&3/7&2/7&1/7&0&0\\7&-2&5&0&1&0\\5&6&-1&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&3/7&2/7&1/7&0&0\\0&-5&3&-1&1&0\\0&27/7&-17/7&-5/7&0&1\end{array}\right]$$

$$\Rightarrow\left[\begin{array}{ccc|ccc}1&3/7&2/7&1/7&0&0\\0&1&-3/5&1/5&-1/5&0\\0&27/7&-17/7&-5/7&0&1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&3/7&2/7&1/7&0&0\\0&1&-3/5&1/5&-1/5&0\\0&0&-4/35&-52/35&27/35&1\end{array}\right]$$

$$\Rightarrow\left[\begin{array}{ccc|ccc}1&3/7&2/7&1/7&0&0\\0&1&-3/5&1/5&-1/5&0\\0&0&1&13&-27/4&-35/4\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&-7&3.75&4.75\\0&1&0&8&-4.25&-5.25\\0&0&1&13&-6.75&-8.75\end{array}\right]$$

Therefore,

$$A^{-1}=\begin{bmatrix}-7&3.75&4.75\\8&-4.25&-5.25\\13&-6.75&-8.75\end{bmatrix}$$

The solution for the equations is given by

$$A^{-1}b=\begin{bmatrix}-7&3.75&4.75\\8&-4.25&-5.25\\13&-6.75&-8.75\end{bmatrix}\begin{bmatrix}2\\0\\4\end{bmatrix}=\begin{bmatrix}5\\-5\\-9\end{bmatrix}.$$

(2.) Define: x_i=the # of employees who start to work on ith day for 5 days. (i=1,2,...,7)

Since some of the x_i employees are forced to work one day for overtime, there are two days can be chosen by some of the x_i employees. For example, for the employees who work from Monday to Friday, Saturday or Sunday may be chosen by some of the x_1 employees.

Hence we may define:

y_i=the # of employees who belong to the group that starts to work on ith day for 5 days, and work on the first available overtime day. (e.g., Saturday of the above example).

z_i=the # of employees who belong to the group that starts to work on ith day for 5 days, and work on the second available overtime day. (e.g., Sunday of the above example). (i=1,2,...,7)

We may summarize the above notations into the following table:

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
x1	x1	x1	x1	x1	y1	z1
z2	x2	x2	x2	x2	x2	y2
y3	z3	x3	x3	x3	x3	x3
x4	y4	z4	x4	x4	x4	x4
x5	x5	y5	z5	x5	x5	x5
x6	x6	x6	y6	z6	x6	y6
x7	x7	x7	x7	y7	z7	x7

Therefore the LP can be formulated as below. From the LINDO outputs, we find that the optimal solution is:

Monday-Friday---------- 8 employees, all of them have to work on Saturday,
Wednesday-Sunday-------- 2 employees,
Thursday-Monday------- 4 employees,
Saturday-Wednesday------- 2 employees, all of them have to work on Thursday,
Sunday-Thursday-------- 3 employees.
The total cost is $5370. The solution is interger-valued.

```
 MIN  250 X1 + 250 X2 + 250 X3 + 250 X4 + 250 X5 + 250 X6
      + 250 X7 + 62 Y1 + 62 Z1 + 62 Y2 + 62 Z2 + 62 Y3 + 62 Z3
      + 62 Y4 + 62 Z4 + 62 Y5 + 62 Z5 + 62 Y6 + 62 Z6 + 62 Y7
      + 62 Z7
 SUBJECT TO
    2)     X1 +  X4 +  X5 +  X6 +  X7 +  Z2 +  Y3 >=   17
    3)     X1 +  X2 +  X5 +  X6 +  X7 +  Z3 +  Y4 >=   13
    4)     X1 +  X2 +  X3 +  X6 +  X7 +  Z4 +  Y5 >=   15
    5)     X1 +  X2 +  X3 +  X4 +  X7 +  Z5 +  Y6 >=   19
    6)     X1 +  X2 +  X3 +  X4 +  X5 +  Z6 +  Y7 >=   14
    7)     X2 +  X3 +  X4 +  X5 +  X6 +  Y1 +  Z7 >=   16
    8)     X3 +  X4 +  X5 +  X6 +  X7 +  Z1 +  Y2 >=   11
    9)     X1 -  Y1 -  Z1 >=   0
   10)     X2 -  Y2 -  Z2 >=   0
   11)     X3 -  Y3 -  Z3 >=   0
   12)     X4 -  Y4 -  Z4 >=   0
   13)     X5 -  Y5 -  Z5 >=   0
   14)     X6 -  Y6 -  Z6 >=   0
   15)     X7 -  Y7 -  Z7 >=   0
 END


: go
```

```
     LP OPTIMUM FOUND  AT STEP     32

          OBJECTIVE FUNCTION VALUE

  1)         5370.00000

 VARIABLE          VALUE          REDUCED COST
        X1         8.000000          0.000000
        X2         0.000000          0.000000
        X3         2.000000          0.000000
        X4         4.000000          0.000000
        X5         0.000000         30.000000
        X6         2.000000          0.000000
        X7         3.000000          0.000000
        Y1         8.000000          0.000000
        Z1         0.000000         30.000000
        Y2         0.000000         30.000000
        Z2         0.000000          0.000000
        Y3         0.000000          0.000000
        Z3         0.000000         30.000000
        Y4         0.000000         30.000000
        Z4         0.000000          0.000000
        Y5         0.000000          0.000000
        Z5         0.000000          0.000000
        Y6         2.000000          0.000000
        Z6         0.000000         30.000000
        Y7         0.000000         30.000000
        Z7         0.000000          0.000000


 ROW         SLACK OR SURPLUS     DUAL PRICES
    2)          0.000000        -62.000000
    3)          0.000000        -32.000000
    4)          0.000000        -62.000000
    5)          0.000000        -62.000000
    6)          0.000000        -32.000000
    7)          0.000000        -62.000000
    8)          0.000000        -32.000000
    9)          0.000000          0.000000
   10)          0.000000          0.000000
   11)          2.000000          0.000000
   12)          4.000000          0.000000
   13)          0.000000          0.000000
   14)          0.000000          0.000000
   15)          3.000000          0.000000

 NO. ITERATIONS=      32
```

(3.) Define: Xi=the # of lb from Chemical i, i=1,2, 3, and 4.
The formulation and outputs of LINDO are shown below.
The optimal solution is:

285 lb of Chemical 1,
100 lb of Chemical 2,
417.5 lb of Chemical, and
197.5 lb of Chemical 4.
Total cost is $10637.5.

```
MIN      8 X1 + 10 X2 + 11 X3 + 14 X4
SUBJECT TO
   2)    0.03 X1 + 0.06 X2 + 0.1 X3 + 0.12 X4 >=   80
   3)    0.02 X1 + 0.04 X2 + 0.03 X3 + 0.09 X4 >=   40
   4)    0.01 X1 + 0.01 X2 + 0.04 X3 + 0.04 X4 >=   20
   5)     X2 >=   100
   6)     X1 +  X2 +  X3 +  X4 =    1000
END


: go
    LP OPTIMUM FOUND  AT STEP     8

         OBJECTIVE FUNCTION VALUE

  1)        10637.5000

VARIABLE         VALUE          REDUCED COST
       X1       285.000000          0.000000
       X2       100.000000          0.000000
       X3       417.500000          0.000000
       X4       197.500000          0.000000


ROW         SLACK OR SURPLUS     DUAL PRICES
   2)            0.000000        -37.500000
   3)            0.000000        -37.499992
   4)            8.449999          0.000000
   5)            0.000000         -0.125000
   6)            0.000000         -6.125000

NO. ITERATIONS=       8
```

(4a.)

-z	x1	x2	s1	s2	RHS	ratio
1	1	1	0	0	0	
0	1	5	1	0	5	5
0	(2)	1	0	1	4	2

-z	x1	x2	s1	s2	RHS	ratio
1	0	1/2	0	-1/2	-2	
0	0	(9/2)	1	-1/2	3	2/3
0	1	1/2	0	1/2	2	4

-z	x1	x2	s1	s2	RHS
1	0	0	-1/9	-4/9	-7/3
0	0	1	2/9	-1/9	2/3
0	1	0	-1/9	5/9	5/3

The optimal solution is (x1*,x2*)=(5/3, 2/3), Z*=7/3.

(4b).

-w	-z	x1	x2	s1	s2	a1	a2	RHS
1	0	0	0	0	0	1	1	0
0	1	1	2	0	0	0	0	0
0	0	1	3	-1	0	1	0	11
0	0	2	1	0	-1	0	1	9

First we force the artificial variables away from the basic.

-w	-z	x1	x2	s1	s2	a1	a2	RHS
1	0	-3	-4	1	1	0	0	-20
0	1	1	2	0	0	0	0	0
0	0	1	(3)	-1	0	1	0	11
0	0	2	1	0	-1	0	1	9

After x2 into basic and a1 out of basic, we have

-w	-z	x1	x2	s1	s2	a1	a2	RHS
1	0	-5/3	0	-1/3	1	4/3	0	-16/3
0	1	1/3	0	2/3	0	-2/3	0	-22/3
0	0	1/3	1	-1/3	0	1/3	0	11/3
0	0	5/3	0	(1/3)	-1	-1/3	1	16/3

s1 into basic and a2 out of basic.

-w	-z	x1	x2	s1	s2	a1	a2	RHS
~~1~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~1~~	~~1~~	~~0~~
0	1	-3	0	0	2	~~0~~	~~-2~~	-18
0	0	2	1	0	-1	~~0~~	~~1~~	9
0	0	5	0	1	-3	~~-1~~	~~3~~	16

We obtain the following LP without artificial variables.

-z	x1	x2	s1	s2	RHS
1	-3	0	0	2	-18
0	2	1	0	-1	9
0	(5)	0	1	-3	16

-z	x1	x2	s1	s2	RHS
1	0	0	3/5	1/5	-42/5
0	0	1	-2/5	1/5	13/5
0	1	0	1/5	-3/5	16/5

which is optimal. Optimal solution is (x1*,x2*)=(16/5,13/5), Z*=42/5.

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Homework #3

1.) Exercise 50, Review Problems, page 120 of Winston, 3rd edition (#36, p. 120 of 2nd edition):

> "To process income tax forms, the IRS first sends each form through the data preparation (DP) department, where information is coded for computer entry. Then the form is sent to data entry (DE), where it is entered into the computer. During the next three weeks, the following number of forms will arrive: Week 1: 40,000; week 2: 30,000; week 3: 60,000. The IRS meets the crunch by hiring employees who work 40 hours per week and are paid $200 per week. Data preparation of a form requires 15 minutes, and data entry of a form requires 10 minutes. Each week, an employee is assigned to either data entry or data preparation. The IRS must complete processing of all forms by the end of week 5 and wants to minimize the cost of accomplishing this goal. Formulate an LP that will determine how many workers should be working each week and how the workers should be assigned over the next five weeks."

Find the optimal solution by LINDO (or other LP software). Is the LP solution integer?

2.) **Revised Simplex Method**: Complete the computations below for the problem:

Maximize $12X_1 + 8X_2 + 0X_3 + 0X_4 + 0X_5$
subject to $5X_1 + 2X_2 + 1X_3 + 0X_4 + 0X_5 = 150$
$2X_1 + 3X_2 + 0X_3 + 1X_4 + 0X_5 = 100$
$4X_1 + 2X_2 + 0X_3 + 0X_4 + 1X_5 = 80$
$X_j \geq 0, j=1,2,3,4,5$

Use the slack variables X_3, X_4, and X_5 to form the initial basis, i.e., B={3,4,5}. Then

$$A^B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left(A^B\right)^{-1}$$

<u>Iteration 1</u>: The basic solution is $X_B = (A^B)^{-1}b = [150, 100, 80]^t$. CB = [0, 0, 0] and so the simplex multiplier vector is $\pi_B = C_B(A^B)^{-1} = [0, 0, 0]$. The relative profits of the nonbasic variables X_1 and X_2 are:

$\overline{C}_1 = C_1 - \pi A^1 =$ **a**

$\overline{C}_2 = C_2 - \pi A^2 =$ **b**

Let's select X_1 to enter the basis. The substitution rates of X_1 for the basic variables are

$$\alpha = \left(A^B\right)^{-1} A^1 = \begin{bmatrix} 5 \\ 2 \\ \mathbf{c} \end{bmatrix}$$

i.e., one unit increase of X_1 will replace _c_ units of the third basic variable, X_5. To determine the first basic variable to reach its lower bound as X_1 increases, we perform the minimum ratio test:

$$\text{Min}\left\{ \frac{150}{5}, \frac{100}{2}, \frac{80}{\mathbf{c}} \right\} = \underline{\mathbf{d}}$$

Therefore, the basic variable which reaches zero first (and leaves the basis) is X_5, and the new basis will be B= {3, 4, _e_ }, with the basis matrix

$$A^B = \begin{bmatrix} 1 & 0 & \underline{\mathbf{f}} \\ 0 & 1 & \underline{\mathbf{g}} \\ 0 & 0 & \underline{\mathbf{h}} \end{bmatrix}$$

To update the basis inverse, we write the pivot column A1 alongside the old basis inverse, and perform the pivot in row 3:

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \underline{\mathbf{i}} & 0 \\ 0 & 1 & \underline{\mathbf{j}} & 0 \\ 0 & 0 & \underline{\mathbf{k}} & 1 \end{bmatrix}$$

so that, for B= {3, 4, _e_ },

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1 & 0 & \underline{\mathbf{i}} \\ 0 & 1 & \underline{\mathbf{j}} \\ 0 & 0 & \underline{\mathbf{k}} \end{bmatrix}$$

Iteration 2:

The new basic solution is

$$X_B = \left(A^B\right)^{-1} b = \begin{bmatrix} \underline{\mathbf{l}} \\ 60 \\ 20 \end{bmatrix}$$

The new simplex multiplier vector is

$$\pi_B = C_B \left(A^B\right)^{-1} = [\underline{\mathbf{m}}, 0, \underline{\mathbf{n}}] \left(A^B\right)^{-1} = [\underline{\mathbf{o}}, 0, \underline{\mathbf{p}}]$$

We next "price" the nonbasic variables, X_2 and X_5:

$$\overline{C}_2 = C_2 - \pi_B A^2 = C_2 - \pi_B \begin{bmatrix} 2 \\ 3 \\ \underline{\mathbf{q}} \end{bmatrix} = \underline{\mathbf{p}}$$

$$\overline{C}_5 = C_5 - \pi_B A^5 = \underline{\mathbf{r}}$$

Since p > 0 (and we are maximizing), we select X_2 to enter the basis. The substitution rates of X_2 for the basic variables are

$$\alpha = \left(A^B\right)^{-1} A^2 = \begin{bmatrix} \underline{\mathbf{s}} \\ 2 \\ \mathbf{t} \end{bmatrix}$$

which means that, if X_2 increases by one unit,

- the first basic variable (X_3) increases/decreases *(circle)* by s units,
- the second basic variable (X_4) increases/decreases *(circle)* by 2 units, and
- the third basic variable (X_1) increases/decreases *(circle)* by t units.

To determine the basic variable which reaches zero first (and leaves the basis), we perform the minimum ratio test:

$$\min\left\{ --, \frac{60}{2}, \frac{20}{\mathbf{t}} \right\} = \underline{\mathbf{u}}$$

This means that X_4 should leave the basis, i.e., we should pivot in the second row. The new basis is therefore B = { 3, _v_ , _w_ } and the new basis inverse matrix is found by updating the old basis inverse matrix:

$$\begin{bmatrix} 1 & 0 & \underline{\mathbf{i}} & \underline{\mathbf{s}} \\ 0 & 1 & \underline{\mathbf{j}} & 2 \\ 0 & 0 & \underline{\mathbf{k}} & \underline{\mathbf{t}} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{4} & -\frac{11}{8} & 0 \\ 0 & \underline{\mathbf{x}} & -\frac{1}{4} & 1 \\ 0 & \underline{\mathbf{y}} & \frac{3}{8} & 0 \end{bmatrix}$$

i.e.,

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{11}{8} \\ 0 & \underline{\mathbf{x}} & -\frac{1}{4} \\ 0 & \underline{\mathbf{y}} & \frac{3}{8} \end{bmatrix}$$

<u>Iteration 3</u>: The new basic solution is

$$X_B = \left(A^B\right)^{-1} b = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{11}{8} \\ 0 & \underline{\mathbf{x}} & -\frac{1}{4} \\ 0 & \underline{\mathbf{y}} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 100 \\ 150 \\ 80 \end{bmatrix} = \begin{bmatrix} 65 \\ 30 \\ \underline{\mathbf{z}} \end{bmatrix}$$

i.e., $X_3 = 65$, $X_2 = 30$, and $X_1 =$ _z_.

The simplex multiplier vector is now

$$\pi_B = C_B \left(A^B\right)^{-1} = \left[0, \underline{\mathbf{aa}}, 12\right] \left(A^B\right)^{-1} = \left[0, \underline{\mathbf{bb}}, \frac{5}{2}\right]$$

We next price the nonbasic variables, X_4 and X_5:

$$\bar{C}_4 = C_4 - \pi_B A^4 = -1$$
$$\bar{C}_5 = C_5 - \pi_B A^5 = -\frac{5}{2}$$

Since the relative profits are both negative, this means that the current basic solution is optimal! The optimal profit is therefore _cc_.

3.) Write the dual LPs of the following primal LPs:

a. Maximize $2X_1 + X_2$
subject to $11X_1 + 3X_2 \geq 33$
$8X_1 + 5X_2 \leq 40$
$7X_1 + 10X_2 \leq 70$
$X_1 \geq 0, X_2 \geq 0$

b. Minimize $22X_1 - X_2 + 5X_3$
subject to $X_1 + 3X_3 = 33$
$2X_1 - X_2 \leq 40$
$X_2 + 5X_3 \geq 70$
X_1 unrestricted in sign, $X_2 \geq 0$, $X_3 \leq 0$

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Solutions:

Define X_i=the # of workers for preparing the data in week i, and

Y_i=the # of workers for entering the data in week i, i=1,2,3,4,5.

The optimal solution is: (Note : not unique solution)

Week 1: hire 250 workers for preparing data and 167 workers for entering data,
Week 2: hire 188 workers for preparing data and 0 workers for entering data,
Week 3: hire 375 workers for preparing data and 0 workers for entering data,
Week 4: hire 0 workers for preparing data and 375 workers for entering data,
Week 5: hire 0 workers for preparing data and 0 workers for entering data,

The LINDO formulation and outputs are as follows.

```
MIN    200 X1 + 200 Y1 + 200 X2 + 200 Y2 + 200 X3 + 200 Y3
      + 200 X4 + 200 Y4 + 200 X5 + 200 Y5
 SUBJECT TO
    2)    160 X1 <=   40000
    3)  - 160 X1 + 240 Y1 <=   0  (# of prepared data≥ # of entered data in week 1)
    4)    160 X1 + 160 X2 <=   70000 (# of prepared data ≤30000+40000)
    5)  - 160 X1 + 240 Y1 - 160 X2 + 240 Y2 <=   0
    6)    160 X1 + 160 X2 + 160 X3 <=   130000
    7)  - 160 X1 + 240 Y1 - 160 X2 + 240 Y2 - 160 X3 + 240 Y3 <=   0
    8)    160 X1 + 160 X2 + 160 X3 + 160 X4 <=   130000
    9)  - 160 X1 + 240 Y1 - 160 X2 + 240 Y2 - 160 X3 + 240 Y3
      - 160 X4 + 240 Y4 <=   0
   10)    160 X1 + 160 X2 + 160 X3 + 160 X4 + 160 X5 =    130000
   11)    240 Y1 + 240 Y2 + 240 Y3 + 240 Y4 + 240 Y5 =    130000
 END


: go
     LP OPTIMUM FOUND  AT STEP     6

          OBJECTIVE FUNCTION VALUE

  1)         270833.344

 VARIABLE         VALUE          REDUCED COST
        X1       250.000000         0.000000
        Y1       166.666672         0.000000
        X2       187.500000         0.000000
        Y2         0.000000         0.000000
        X3       375.000000         0.000000
        Y3         0.000000         0.000000
        X4         0.000000         0.000000
        Y4       375.000000         0.000000
        X5         0.000000         0.000000
        Y5         0.000000         0.000000


 ROW         SLACK OR SURPLUS     DUAL PRICES
    2)           0.000000         0.000000
    3)           0.000000         0.000000
    4)           0.000000         0.000000
    5)       30000.000000         0.000000
    6)           0.000000         0.000000
    7)       90000.000000         0.000000
    8)           0.000000         0.000000
    9)           0.000000         0.000000
   10)           0.000000        -1.250000
```

```
   11)            0.000000         -0.833333

 NO. ITERATIONS=       6
```

2.) **Revised Simplex Method**: Complete the computations below for the problem:

$$\text{Maximize } 12X_1 + 8X_2 + 0X_3 + 0X_4 + 0X_5$$
$$\text{subject to } 5X_1 + 2X_2 + 1X_3 + 0X_4 + 0X_5 = 150$$
$$2X_1 + 3X_2 + 0X_3 + 1X_4 + 0X_5 = 100$$
$$4X_1 + 2X_2 + 0X_3 + 0X_4 + 1X_5 = 80$$
$$X_j \geq 0,\ j=1,2,3,4,5$$

Use the slack variables X_3, X_4, and X_5 to form the initial basis, i.e., B={3,4,5}. Then

$$A^B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left(A^B\right)^{-1}$$

Iteration 1: The basic solution is $X_B = (A^B)^{-1}b = [150, 100, 80]^t$. CB = [0, 0, 0] and so the simplex multiplier vector is $\pi_B = C_B(A^B)^{-1} = [0, 0, 0]$. The relative profits of the nonbasic variables X_1 and X_2 are:

$$\overline{C}_1 = C_1 - \pi A^1 = \underline{\mathbf{12}}$$
$$\overline{C}_2 = C_2 - \pi A^2 = \underline{\mathbf{8}}$$

Let's select X_1 to enter the basis. The substitution rates of X_1 for the basic variables are

$$\alpha = \left(A^B\right)^{-1} A^1 = \begin{bmatrix} 5 \\ 2 \\ \underline{\mathbf{4}} \end{bmatrix}$$

i.e., one unit increase of X_1 will replace **_4_** units of the third basic variable, X_5. To determine the first basic variable to reach its lower bound as X_1 increases, we perform the minimum ratio test:

$$\text{Min}\left\{\frac{150}{5}, \frac{100}{2}, \frac{80}{\underline{\mathbf{4}}}\right\} = \underline{\mathbf{20}}$$

Therefore, the basic variable which reaches zero first (and leaves the basis) is X_5, and the new basis will be B= {3, 4, **_1_** }, with the basis matrix

$$A^B = \begin{bmatrix} 1 & 0 & \underline{\mathbf{5}} \\ 0 & 1 & \underline{\mathbf{2}} \\ 0 & 0 & \underline{\mathbf{4}} \end{bmatrix}$$

To update the basis inverse, we write the pivot column A1 alongside the old basis inverse, and perform the pivot in row 3:

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & \underline{\mathbf{-5/4}} & 0 \\ 0 & 1 & \underline{\mathbf{-1/2}} & 0 \\ 0 & 0 & \underline{\mathbf{1/4}} & 1 \end{bmatrix}$$

so that, for B= {3, 4, **_1_** },

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1 & 0 & \underline{\mathbf{-5/4}} \\ 0 & 1 & \underline{\mathbf{-1/2}} \\ 0 & 0 & \underline{\mathbf{1/4}} \end{bmatrix}$$

Iteration 2:

The new basic solution is

$$X_B = \left(A^B\right)^{-1} b = \begin{bmatrix} \underline{\mathbf{50}} \\ 60 \\ 20 \end{bmatrix}$$

The new simplex multiplier vector is

$$\pi_B = C_B \left(A^B\right)^{-1} = [\ \underline{\mathbf{0}}\ ,\ 0,\ \underline{\mathbf{12}}\ ] \left(A^B\right)^{-1} = [\ \underline{\mathbf{0}}\ ,\ 0\ ,\ \underline{\mathbf{3}}\ ]$$

We next "price" the nonbasic variables, X_2 and X_5:

$$\overline{C}_2 = C_2 - \pi_B A^2 = C_2 - \pi_B \begin{bmatrix} 2 \\ 3 \\ \underline{\mathbf{2}} \end{bmatrix} = \underline{\mathbf{2}}$$

$$\overline{C}_5 = C_5 - \pi_B A^5 = \underline{\mathbf{-3}}$$

Since **p** > 0 (and we are maximizing), we select X_2 to enter the basis. The substitution rates of X_2 for the basic variables are

$$\alpha = \left(A^B\right)^{-1} A^2 = \begin{bmatrix} \underline{\mathbf{-1/2}} \\ 2 \\ \underline{\mathbf{1/2}} \end{bmatrix}$$

which means that, if X_2 increases by one unit,

- the first basic variable (X_3) **increases** *(circle)* by **1/2** units,
- the second basic variable (X_4) **decreases** *(circle)* by 2 units, and
- the third basic variable (X_1) **decreases** *(circle)* by **1/2** units.

To determine the basic variable which reaches zero first (and leaves the basis), we perform the minimum ratio test:

$$\min\left\{ -- ,\ \frac{60}{2} ,\ \frac{20}{\underline{\mathbf{1/2}}} \right\} = \underline{\mathbf{30}}$$

This means that X_4 should leave the basis, i.e., we should pivot in the second row. The new basis is therefore B = { 3, **2** , **1** } and the new basis inverse matrix is found by updating the old basis inverse matrix:

$$\begin{bmatrix} 1 & 0 & \underline{\mathbf{-5/4}} & \underline{\mathbf{-1/2}} \\ 0 & 1 & \underline{\mathbf{-1/2}} & 2 \\ 0 & 0 & \underline{\mathbf{1/4}} & \underline{\mathbf{1/2}} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{4} & -\frac{11}{8} & 0 \\ 0 & \underline{\mathbf{1/2}} & -\frac{1}{4} & 1 \\ 0 & \underline{\mathbf{-1/4}} & \frac{3}{8} & 0 \end{bmatrix}$$

i.e.,

$$\left(A^B\right)^{-1} = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{11}{8} \\ 0 & \underline{\mathbf{1/2}} & -\frac{1}{4} \\ 0 & \underline{\mathbf{-1/4}} & \frac{3}{8} \end{bmatrix}$$

Iteration 3: The new basic solution is

$$X_B = \left(A^B\right)^{-1} b = \begin{bmatrix} 1 & \frac{1}{4} & -\frac{11}{8} \\ 0 & \underline{1/2} & -\frac{1}{4} \\ 0 & \underline{-1/4} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} 150 \\ 100 \\ 80 \end{bmatrix} = \begin{bmatrix} 65 \\ 30 \\ \underline{5} \end{bmatrix}$$

i.e., $X_3 = 65$, $X_2 = 30$, and $X_1 = \underline{5}$.
The simplex multiplier vector is now

$$\pi_B = C_B \left(A^B\right)^{-1} = [0, \underline{8}, 12] \left(A^B\right)^{-1} = \left[0, \underline{1}, \frac{5}{2}\right]$$

We next price the nonbasic variables, X_4 and X_5:

$$\overline{C}_4 = C_4 - \pi_B A^4 = -1$$
$$\overline{C}_5 = C_5 - \pi_B A^5 = -\frac{5}{2}$$

Since the relative profits are both negative, this means that the current basic solution is optimal! The optimal profit is therefore $\underline{300}$.

3a.) Dual problem:

Min 33Y1+40Y2+70Y3
st. 11Y1+8Y2+7Y3≥2
3Y1+5Y2+10Y3≥1
Y1≤0,Y2≥0,Y3≥0

3b) Dual problem:

Max 33Y1+40Y2+70Y3
st Y1+2Y2=22
-Y2+Y3≤-1
3Y1+5Y3≥5
Y1 urs, Y2≤0, Y3≥0

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Homework #4

1.) Exercise 2, §5.2, page 211 of Winston, 3rd edition:

"Carco manufactures cars and trucks. Each car contributes $300 to profit, and each truck contributes $400. The resources required to manufacture a car and a truck are shown in the table below.

Vehicle Type	Days on Type 1 Machine	Days on Type 2 Machine	Tons of Steel
Car	0.8	0.6	2
Truck	1.0	0.7	3

Each day, Carco can rent up to 98 type 1 machines at a cost of $50 per machine. At present, the company has 73 type 2 machines and 260 tons of steel available. Marketing considerations dictate that at least 88 cars and at least 26 trucks be produced. Let x_1 = number of cars produced daily; x_2 = number of trucks produced daily; and m_1 = type 1 machines rented daily.
To maximize profit, Carco should solve the LP

```
MAX 300 X1 + 400 X2 - 50 M1
ST
0.8 X1 + X2 - M1 <= 0
M1 <= 98
0.6 X1 + 0.7 X2 <= 73
2 X1 + 3 X2 <= 260
```

```
X1 >= 88
X2 >= 26
END
```

Use the LINDO output (given in the textbook, page 212) to answer the following questions:

a. If each car contributed $310 to profit, what would be the new optimal solution to the problem?

b. If Carco were required to produce at least 86 cars, what would Carco's profit become?"

2.) Exercise 6, Review Problems, page 227 of Winston, 3rd edition:

Gepbab Production Company uses labor and raw material to produce three products. The resource requirements and sales price for the three products are as shown in the tale below:

	Product 1	Product 2	Product 3	
Labor	3 hours	4 hours	6 hours	
Raw material	2 units	2 units	5 units	
Sales price		$6	$8	$13

At present, 60 units of raw material are available. Up to 90 hours of labor can be purchased at $1 per hour. To maximize Gepbab profits, solve the following LP:

```
MAX 6 X1 + 8 X2 + 13 X3 - L
ST
3 X1 + 4 X2 + 6 X3 - L <= 0
2 X1 + 2 X2 + 5 X3 <= 60
L <= 90
END
```

Here, x_i = units of product i produced, and L = number of labor hours purchased. Use the LINDO output in figure 21 (page 228 of the text) to answer the following questions:

a. What is the most the company would be willing to pay for another unit of raw material?
b. What is the most the company would be willing to pay for another hour of labor?
c. What would product 1 have to sell for to make it desirable for the company to produce it?
d. If 100 hours of labor could be purchased, what would the company's profit be?
e. Find the new optimal solution if product 3 sold for $15.

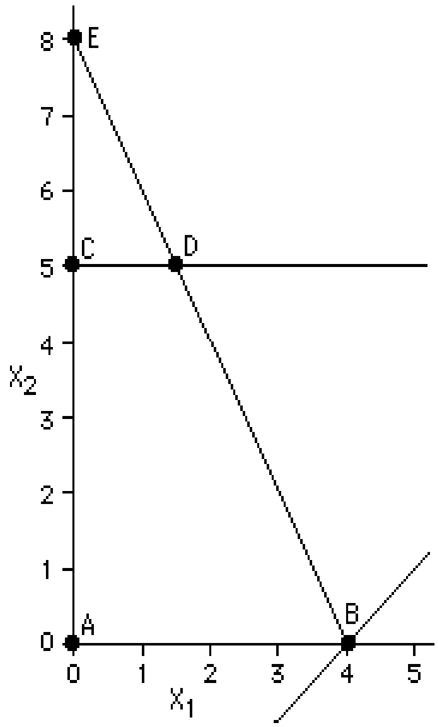
3.) Write the dual LP of the following primal LP:

Maximize $12X_1 + X_2 + 5X_3$
subject to $X_1 - 3X_3 = 33$
$2X_1 + X_2 \le 40$
$X_2 + 5X_3 \ge 70$
X_1 unrestricted in sign, $X_2 \ge 0$, $X_3 \le 0$

4.) Consider the LP:

Minimize $4X_1 - X_2$
subject to $2X_1 + X_2 \le 8$
$X_2 \le 5$
$X_1 - X_2 \le 4$
$X_1 \ge 0$, $X_2 \ge 0$

a. Indicate on the graph below the feasible region:



b. For each of points A, B, C, D, and E, representing basic solutions of the LP, identify which variables (including slack variables) are basic.

c. Which if any of the five points above are "degenerate" basic solutions? Which are infeasible?

d. Compute the objective values of the feasible extreme points and determine the optimal solution.

e. Write the dual of the LP above. (Warning: note the direction of the inequalities!)

f. According to the complementary slackness theorem, knowing the positive variables in the primal optimal solution, which dual constraint(s) must be "tight" (i.e., slack or surplus equals zero)? By the same theorem, knowing which primal constraints are "slack" at the optimum, which dual variable(s) must be zero?

g. Write the equation(s) determined in (f), substitute zero values which were determined for the dual variable(s) and solve for the remaining dual variable(s).

h. What is the dual objective value for the solution determined in (g)? Is it optimal?

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Solution:

1a.) If each car contributed $310 to profit, what would be the new optimal solution to the problem?

Solution. Since the reduced cost for X1 is 0 and the allowable increase for X1 is 20, the basic variable remains the same. The objective value = 32540+(310-300)(88)=33420.

1b.) If Carco were required to produce at least 86 cars, what would Carco's profit become?"

Solution. The dual price for constraint #X1 >= 88 is -20 and allowed decrease is 3, therefore, the profit becomes 32540+(86-88)(-20)=32580.

2) a. What is the most the company would be willing to pay for another unit of raw material?

Solution. $0.5. (Since the dual price for constraint #3 is $0.5)

b. What is the most the company would be willing to pay for another hour of labor?
Solution. \$1.75. (There are two ways to obtain this value:
i. The dual price for row #4 is \$0.75, implying that they could afford to spend up to 75¢ per hour to expand the labor supply, which they could then purchase for \$1.00.
ii. The dual price for row #2 is \$1.75, implying that using one more hour than is purchased would increase their profits by \$1.75.)

c. What would product 1 have to sell for to make it desirable for the company to produce it?
Solution. \$6.25. (The reduced cost for product #1 is \$0.25. Thus, in order to balance, the sell price is at least \$6+\$0.25=\$6.25)

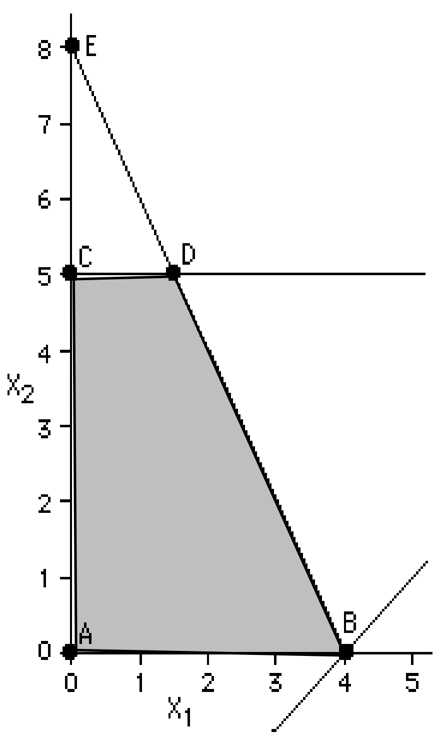
d. If 100 hours of labor could be purchased, what would the company's profit be?
Solution. \$105. (Changing the RHS of constraint #4 from 90 to 100. Since the allowable increase is 30, the profit becomes \$97.5+\$0.75(100-90)=\$105.

e. Find the new optimal solution if product 3 sold for \$15.
Solution. \$112.5. (Since the allowable increase for X3 is 3, the basics remain the same. The objective value becomes \$97.5+(\$7.5)(2)=\$112.5.

3.) (Dual)

$$\begin{aligned} &\text{Min } 33Y1 + 40Y2 + 70Y3 \\ &\text{s.t. } Y1 + 2Y2 = 12 \\ &\quad Y2 + Y3 \geq 1 \\ &\quad -3Y1 + 5Y3 \leq 5 \\ &Y1 \text{ urs}, Y2 \geq 0, Y3 \leq 0. \end{aligned}$$

4.) a. Indicate on the graph below the feasible region:



b. Inserting the slack variables to primal problem, we obtain

$$\begin{aligned} &\text{Minimize } 4X_1 - X_2 \\ &\text{subject to } 2X_1 + X_2 + S1 = 8 \\ &\quad X_2 + S2 = 5 \\ &\quad X_1 - X_2 + S3 = 4 \end{aligned}$$

$X_1 \geq 0$, $X_2 \geq 0$, $S1 \geq 0$, $S2 \geq 0$, $S3 \geq 0$.

point	basic variable
A	S1, S2, S3 (since X1=X2=0 and S1>0, S2>0, S3>0)
B	X1, S1, S2 (since X1>0, S1>0. We arbitrarily choose the other one basic variable from X2, S2, S3)
C	X2, S1, S3 (since X1=S2=0, X2>0, S1>0, S3>0)
D	X1, X2, S3 (since S1=0, S2=0)
E	X2, S2, S3 (since X1=S1=0)

c. Point B is degenerate, since some element of basic variables is 0 (e.g. S2=0).
Point E is infeasible, since S2=-3<0.

d.

Point	Objective value
A	0
B	16
C	-5
D	1

Thus, the optimal solution is point C with objective value -5.

e. Max 8Y1+5Y2+4Y3

s.t. $2Y1 \quad +Y3 \leq 4$

$Y1 + Y2 - Y3 \leq -1$

$Y1 \leq 0$, $Y2 \leq 0$, $Y3 \leq 0$

f. The optimal solution for primal is point C=(0, 5) in which X1=0,X2=5, S1=3, S2=0, S3=9. Hence the constraints #1 and #3 in primal are tight, which imply Y1=Y3=0.

g. As above, since X2=5>0, constraint #2 in dual problem must be tight, i.e., Y1+Y2-Y3=-1. Because Y1=Y3=0 by (f), we get Y2=-1.

h. Objective value for dual is 8(0)+5(-1)+4(0)=-5.
Since primal objective value=dual objective value=-5, Y=(0,-1,0) is the optimal.

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Homework #5

1.) Continuation of second exercise from HW#4 (Gepbab Prod'n Co., Exercise 6, Review Problems, page 227 of Winston, 3rd edition):

Gepbab Production Company uses labor and raw material to produce three products. The resource requirements and sales price for the three products are as shown in the tale below:

	Product 1	Product 2	Product 3	
Labor	3 hours	4 hours	6 hours	
Raw material	2 units	2 units	5 units	
Sales price		$6	$8	$13

At present, 60 units of raw material are available. Up to 90 hours of labor can be purchased at $1 per hour. To maximize Gepbab profits, solve the following LP:

```
MAX 6 X1 + 8 X2 + 13 X3 - L
ST
3 X1 + 4 X2 + 6 X3 - L <= 0
2 X1 + 2 X2 + 5 X3 <= 60
L <= 90
END
```

Here, x_i = units of product i produced, and L = number of labor hours purchased.

```
      OBJECTIVE FUNCTION VALUE
      1)     97.5000000

 VARIABLE        VALUE          REDUCED COST
       X1          .000000           .250000
       X2        11.250000           .000000
       X3         7.500000           .000000
        L        90.000000           .000000

      ROW   SLACK OR SURPLUS     DUAL PRICES
       2)          .000000          1.750000
       3)          .000000           .500000
       4)          .000000           .750000

RANGES IN WHICH THE BASIS IS UNCHANGED:
                          OBJ COEFFICIENT RANGES
VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                  COEF          INCREASE         DECREASE
      X1        6.000000          .250000         INFINITY
      X2        8.000000          .666667          .666667
      X3       13.000000         3.000000         1.000000
       L       -1.000000         INFINITY          .750000

                          RIGHTHAND SIDE RANGES
     ROW         CURRENT        ALLOWABLE        ALLOWABLE
                   RHS          INCREASE         DECREASE
       2          .000000       30.000000        18.000000
       3        60.000000       15.000000        15.000000
       4        90.000000       30.000000        18.000000


THE TABLEAU
ROW  (BASIS)    X1    X2    X3     L  SLK 2  SLK 3  SLK 4
  1      ART  .250  .000  .000  .000  1.750   .500   .750    97.500
  2       X2  .375 1.000  .000  .000   .625  -.750   .625    11.250
  3       X3  .250  .000 1.000  .000  -.250   .500  -.250     7.500
  4        L  .000  .000  .000 1.000   .000   .000  1.000    90.000
```

a. If the quantity of product 1 which is produced were to increase by 20 units, then according to the "substitution rates" in the optimal tableau above, what is
   - the quantity of product 2 produced? ______
   - the quantity of product 3 produced? ______
   - the labor hours purchased ? ______

b. Is the (nonoptimal) solution which you find in (a) *basic* or *nonbasic* ?
   If nonbasic, how much of product 1 should be produced in order to obtain a basic solution?

c. What is the equation form of the raw material availability inequality? What is the name of the slack variable? What is the value of this slack variable in the optimal solution? Is it basic or nonbasic?

d. If the RHS remains unchanged, but the amount of raw material *used* were to increase to 70, what would be the value of the slack variable? (Note that this value would be infeasible according to the original problem definition!)

e. According to the substitution rates in the tableau above, if the slack variable (unused raw material) were to change from its current value to the value which you specified in (d), what would be
   - the quantity of product 1 produced? ______
   - the quantity of product 2 produced? ______
   - the quantity of product 3 produced? ______
   - the labor hours purchased ? ______

2.) Exercise 16, Chapter 5 Review Problems, pages 231-232 of Winston, 3rd edition:

Cornco produces two products: PS and QT. The sales price for each product and the maximum quantity of each that can be sold during each of the next three months are:

	Month 1		Month 2		Month 3	
Product	Price	Demand	Price	Demand	Price	Demand
PS	$40	50	$60	45	$55	50
QT	$35	43	$40	50	$44	40

Each product must be processed through two assembly lines: 1 and 2. The number of hours required by each product on each assembly line are:

Produce	Line 1	Line 2
PS	3 hours	2 hours
QT	2 hours	2 hours

The number of hours available on each assembly line during each month are:

Line	Month 1	Month 2	Month 3
1	1200	160	190
2	2140	150	110

Each unit of PS requires 4 pounds of raw material, while each unit of QT requires 3 pounds. Up to 710 units of raw material can be purchased at $3 per pound. At te beginning of month 1, 10 units of PS and 5 units of QT are available. It costs $10 to hold a unit of either product in inventory for a month. Solve this LP on LINDO and use your output to answer the following questions:

a. Find the new optimal solution if it costs $11 to hold a unit of PS in inventory at the end of month 1.
b. Find the company's new optimal solution if 210 hours on line 1 are available during month 1.
c. Find the company's new profit level if 109 hours are available on line 2 during month 3.
d. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 2?
e. What is the most Cornco should be willing to pay for an extra pound of raw material?
f. What is the most Cornco should be willing to pay for an extra hour of line 1 time during month 3?
g. Find the new optimal solution if PS sells for $50 during month 2.
h. Find the new optimal solution if QT sells for $50 during month 3.
i. Suppose spending $20 on advertising would increase demand for QT in month 2 by 5 units. Should the advertising be done?

3. Consider the transportation problem with the tableau:

source \ destination	1	2	3	4	supply
1	4	7	9	2	8
2	3	5	7	3	6
3	2	4	8	4	4
demand	5	6	4	3	

a. Is this a "balanced" transportation problem? (If not, modify it so as to obtain an equivalent balanced problem.)
b. How many basic variables will this problem have?

c. Use the "Northwest-Corner Method" to obtain an initial feasible solution. Is it basic? Is it degenerate? What is its cost?

d. Apply the transportation simplex method to this problem, using dual variables in order to evaluate the reduced costs at each iteration.

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Solutions:

1.) a.

the quantity of product 2 produced? _**3.75**_
the quantity of product 3 produced? __**2.5**__
the labor hours purchased ? __90__

b. Nonbasic. Since there are three constraints, at most three variables are positive. But now, four variables are positive (x1=20, x2=3.75, x3=2.5, L=90).

Since $\begin{bmatrix} profit \\ x_2 \\ x_3 \\ L \end{bmatrix} = \begin{bmatrix} 97.5 \\ 11.25 \\ 7.5 \\ 90 \end{bmatrix} - \begin{bmatrix} 0.25 \\ 0.375 \\ 0.25 \\ 0 \end{bmatrix} x_1$, to get a new basic solution we have to increase x_1 such that one of the basic variables (i.e. x2,x3,L) leaves the basic (i.e., such that one of these three variable are zero). Therefore, increasing $x_1 = \min\left\{\frac{11.25}{0.375}, \frac{7.5}{0.25}\right\} = 30$, we will obtain a new basic solution (x1,x2,L)=(30,0,90) or (x1,x3,L)=(30,0,90). Note that the new basic is degenerate.

c. 2x1+2x2+5x3+SLK3=60. SLK3=0 at optimal. It is nonbasic (since basic variables are x1,x2,x3,L at optimal).

d. Change the RHS from 60 into 70. The SLK3 is changed from 0 into -10.

e.

the quantity of product 1 produced? _**0**___
the quantity of product 2 produced? _**3.75**_
the quantity of product 3 produced? _**12.5**_
the labor hours purchased ? _**90**___

2.) Define: PSi=# of products of PS produced in month i, i=1,2,3.
PSiI=inventory of PS in month i, i=1,2,3.
QTi=# of products of QT produced in month i, i=1,2,3.
QTiI=inventory of QT in month i, i=1,2,3.

Note: For assembly line, there need 3 hrs in Line 1 and 2 hrs in Line 2 to produce one PS unit (i.e., total = 5 hrs) etc.

LP model and LINDO outputs are as below:

```
MAX   28 PS1 + 48 PS2 + 43 PS3 + 26 QT1 + 31 QT2 + 35 QT3
      - 10 PS1I - 10 PS2I - 10 PS3I - 10 QT1I - 10 QT2I - 10 QT3I
 SUBJECT TO
    2)     PS1 -  PS1I <=   40
    3)     PS2 +  PS1I -  PS2I <=   45
    4)     PS3 +  PS2I -  PS3I <=   50
    5)     QT1 -  QT1I <=   38
    6)     QT2 +  QT1I -  QT2I <=   50
    7)     QT3 +  QT2I -  QT3I <=   40
```

```
   8)     3 PS1 + 2 QT1 <=   1200
   9)     2 PS1 + 2 QT1 <=   2140
  10)     3 PS2 + 2 QT2 <=   160
  11)     2 PS2 + 2 QT2 <=   150
  12)     3 PS3 + 2 QT3 <=   190
  13)     2 PS3 + 2 QT3 <=   110
  14)     4 PS1 + 4 PS2 + 4 PS3 + 3 QT1 + 3 QT2 + 3 QT3 <=   710
 END


: go
     LP OPTIMUM FOUND  AT STEP     8

          OBJECTIVE FUNCTION VALUE

  1)         6999.16650

 VARIABLE         VALUE          REDUCED COST
       PS1        40.000000          0.000000
       PS2        45.000000          0.000000
       PS3        50.000000          0.000000
       QT1        39.166668          0.000000
       QT2        12.500000          0.000000
       QT3         5.000000          0.000000
      PS1I         0.000000          7.500000
      PS2I         0.000000          8.500000
      PS3I         0.000000          7.333334
      QT1I         1.166667          0.000000
      QT2I         0.000000         10.000000
      QT3I         0.000000         10.000000


 ROW         SLACK OR SURPLUS     DUAL PRICES
   2)         0.000000          6.666666
   3)         0.000000          4.166666
   4)         0.000000          2.666666
   5)         0.000000         10.000000
   6)        36.333332          0.000000
   7)        35.000000          0.000000
   8)      1001.666687          0.000000
   9)      1981.666626          0.000000
  10)         0.000000          7.500000
  11)        35.000000          0.000000
  12)        30.000000          0.000000
  13)         0.000000          9.500000
  14)         0.000000          5.333333

 NO. ITERATIONS=       8

  DO RANGE(SENSITIVITY) ANALYSIS?
? y


    RANGES IN WHICH THE BASIS IS UNCHANGED

                          OBJ COEFFICIENT RANGES
 VARIABLE         CURRENT        ALLOWABLE        ALLOWABLE
                   COEF          INCREASE         DECREASE
       PS1        28.000000         7.500000         6.666666
```

PS2	48.000000	8.500000	4.166666
PS3	43.000000	7.333334	2.666666
QT1	26.000000	5.000000	5.000000
QT2	31.000000	2.777777	5.666667
QT3	35.000000	2.666666	7.333334
PS1I	-10.000000	7.500000	INFINITY
PS2I	-10.000000	8.500000	INFINITY
PS3I	-10.000000	7.333334	INFINITY
QT1I	-10.000000	5.000000	5.000000
QT2I	-10.000000	10.000000	INFINITY
QT3I	-10.000000	10.000000	INFINITY

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	40.000000	0.875000	27.250000
3	45.000000	8.333333	7.000000
4	50.000000	3.500000	35.000000
5	38.000000	1.166667	36.333332
6	50.000000	INFINITY	36.333332
7	40.000000	INFINITY	35.000000
8	1200.000000	INFINITY	1001.666687
9	2140.000000	INFINITY	1981.666626
10	160.000000	2.333333	25.000000
11	150.000000	INFINITY	35.000000
12	190.000000	INFINITY	30.000000
13	110.000000	2.333333	10.000000
14	710.000000	109.000000	3.500000

a. Remain the same. Since the allowable decreasing for PS1I is infinity.

b. Remain the same.

c. The dual price for row 13 is 9.5 and allowable decrease is 2.333, thus, new profit=6999.1665-9.5(1)=6989.665.

d. 7.5 (since dual price for row 10 is 7.5)

e. 5.3333 (the dual price for row 14).

f. Zero. (since the dual price for row 12 is 0).

g. We cannot find. (since the allowable decrease for PS2 is 4.166 which is less than 10).

h. We cannot find. (since the allowable increase for QT3 is 2.666 which is less than 6).

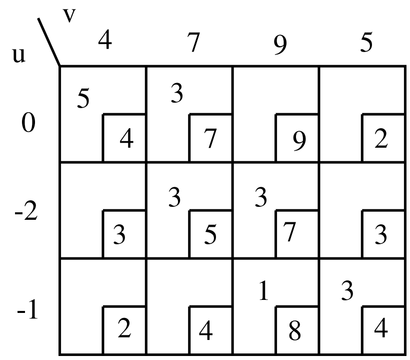
i. No. Since the dual price for row 6 is zero and the allowable increase is infinity.

3.

a. It's balanced.

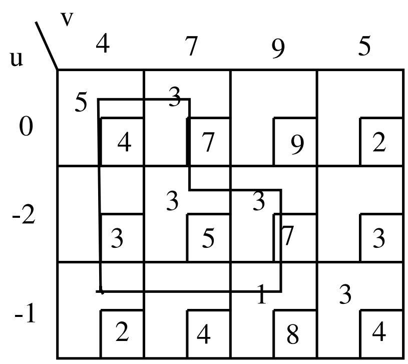
b. 4+3-1=6.

c.



It is basic and nondegenerate. Cost=5(4)+3(7)+3(5)+3(7)+1(8)=3(4)=97.

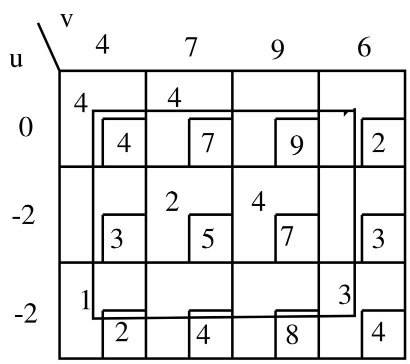
d.



$\bar{c}31=2-(4-1)<0$

x31 enters into bassic

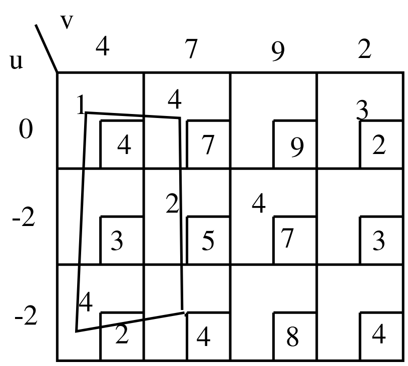
x33 leaves the basic



$\bar{c}14=2-(6+0)<0$

x14 enters into basic

x34 leaves the basic



$\bar{c}32=4-(7-2)<0$

x32 enters into basic

x12 leaves the basic

u \ v	4	6	8	2
0	5 [4]	[7]	[9]	3 [2]
-1	[3]	2 [5]	4 [7]	[3]
-2	0 [2]	4 [4]	[8]	[4]

All reduced costs are nonnegative.
Stop the procedure.

Optimal solution is x11=5, x14=3, x22=2, x23=4, x31=0, x32=4.
Total cost = 5(4)+3(2)+2(5)+4(7)+0(2)+4(4)=80.

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Homework #6

1.) **Assignment Problem.** Suppose that 4 jobs are to assigned to 4 machines, with the cost matrix determined as follows:

Your 9-digit student ID#	C_{11}	C_{12}	C_{13}	C_{14}	C_{21}	C_{22}	C_{23}	C_{24}	C_{31}
Your 7-digit telephone #	C_{32}	C_{33}	C_{34}	C_{41}	C_{42}	C_{43}	C_{44}		

Write your ID# and phone # above, and then transfer to the 4x4 cost matrix to the right:

MACHINE j

C_{ij} (JOB i)	1	2	3	4
1				
2				
3				
4				

a. Perform row & column reduction, and write the resulting matrix (reduced matrix #1) on the left below .
b. How many (horizontal &/or vertical) lines are required to "cover" all of the zeroes in reduced matrix #1?
c. Is there a zero-cost assignment possible in reduced matrix #1? ____
d. If the answer in (c) is "yes", what is that assignment? *(write the solution below and stop.)* Otherwise, perform another reduction (which will alter the location of the zeroes, perhaps increasing the number of zeroes) and write the result in reduced matrix #2 below.
e. How many (horizontal &/or vertical) lines are required to "cover" all of the zeroes in reduced matrix #2?
f. Is there a zero-cost assignment possible in reduced matrix #2? ____

MACHINE j

JOB i	1	2	3	4
1				
2				
3				
4				

Reduced matrix #1

MACHINE j

JOB i	1	2	3	4
1				
2				
3				
4				

Reduced matrix #2

MACHINE j

JOB i	1	2	3	4
1				
2				
3				
4				

Reduced matrix #3

g. If the answer in (f) is yes, write the solution below and stop; otherwise, perform another reduction and write the result in reduced matrix #3 on the right below.

h. How many (horizontal &/or vertical) lines are required to "cover" all of the zeroes in reduced matrix #3?

i. Is there a zero-cost assignment possible in reduced matrix #3? ____

j. If the answer in (i) is yes, write the solution below.

Solution:

Job	1	2	3	4
Machine				

Total cost: ____

2. **Decision Analysis.** The following matrix gives the expected profit in thousands of dollars for five marketing strategies and five potential levels of sales:

		Level of sales				
		1	2	3	4	5
Strategy	1	10	20	30	40	50
	2	20	25	25	30	35
	3	50	40	5	15	20
	4	40	35	30	25	25
	5	10	20	25	30	20

a. What marketing strategy would be chosen according to the maximin rule? ____

b. What marketing strategy would be chosen according to the maximin rule? ____

c. Compute, for each combination of strategy/sales level, the "regret":

		Level of sales				
		1	2	3	4	5
Strategy	1	___	___	___	___	___
	2	___	___	___	___	___
	3	___	___	___	___	___
	4	___	___	___	___	___
	5	___	___	___	___	___

d. What marketing strategy would be chosen according to the minimax regret rule? ____

3. **Decision Tree.** (Exercise 2, §13.3, page 753 of Winston's text, 3rd edition.)

The decision sciences department is trying to determine which of two copying machines to purchase. Both machines will satisfy the department's needs for the next ten years.

Machine 1 costs $2000 and has a maintenance agreement, which, for an annual fee of $150, covers all repairs. Machine 2 costs $3000, and its annual maintenance cost is a random variable. At present, the decision sciences department believes there is a 40% chance that the annual cost for machine 2 will be $0, a 40% chance it will be $100, and a 20% chance it will be $200.

Before the purchase decision is made, the department can have a trained repairer evaluate the quality of machine 2. If the repairer believes that machine 2 is satisfactory, there is a 60% chance that its annual maintenance cost will be $0 and a 40% chance that it will be $100. If the repairer believes that machine 2 is unsatisfactory, there is a 20% chance that the annual maintenance cost will be $0, a 40% chance it will be $100, and a 40% chance it will be $200. If there is a 50% chance that the repairer will give a satisfactory report, what is the EVSI (i.e., the expected value of the repairer's report)? If the repairer charges $40, what should the decision sciences department do? What is EVPI (expected value of perfect information)?

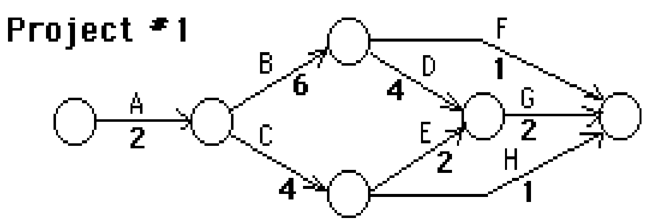
Homework #7

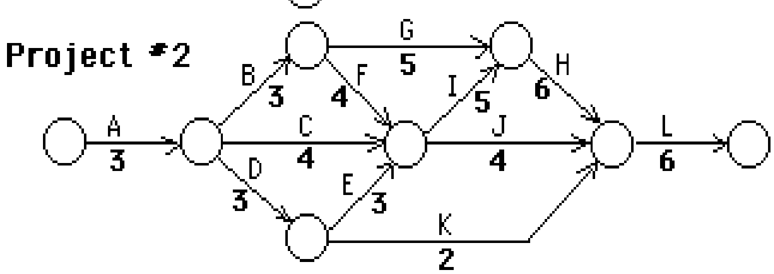
1.) **Decision Tree.** (Exercise #9, §13.4, page 762 of Winston, O.R., 3rd edition)

The government is attempting to determine whether immigrants should be tested for a contagious disease. Let's assume that the decision will be made on a financial basis. Assume that each immigrant who is allowed into the country and has the disease costs the United States $100,000, and each immigrant who enters and does not have the disease will contribute $10,000 to the national economy. Assume that 10% of all potential immigrants have the disease. The government may admit all immigrants, admit no immigrants, or test immigrants for the disease before determining whether they should be admitted. It costs $100 to test a person for the disease; the test result is either positive or negative. If the test result is positive, the person definitely has the disease. However, 20% of all people who do have the disease test negative. A person who does not have the disease always tests negative. The government's goal is to maximize (per potential immigrant) expected benefits minus expected costs.

a. Compute the probability that a potential immigrant has the disease if the test result is negative.
b. Draw the decision tree for this problem.
c. Find the optimal decision.
d. What is the expected value of the test (EVSI)?
e. What is the expected value of perfect information (EVPI)?

2. **Project Scheduling** (Exercises 16 & 17, §8.4, pages 434-435, Winston, O.R., 3rd edition, with modifications)





For each project network above:

a. Using the forward pass & backward pass procedures, compute the early time (ET) and late time (LT) for each node.

b. For each activity, find the early start time (ES), early finish time (EF), late start time (LS), late finish time (LF), and the total float (slack).

Project #1:

Activity	ES	EF	LS	LF	TF
A	___	___	___	___	___
B	___	___	___	___	___
C	___	___	___	___	___
D	___	___	___	___	___
E	___	___	___	___	___
F	___	___	___	___	___
G	___	___	___	___	___
H	___	___	___	___	___

Project #2:

Activity	ES	EF	LS	LF	TF
A	___	___	___	___	___
B	___	___	___	___	___
C	___	___	___	___	___
D	___	___	___	___	___
E	___	___	___	___	___
F	___	___	___	___	___
G	___	___	___	___	___
H	___	___	___	___	___
I	___	___	___	___	___
J	___	___	___	___	___
K	___	___	___	___	___
L	___	___	___	___	___

c. Find the critical path.

d. Draw the corresponding A-O-N (activity on node) network.

In the case of project #2, assume that the durations specified on the network are the *expected* values (in days), but that the actual durations are random variables. Also assume that the standard deviations of all of the activities are 25% of the expected values. (E.g., the duration of activity L has mean 6 days and standard deviation 1.5 days.)

e. Assuming (as does PERT) that the critical path found in (c) is always critical, what is the expected length and the standard deviation of the length of the critical path?

f. Assuming (as does PERT) that the length of the critical path is normally distributed, what is the probability that project #2 is completed within 30 days? Use the table on pages 632-633 of Winston, <u>O.R.</u> (3rd edition) or a similar table from a probability & statistics textbook.

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Solutions:

1.)

a. Let D: disease, ND: no disease, -: negative test, and +: positive test.

$$P(D|-) = \frac{P(-|D)P(D)}{P(-)} = \frac{P(-|D)P(D)}{P(-|D)P(D) + P(-|ND)P(ND)}$$

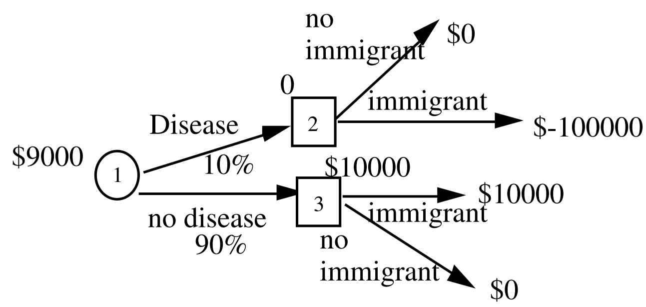
$$= \frac{0.2(0.1)}{0.2(0.1) + 1.0(0.9)} = \frac{0.02}{0.92} = \frac{1}{46}$$

b. See next page.

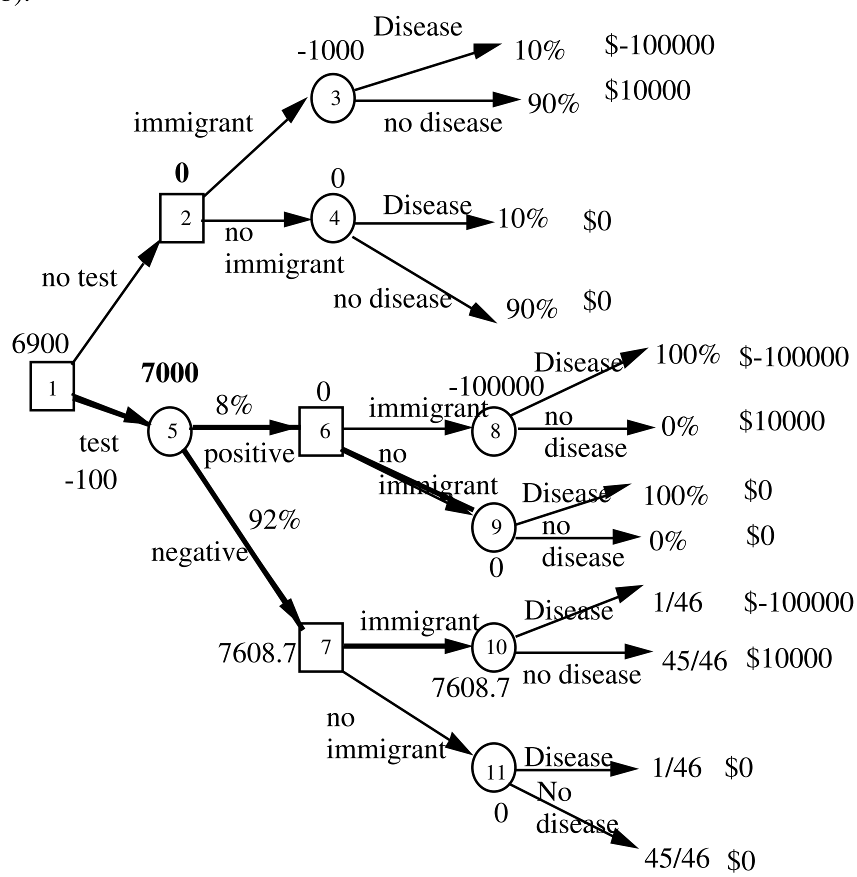
c. From the decision tree in (b), the optimal decision is : Do the test. Accept the immigrants whose test results are negative and reject the immigrants whose test results are positive.

d. EVSI=EVWSI-EVWOI=7000 - 0 =7000.

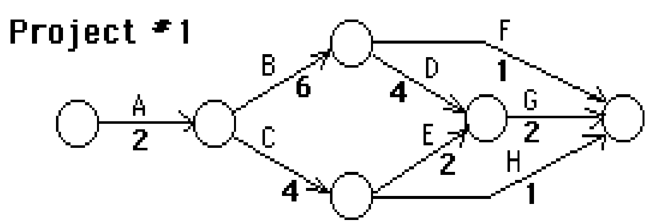
e. EVPI=EVWPI-EVWOI=9000-0=9000.

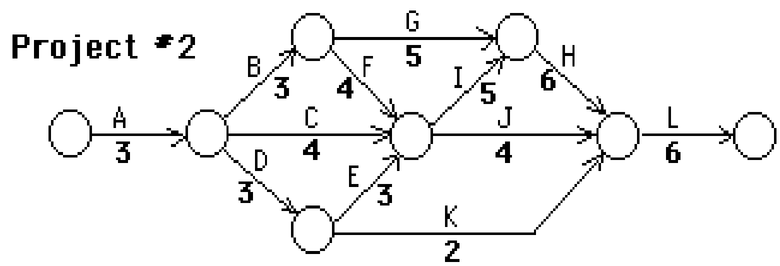


Solution for (b).



2.





For each project network above:

a. Using the forward pass & backward pass procedures, compute the early time (ET) and late time (LT) for each node.
b. For each activity, find the early start time (ES), early finish time (EF), late start time (LS), late finish time (LF), and the total float (slack).

Project #1:

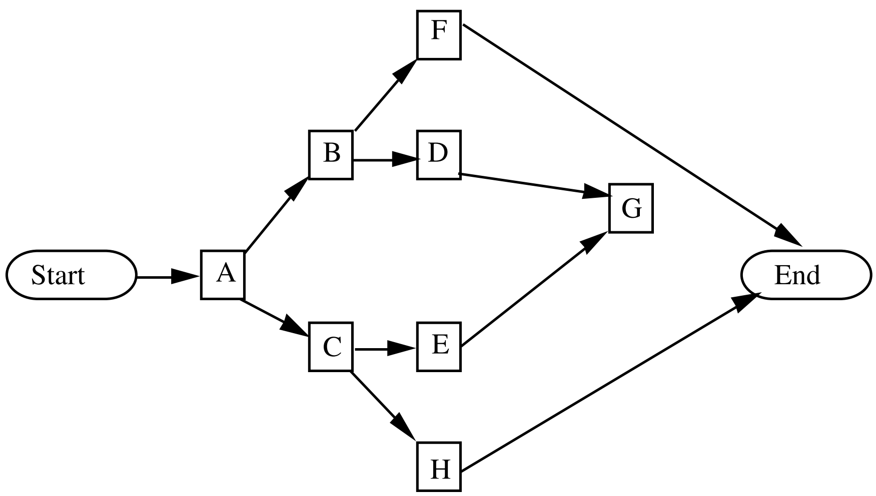
Activity	ES	EF	LS	LF	TF
A	**0**	**2**	**0**	**2**	**0**
B	**2**	**8**	**2**	**8**	**0**
C	**2**	**6**	**6**	**10**	**4**
D	**8**	**12**	**8**	**12**	**0**
E	**6**	**8**	**10**	**12**	**4**
F	**8**	**9**	**13**	**14**	**5**
G	**12**	**14**	**12**	**14**	**0**
H	**6**	**7**	**13**	**14**	**7**

Project #2:

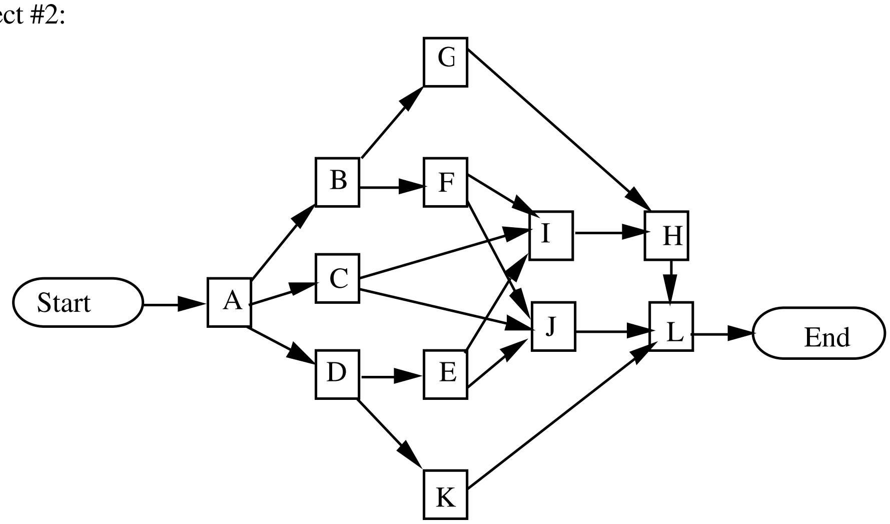
Activity	ES	EF	LS	LF	TF
A	**0**	**3**	**0**	**3**	**0**
B	**3**	**6**	**3**	**6**	**0**
C	**3**	**7**	**6**	**10**	**3**
D	**3**	**6**	**4**	**7**	**1**
E	**6**	**9**	**7**	**10**	**1**
F	**6**	**10**	**6**	**10**	**0**
G	**6**	**11**	**10**	**15**	**4**
H	**15**	**21**	**15**	**21**	**0**
I	**10**	**15**	**10**	**15**	**0**
J	**10**	**14**	**17**	**21**	**7**
K	**6**	**8**	**19**	**21**	**13**
L	**21**	**27**	**21**	**27**	**0**

c. Project #1: A-B-D-G, Project#2: A-B-F-I-H-L.

d. Project #1:



Project #2:



e. Let T, a random variable, be the competition time of the project.

Critical path is : A-B-F-I-H-L, thus, $E(T) = 27$ and

$Var(T) = (3 \times 25\%)^2 + (3 \times 25\%)^2 + (4 \times 25\%)^2 + (6 \times 25\%)^2 + (5 \times 25\%)^2 + (6 \times 25\%)^2$

$= 8.8175$

That is $\sigma(T) = \sqrt{8.8175}$.

f. $P(T \le 30) = P\left(\frac{T-27}{\sqrt{8.1875}} \le \frac{30-27}{\sqrt{8.1875}}\right) = P\left(\frac{T-27}{\sqrt{8.1875}} \le 1.05\right) = 85\%$.

Note: $\frac{T-27}{\sqrt{8.1875}} \approx$ Normal (0,1).

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Homework #8

(1.) **Integer Programming Formulation.** (Exercise #17, §9.2, page 494 of Winston, O.R., 3rd edition)

A product can be produced on four different machines. Each machine has a fixed setup cost, variable production costs per unit processed, and a production capacity as follows:

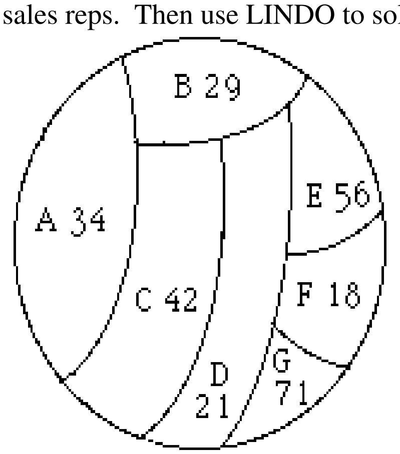
Machine	Fixed cost	Variable cost/unit	Capacity
1	\$1000	\$20	900
2	\$920	\$24	1000
3	\$800	\$16	1200
4	\$700	\$28	1600

A total of 2000 units of the product must be produced. Formulate an integer linear programming model to tell us how to minimize total costs. Solve with LINDO (or other software package.)

Hint: Define X_i = number of units produced on machine #i and Y_i = 1 if machine #i is used, 0 otherwise.

(2.) **Integer Programming Formulation.** (Exercise #20, §9.2, page 494 of Winston, O.R., 3rd edition)

WSP Publishing sells textbooks to college students. WSP has two sales representatives available to assign to the A-G state area. The number of college students (in thousands) in each state is given in the figure below. Each sales representative must be assigned to two adjacent states. For example, a sales rep could be assigned to A and B, but not A and D. WSP's goal is to maximize the total number of students in the states assigned to the sales reps. Formulate an integer linear programming model whose solution will tell you where to assign the sales reps. Then use LINDO to solve your IP.



(3.) **Integer Programming Formulation.** (Exercise #37, §9.2, page 499 of Winston, O.R., 3rd edition)

The Indiana University Business School has two rooms that seat 50 students, one room that seats 100 students, and one room that seats 150 students. Classes are held five hours a day. At present the four types of requests for rooms are listed in the table below:

Type	Size room requested	Hours requested	# of requests
1	50 seats	2,3,4	3
2	150 seats	1,2,3	1
3	100 seats	5	1
4	50 seats	1,2	2

(For example, the type 1 request is for three consecutive hours, namely hours 2,3, & 4, and there are three such requests.)

The business school must decide how many requests of each type should be assigned to each type of room. Penalties per hour for each type of assignment are:

Size	Sizes Used to Satisfy Request		
Requested	50	100	150
50	0	2	4
100	X	0	1
150	X	X	0

(For example, if a type 1 request is assigned to the room with 100 seats, then the penalty will be 2x3=6.)

An "X" means that a request must be satisfied by a room of adequate size. Formulate an integer linear programming model whose solution will tell the business school how to assign classes to rooms in a way that minimizes total penalties. Use LINDO or other software to solve the problem.

Hint: Number the requests (#1-7), and label the rooms A-D (A & B with 50 seats, C with 100, etc.) Define variables such as X_{1A} = 1 if request #1 is assigned to room A, and 0 otherwise.

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Solutions:

1.) See the LINDO outputs are below. Producing 800 units by machine #1 and 1200 units by machine #3, and the total cost is $37000.

```
 MIN     1000 Y1 + 920 Y2 + 800 Y3 + 700 Y4 + 20 X1 + 24 X2 + 16 X3
      + 28 X4
 SUBJECT TO
    2)      X1 +  X2 +  X3 +  X4 =    2000
    3)  -  900 Y1 +  X1 <=   0
    4)  - 1000 Y2 +  X2 <=   0
    5)  - 1200 Y3 +  X3 <=   0
    6)  - 1600 Y4 +  X4 <=   0
 END
 INTEGER-VARIABLES=       4


: go
     LP OPTIMUM FOUND  AT STEP     9

          OBJECTIVE FUNCTION VALUE

  1)         36888.8906

 VARIABLE         VALUE          REDUCED COST
        Y1          0.888889          0.000000
        Y2          0.000000        920.000000
        Y3          1.000000      -5333.333008
        Y4          0.000000        700.000000
        X1        800.000000          0.000000
        X2          0.000000          2.888889
        X3       1200.000000          0.000000
```

```
       X4         0.000000          6.888889


 ROW         SLACK OR SURPLUS     DUAL PRICES
    2)            0.000000        -21.111111
    3)            0.000000          1.111111
    4)            0.000000          0.000000
    5)            0.000000          5.111111
    6)            0.000000          0.000000

 NO. ITERATIONS=       9
  BRANCHES=    0 DETERM.= -9.000E  2
    FIX ALL VARS.(    2)  WITH RC >   700.000
 SET       Y1 TO    1 AT    1 BND=  -37000.000     TWIN=  -37000.000

 NEW INTEGER SOLUTION OF   37000.0     AT BRANCH      1 PIVOT     10

         OBJECTIVE FUNCTION VALUE

  1)         37000.0000

 VARIABLE         VALUE          REDUCED COST
        Y1         1.000000       1000.000000
        Y2         0.000000        920.000000
        Y3         1.000000      -4000.000000
        Y4         0.000000        700.000000
        X1       800.000000          0.000000
        X2         0.000000          4.000000
        X3      1200.000000          0.000000
        X4         0.000000          8.000000


 ROW         SLACK OR SURPLUS     DUAL PRICES
    2)            0.000000        -20.000000
    3)          100.000000          0.000000
    4)            0.000000          0.000000
    5)            0.000000          4.000000
    6)            0.000000          0.000000

 NO. ITERATIONS=      10
  BRANCHES=    1 DETERM.=  1.000E  0
  BOUND ON OPTIMUM:  36888.89
  DELETE       Y1 AT LEVEL     1
    RELEASE FIXED VARS.
     ENUMERATION COMPLETE. BRANCHES=     1  PIVOTS=    13

 LAST INTEGER SOLUTION IS THE BEST FOUND
```

(2.)
Define: XAB=1 if states both A and B are served by one of the representatives, otherwise 0
XAC=1 if states both A and C are served by one of the representatives, otherwise 0
and so on.
The LINDO outputs are shown as below. The optimal solution is to assign these two representatives for states B & E, and D & G with 177 students being served.

```
 MAX      63 XAB + 76 XAC + 71 XBC + 50 XBD + 85 XBE + 63 XCD
      + 77 XDE + 39 XDF + 92 XDG + 74 XEF + 89 XFG
```

```
SUBJECT TO
 2)      XAB +   XAC +   XBC +   XBD +   XBE +   XCD +   XDE +   XDF
     +   XDG +   XEF +   XFG =       2  (two representatives are available)
 3)      XAB +   XAC <=     1  (two representatives cannot serve state A simultaneously)
 4)      XAB +   XBC +   XBD +   XBE <=     1   (the same reason for state B)
 5)      XAC +   XBC +   XCD <=     1   (the same reason for state C)
 6)      XBD +   XCD +   XDE +   XDF +   XDG <=   1  (the same reason for state D)
 7)      XBE +   XDE +   XEF <=     1   (the same reason for state E)
 8)      XDF +   XEF +   XFG <=     1   (the same reason for state F)
 9)      XDG +   XFG <=     1  (the same reason for state G)
END
INTEGER-VARIABLES=       11


: go
    LP OPTIMUM FOUND  AT STEP      5

         OBJECTIVE FUNCTION VALUE

 1)         177.000000

VARIABLE          VALUE          REDUCED COST
      XAB          0.000000         22.000000
      XAC          0.000000          9.000000
      XBC          0.000000         14.000000
      XBD          0.000000         35.000000
      XBE          1.000000          0.000000
      XCD          0.000000         22.000000
      XDE          0.000000          8.000000
      XDF          0.000000         46.000000
      XDG          1.000000         -3.000000
      XEF          0.000000         11.000000
      XFG          0.000000          0.000000


ROW         SLACK OR SURPLUS      DUAL PRICES
   2)          0.000000         85.000000
   3)          1.000000          0.000000
   4)          0.000000          0.000000
   5)          1.000000          0.000000
   6)          0.000000          0.000000
   7)          0.000000          0.000000
   8)          1.000000          0.000000
   9)          0.000000          4.000000

NO. ITERATIONS=        5
 BRANCHES=    0 DETERM.=  1.000E  0
   FIX ALL VARS.(    8)  WITH RC >   3.00000

LP OPTIMUM IS IP OPTIMUM

LAST INTEGER SOLUTION IS THE BEST FOUND
```

(3.) As the LINDO outputs, the optimal solution is:

Assign the request #1 to room A,
Assign the request #2 to room D,
Assign the request #3 to room C,

Assign the request #4 to room B,
and the total penalties is zero.

```
 MIN      6 X1C + 12 X1D +  X3D + 6 X4C + 12 X4D  (total penalties)
 SUBJECT TO
  2) X1C +  X1D +  X1A +  X1B =  1  (request #1 must be satisfied)
  3) X2A +  X2B +  X2C +  X2D =  1  (request #2 must be satisfied)
  4) X3D +  X3A +  X3B +  X3C =  1  (request #3 must be satisfied)
  5) X4C +  X4D +  X4A +  X4B =  1  (request #4 must be satisfied)
  6) X2A =  0  (request #2 cannot be assigned to room A)
  7) X2B =  0  (request #2 cannot be assigned to room B)
  8) X2C =  0  (request #2 cannot be assigned to room C)
  9) X3A =  0  (request #3 cannot be assigned to room A)
 10) X3B =  0  (request #3 cannot be assigned to room B)
 11) X1A +  X2A +  X4A <=  1  (only one of requests 1,2 and 4 can be assigned to room A
      simultaneously)
 12) X1B +  X2B +  X4B <=  1  (only one of requests 1,2 and 4 can be assigned to room B
      simultaneously)
 13) X1C +  X4C +  X2C <=  1  (only one of requests 1,2 and 4 can be assigned to room C
      simultaneously)
 14) X1D +  X4D +  X2D <=  1  (only one of requests 1,2 and 4 can be assigned to room D
      simultaneously)
 END
 INTEGER-VARIABLES=      16

: go

          OBJECTIVE FUNCTION VALUE

  1)       0.000000000

 VARIABLE        VALUE          REDUCED COST
      X1C         0.000000          6.000000
      X1D         0.000000         12.000000
      X3D         0.000000          1.000000
      X4C         0.000000          6.000000
      X4D         0.000000         12.000000
      X1A         1.000000          0.000000
      X1B         0.000000          0.000000
      X2A         0.000000          0.000000
      X2B         0.000000          0.000000
      X2C         0.000000          0.000000
      X2D         1.000000          0.000000
      X3A         0.000000          0.000000
      X3B         0.000000          0.000000
      X3C         1.000000          0.000000
      X4A         0.000000          0.000000
      X4B         1.000000          0.000000


 ROW         SLACK OR SURPLUS     DUAL PRICES
   2)             0.000000          0.000000
   3)             0.000000          0.000000
   4)             0.000000          0.000000
   5)             0.000000          0.000000
   6)             0.000000          0.000000
```

```
    7)            0.000000          0.000000
    8)            1.000000          0.000000
    9)            0.000000          0.000000
   10)            0.000000          0.000000
   11)            0.000000          0.000000
   12)            0.000000          0.000000
   13)            0.000000          0.000000
   14)            0.000000          0.000000

 NO. ITERATIONS=       7
  BRANCHES=    0 DETERM.=  1.000E  0
  BOUND ON OPTIMUM: 0.0000000
    RELEASE FIXED VARS.
     ENUMERATION COMPLETE. BRANCHES=     0  PIVOTS=    11

 LAST INTEGER SOLUTION IS THE BEST FOUND
```

❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍❍

Homework #9

(1.) **Integer Programming Formulation.** (Exercise #24, Chapter 9 Review Problems, page 550 of Winston, O.R., 3rd edition)

PSI believes they will need the amounts of generating capacity shown in the table on the left below during the next five years. The company has a choice of building (and then operating) power plants with the specifications shown in the table on the right below. Formulate an IP to minimize the total costs of meeting the generating capacity requirements of the next five years.

Year	Generating Capacity (million kwh)
1	80
2	100
3	120
4	140
5	160

Plant	Generating Capacity (million kwh)	Construction Cost ($million)	Annual Operating Cost ($million)
A	70	20	1.5
B	50	16	0.8
C	60	18	1.3
D	40	14	0.6

Hints:

- Define the binary variables
  X_{A1} = 1 if Plant A construction is finished at the beginning of year 1,
  0 otherwise
  , etc.

- Constraints must be specified so that a plant cannot contribute generating capacity to the requirements for a year unless its construction was finished at the beginning of that or an earlier year.
- Assume that a plant will be operated each year from its construction until the end of year 5, so that the cost of X_{A1}, for example, would be 20+5(1.5)=27.5 million

(2.) **Markov Chain.** (Modified Exercise #2, Chapter 19 Review Problems, page 999 of Winston, O.R., 3rd edition)

Customers buy cars from three auto companies. Given the company from which a customer last bought a car, the probability that she will buy her next car from each company is as follows:

Last Bought From	Will Buy Next From: Co. 1	Co. 2	Co.3
Company 1	.80	.10	.10

Company 2	.05	.85	.10
Company 3	.05	.15	.80

a. Draw a diagram for a Markov chain model of a Jane Doe's automobile purchase.
b. If Jane currently owns a Company 1 car, what is the probability that ...
   - the car *following* her next car is a Company 1 car?
   - *at least one* of the next two cars she buys will be a Company 1 car?
c. Write the linear equations which determine the steady-state distribution of Jane's automobile ownership.
d. Solve the equations which you have specified. What are the values of π_1, π_2, and π_3?
e. What fraction of the total market (M) would you expect Company 1 to have over a "long" period of time?

At present, it costs Company 1 an average of $5000 to produce a car, and the average price a customer pays for one is $8000. Company 1 is considering instituting a five-year warranty. It estimates that this will increase the cost per car by $300, but a market research survey indicates that the probabilities will change as follows:

	Will Buy Next From		
Last Bought From	Co. 1	Co. 2	Co.3
Company 1	.85	.05	.10
Company 2	.15	.80	.05
Company 3	.15	.10	.75

f. What is the steady-state distribution of Jane Doe's automobile ownership if the five-year warranty were instituted?
g. What fraction of the total market (M) would you expect Company 1 to have over a "long" period of time if they institute the five-year warranty?
h. Should Company 1 institute the five-year warranty?

(Note that the data given in the textbook is invalid, since the sum of the probabilities in each row must be 1.0 but 0.10+0.20+0.75 > 1! Also, since the solution to this problem is included in the back of the book, I have revised some of the probabilities slightly.)

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Homework #10

(1.) **Markov Chain.** (Modification of Exercise #4, §19.5, page 982 of Winston, O.R., 3rd edition)

At the beginning of each year, my car is in good, fair, or broken-down condition. A good car will be good at the beginning of next year with probability 85%, fair with probability 10%, or broken-down with probability 5%. A fair car will be fair at the beginning of the next year with probability 70%, or broken-down with probability 30%. It costs $6000 to purchase a good car; a fair car can be traded in for $2000; and a broken-down car has no trade-in value and must immediately be replaced by a good car. It costs $1000 per year to operate a good car and $1500 to operate a fair car. Should I replace my car as soon as it becomes a fair car, or should I drive my car until it breaks down? Assume that the cost of operating a car during a year depends on the type of car on hand at the beginning of the year (after a new car, if any, arrives).

Define a Markov chain model with three states (Good, Fair, & Broken-down). Assume, as implied by the problem statement, that a car breaks down at the end of a year, and then (at the beginning of the next year) "must immediately be replaced". For each of the two replacement policies mentioned, answer the following:

a. Draw a diagram of the Markov chain and write down the transition probability matrix.

b. Write down the equations which could be solved to obtain the steadystate probabilities.

c. Solve the equations, either manually or using appropriate computer software.

d. Compute the average cost per year for the replacement policy.

Consider now a third replacement policy, in which a broken-down car is replaced with a fair car, costing $2500. In this case, the state "Good" would be a transient state, and so for the purposes of calculating the steadystate cost/year, omit that state and define a Markov chain model with states Fair & Broken-Down. Repeat the above steps for this third policy.

What is the best policy of these three?

(2.) **Markov Chain.** (Modification of Exercise #3, Chapter 19 Review Problems, page 999 of Winston, O.R., 3rd edition)

A baseball team consists of 2 stars, 13 starters, and 10 substitutes. For tax purposes, the team owner must place a value on the players. The value of each player is defined to be the total value of the salary he will earn until retirement. At the beginning of each season, the players are classified into one of four categories:

Category 1: Star (earns $1 million per year).
Category 2: Starter (earns $400,000 per year).
Category 3: Substitute (earns $100,000 per year).
Category 4: Retired (earns no more salary).

Given that a player is a star, starter, or substitute at the beginning of the current season, the probabiliites that he will be a star, starter, substitute, or retired at the beginning of the next season are as follows:

This Season	Next Season: Star	Starter	Substitute	Retired
Star	0.50	0.30	0.15	0.05
Starter	0.20	0.50	0.20	0.10
Substitute	0.05	0.15	0.50	0.30
Retired	0	0	0	1

a. Determine, for each of the current players on the team, the expected length of their playing career, and the expected number of years in each category.

b. Determine the value of each of the team's current players, and the total value of the team.

c. Assuming that the total number of players on the team must remain constant (at 25), a player must be replaced when he retires. Suppose that the team owner's policy is to replace a retiring player with a player in the "Starter" category. Assuming a steadystate condition, what is the fraction of the team in each category? Will the average total annual salary for the team be greater or smaller than that of the current team?

Hint: Why does your original Markov chain model have no steady state? Define a new Markov chain model having only three states (categories 1, 2, & 3), with a retirement resulting in a transition into category #2. Think of this as being the state of a uniform number, rather than of the individual wearing that number. Why does this second Markov chain model have a steady state? Which Markov chain is "regular"?

d. What is the expected number of years required for a "substitute" player (i.e., that uniform number) to develop into a "star" player?

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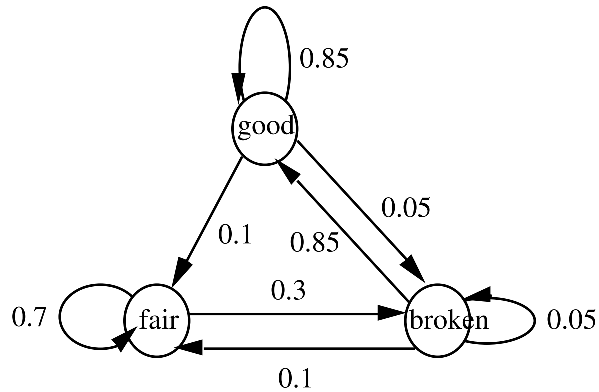
Solutions:

(1.) Throughout this problem, we assume, **state 1: Good, state 2: Fair, and state 3: Brokendown**.

For Policy 1: Replace the old car until it is brokendown.

(a).

```
Transition Probability Matrix
-----------------------------
 f|    to
 r|    --
 o|  1   2    3
 m|---- --- ----
 1|0.85 0.1 0.05
 2|0    0.7 0.3
 3|0.85 0.1 0.05
```



(b).

$$\pi_1 = 0.85\pi_1 + 0.85\pi_2$$
$$\pi_2 = 0.1\pi_1 + 0.7\pi_2 + 0.1\pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$

(c).

Steady State Distribution

i	P{i}
1	0.6375
2	0.25
3	0.1125

(d).

i	Pi	C	Pi*C
1	0.6375	1000	637.5
2	0.25	1500	375
3	0.1125	7000	787.5

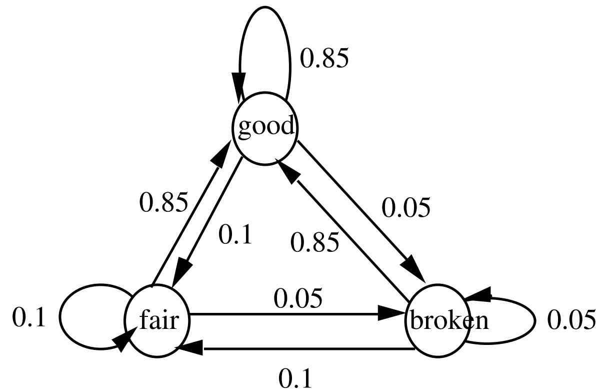
the average cost/period in steady state is 1800.

For Policy 2: Replace the car when it is fair.

(a).

Transition Probability Matrix

from	1	2	3
1	0.85	0.1	0.05
2	0.85	0.1	0.05
3	0.85	0.1	0.05



(b).

$$\pi_1 = 0.85\pi_1 + 0.85\pi_2 + 0.85\pi_3$$
$$\pi_2 = 0.1\pi_1 + 0.1\pi_2 + 0.1\pi_3$$
$$\pi_1 + \pi_2 + \pi_3 = 1$$

(c).

Steady State Distribution

i	P{i}
1	0.85
2	0.1
3	0.05

(d).

P1

i	Pi	C	Pi*C
1	0.85	1000	850
2	0.1	5000	500
3	0.05	7000	350

the average cost/period in steady state is 1700.

Consider now a third replacement policy, in which a broken-down car is replaced with a fair car, costing $2500. In this case, the state "Good" would be a transient state, and so for the purposes of calculating the steady state cost/year, omit that state and define a Markov chain model with states Fair & Broken-Down. Repeat the above steps for this third policy.

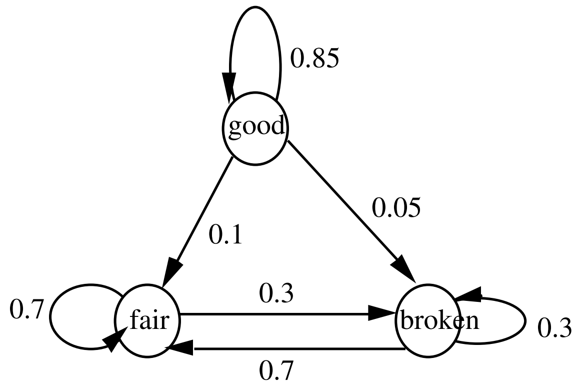
Solutions.

For policy 3:

(a).

Transition Probability Matrix

from	1	2	3
1	0.85	0.1	0.05
2	0	0.7	0.3
3	0	0.7	0.3



(b). Since state 1 is transient state, therefore $\pi_1 = 0$.

$$\pi_2 = 0.7\pi_2 + 0.7\pi_3$$

$$\pi_2 + \pi_3 = 1$$

(c).

```
                 P1
                 --
     Steady State Distribution
     -------------------------
          i P{i}
          1 3.27386909E-16
          2 7.00000000E-1
          3 3.00000000E-1
```

(d).

```
                      P1
                      --
 i       Pi          C       Pi*C
 - -------------- ---- --------------
 1 3.27386909E-16 1000 3.27386909E-13
 2 7.00000000E-1  1500 1.05000000E3
 3 3.00000000E-1  4000 1.20000000E3
the average cost/period in steady state is 2250
```

What is the best policy of these three?

Solutions. Policy 2 is the best one, i.e, to replace the fair car with a good one.

(2.) By the outputs of the software in HP machine,

```
A = Absorption Probabilities
____________________________
            |
           f|
           r|
           o|  4
           m|  -
           1|  1
           2|  1
           3|  1
E = Expected No. Visits to Transient States
-------------------------------------------
       f|
```

r			
o	1	2	3
m	---	-----------	-----------
1	3.2	2.509090909	1.963636364
2	1.6	3.527272727	1.890909091
3	0.8	1.309090909	2.763636364

Thus,
the expected length for star is 3.2+2.51+1.96=7.67 (years),
the expected length for starter is 1.6+3.53+1.89=7.02, and
the expected length for substitute is 0.8+1.31+2.76=4.87.

b.
The value of star =1(3.2)+(0.4)2.51+(0.1)1.96=4.4 (million).
The value of starter =1(1.6)+(0.4)3.53+(0.1)1.89=3.143 (million).
The value of substitute =1(0.8)+(0.4)1.31+(0.1)2.76=1.6 (million).
Thus the total value for the team = 2(4.4)+13(3.143)+10(1.6)=66 (million)

c.

Transition Probability Matrix

f r o m	1	2	3
1	0.5	0.35	0.15
2	0.2	0.6	0.2
3	0.05	0.45	0.5

Steady State Distribution

i	P{i}
1	0.2279792746
2	0.5025906736
3	0.2694300518

That is, 22.79% for category 1, 50.26% for category 2, and 26.94% for category 3.
The average annual salary for the current team is

$$\frac{4.4\times 2}{7.67}+\frac{3.143\times 13}{7.02}+\frac{1.6\times 10}{4.87}=10.3 \text{ (million)}$$

On the other hand, for the new model with assumption in (c), the average annual salary for the team is

$$25\pi_1(1)+25\pi_2(0.4)+25\pi_3(0.1)=11.5 \text{ (million).}$$

Hence, the annual salary for new model is greater than that of the current team.

d.

Mean First Passage Times

f r o m	to -- 1	2	3

1	4.386363636	2.680412371	5.769230769
2	6.363636364	1.989690722	5.384615385
3	7.727272727	2.268041237	3.711538462

Thus, the answer is 7.72727 years.

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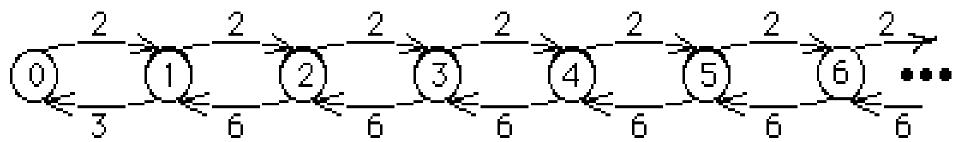
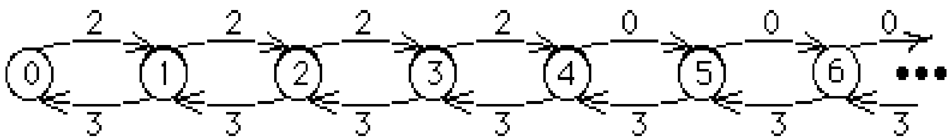
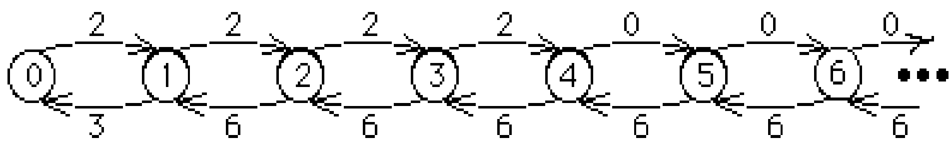
Homework #11

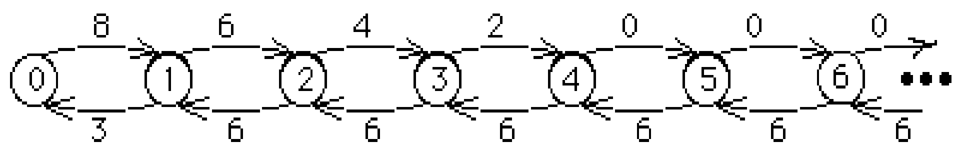
(1.) For each diagram of a Markov model of a queue in (a) through (f) below, indicate the correct Kendall's classification from among the following choices :

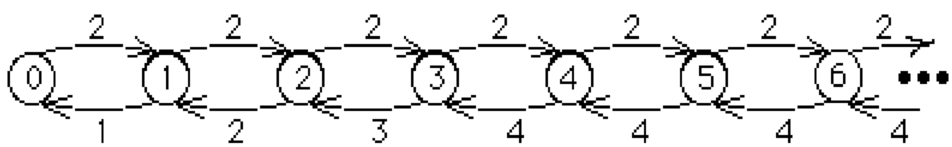
(1) M/M/1	(2) M/M/2	(3) M/M/1/4
(4) M/M/4	(5) M/M/2/4	(6) M/M/2/4/4
(7) M/M/1/2/4	(8) M/M/4/2	(9) M/M/4/4
	(10) none of the above	

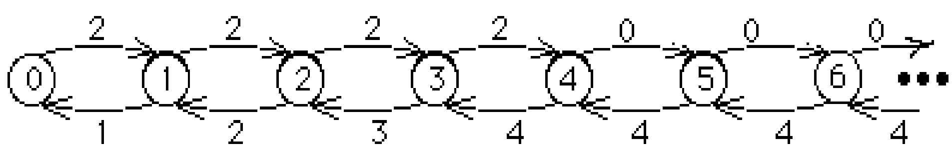
____(a) 

____(b) 

____(c) 

____(d) 

____(e) 

____(f) 

(2.) Two mechanics work in an auto repair shop, with a capacity of 3 cars. If there are 2 or more cars in the shop, each mechanic works individually, each completing the repair of a car in an average of 4 hours (the actual time being random with exponential distribution). If there is only one car in the shop, both mechanics work together on it, completing the repair in an average time of 3 hours (also exponentially distributed). Cars arrive randomly, according to a Poisson process, at the rate of one every two hours when there is at least one idle mechanic, but one every 4 hours when both mechanics are busy. If 3 cars are in the shop, no cars arrive.

a. Draw a transition diagram, with rates included, for this system. Is it a birth-death process?
b. Compute the steady-state probabilities.
c. What fraction of the day will both mechanics be idle?
d. What fraction of the day will both mechanics be working on the same car?

e. What is the average number of cars in the shop?
f. How many cars can be expected to arrive during an 8-hour day?
g. What is the average total time spent by a car in the shop (including both waiting and repair time)?
h. What is the average waiting time spent by a car in the shop?

(3.) A small grocery store has only one check-out counter. Customers arrive at the check-out at a rate of one per 2 minutes. The grocery store clerk requires an average of one minute and 30 seconds to serve each customer. However, as soon as the waiting line exceeds 2 customers, including the customer being served, the manager helps by packing the groceries, which reduces the average service time to one minute. Assume a Poisson arrival process and exponentially-distributed service times.

a. Draw the flow diagram for a birth-death model of this system.
b. Compute the steady-state distribution of the number of customers at the check-out.
c. What fraction of the time will the check-out clerk be idle?
d. What is the expected number of customers in the check-out area?
e. What is the expected length of time that a customer spends in the check-out area?
f. Suppose that the store is being remodeled, and space is being planned so that the waiting line does not overflow the space allocated to it more than 1 percent of the time, and that 4 feet must be allocated per customer (with cart). How much space should be allocated for the waiting line?
g. What fraction of the time will the manager spend at the check-out area?

(4.) A small bank has two tellers, one for deposits and one for withdrawals. The service time for each teller is exponentially distributed, with a mean of 1 minute. Customers arrive at the bank according to a Poisson process, with mean rate of 40 per hour. Each customer is equally likely to be a depositer or a withdrawer (but not both!) The bank is thinking of changing the current arrangement to allow each teller to handle both deposits and withdrawals. The bank would expect that each teller's mean service time would increase to 1 minute and 15 seconds, but it hopes that the new arrangement would prevent long lines in front of one teller when the other is idle, a situation that occurs from time to time under the current setup.

a. From the data given for the current setup with separate tellers for deposits and withdrawals, estimate:
   (i.) the fraction of the time which each teller is idle
   (ii.) the expected time which a customer spends in a queue
b. Estimate the same values (i) and (ii) above for the proposed setup.

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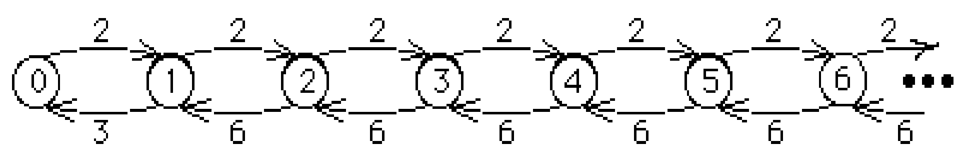
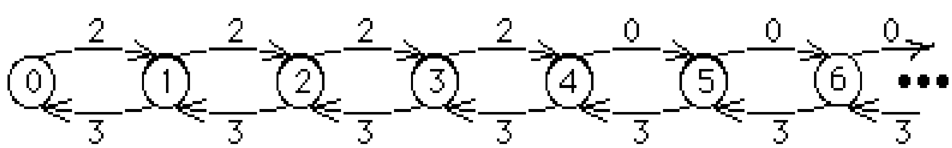
Solutions:

(1.)

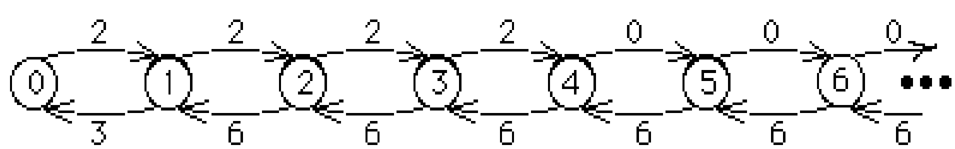
__2__ (a)

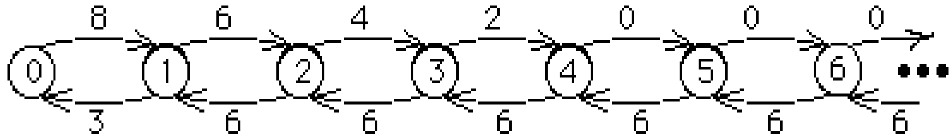


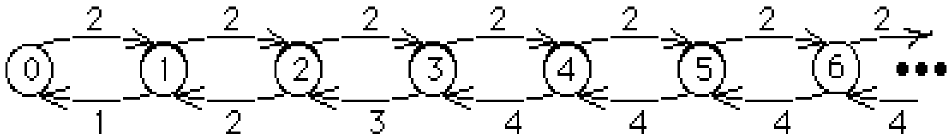
__3__ (b)

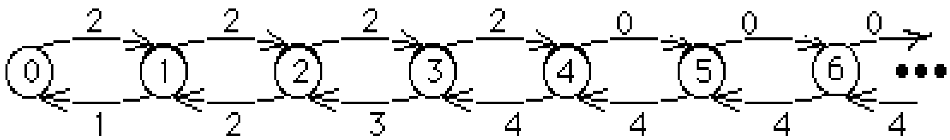


__5__ (c)

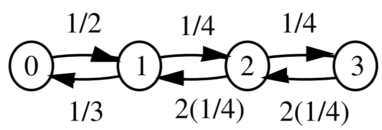


6 (d) 

4 (e) 

9 (f) 

(2.) a.



It is a birth-death process.

b.
Using the formula for birth-death process, we obtain
$\pi_0 = 0.2759,\ \pi_1 = 0.4138,\ \pi_2 = 0.2069,\ \pi_3 = 0.1034$.

c. 27.59%.

d. 41.38%.

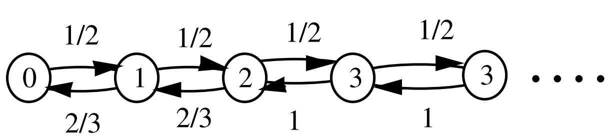
e. $L = 0\pi_0 + 1\pi_1 + 2\pi_2 + 3\pi_3 = 1.14$ (cars).

f. $\underline{\lambda} = (1/2)\pi_0 + (1/4)\pi_1 + (1/4)\pi_2 = 0.293$ cars/per hour.
$8\underline{\lambda} = 8(0.293) = 2.345$ cars.

g. $W = \dfrac{L}{\underline{\lambda}} = \dfrac{1.14}{0.293} = 3.89$ (hours).

h. $W_q = \dfrac{L_q}{\underline{\lambda}} = \dfrac{0\pi_0 + 0\pi_1 + 0\pi_2 + 1\pi_3}{0.293} = 0.353$ (hour).

(3.)
a.



b. By formula we have

$$\pi_0 = 0.347826,\ \pi_1 = 0.260870,\ \pi_2 = 0.195652$$
$$\pi_3 = 0.097826,\ \pi_4 = 0.048913,\ \pi_5 = 0.024457$$

etc.

c. 34.78%.

d. $L = \sum_{i=0}^{\infty} i\pi_i = 1.4267$ (By the software in HP workstation).

e. $W = \frac{L}{\lambda} = \frac{1.4267}{1/2} = 2.85$ (min).

f. Since $P(\#\text{ of customers in system} \le 6) = 98.78\%$ and $P(\#\text{ of customers in system} \le 7) = 99.39\%$, therefore 6(4)=24 feet is needed for the capacity of queue (not including the one being served).

g. $1 - \pi_0 - \pi_1 - \pi_2 = 19.56\%$.

(4.)

a. From the data given for the current setup with separate tellers for deposits and withdrawals, estimate:
   (i.) the fraction of the time which each teller is idle
   (ii.) the expected time which a customer spends in a queue

b. Estimate the same values (i) and (ii) above for the proposed setup.

Solution.

For the current system, we have **two M/M/1** with $\lambda = 20/\text{hr}$ and $\mu = 40/hr$, respectively.

a. (i) $P(\text{teller is idle}) = \pi_0 = 66.67\%$, $P(\text{Both tellers are idle}) = (\pi_0)^2 = 44.44\%$.

(ii) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1}{120}$ hour $= 0.5$ (min).

For proposed system, we have **one M/M/2** with $\lambda = 40/\text{hr}$ and $\mu = 48/\text{hr}$.

(i) $P(\text{One teller is idle and another is busy}) = \pi_1 = 34.31\%$, and
$P(\text{Both tellers are idle}) = \pi_0 = 41.17\%$

(ii) $W_q = \frac{L_q}{\lambda} = \frac{0.1751}{0.67} = 0.263$ (min).

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Homework #12

1. *Match Problem.* Suppose that there are 27 matches originally on the table, and you are challenged by your dinner partner to play this game. Each player must pick up either 1, 2, 3, or 4 matches, with the player who picks up the last match pays for dinner.

   a. If you have the choice, would you choose to go first or second?

   b. Assuming that you may choose whether to take the first move or to let your opponent go first, can you be certain of winning the game?

   c. What is your optimal strategy? (Describe your decision rule as concisely as you can.)

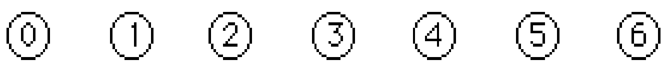
2. *Auto Replacement Problem.* Suppose that a new car costs $15,000 and that the annual operating cost and resale value of the car are as shown in the table below:

Age of Car (years)	Resale Value	Operating Cost
1	$11000	$400 (year 1)
2	$9000	$600 (year 2)
3	$7500	$900 (year 3)
4	$5000	$1200 (year 4)
5	$4000	$1600 (year 5)
6	$3000	$2200 (year 6)

(The operating cost specified above is for the year which is ending.) If I have a new car now (time 0, and this initial car is assumed to be "free", i.e. a "sunk" cost), determine a replacement policy that minimizes the net cost of owning and operating a car for the next six years. *(Hint: See solutions of HW#12 of last year, which solved the same problem but with different costs.)*

a. What is my minimum total cost for the six year period?

b. At what age should I replace my initial car? My second car? Indicate the optimal "path" from "0" to "6" on the diagram below, where, as in the notes, visiting a node "t" means that you replace your car at the end of the t^{th} year.



c. If I were to replace my *initial* car one year earlier than is optimal (but thereafter behave rationally and do the best I can under the new circumstances), how much additional cost must I pay? *(Hint: this should not require any additional computations!)*

3. **Deterministic Production Planning:** Consider a production/inventory system with the following characteristics:
   - Maximum inventory level is 8
   - Storage costs are $1/week per unit in inventory at the beginning of the week
   - Initially, the inventory contains 2 units.
   - Maximum production level is 6/week
   - Setup cost for production is $10 in each week in which production is scheduled
   - Marginal production costs (costs in excess of setup cost) are $2 per unit
   - Demand in each of the next 8 weeks is assumed to be known and must be satisfied They are (where t =stage="weeks remaining", i.e., D[8] = first week demand, ... D[1]= 8th week demand):

   Demands

t	8	7	6	5	4	3	2	1
D[t]	5	1	4	4	3	2	3	1

   - Anything produced during a certain week (plus anything in inventory at the beginning of the week) may be used to satisfy demand during that week, while anything in excess of the maximum inventory level (8) at the end of the week, after demand is satisfied, is discarded)

- At the end of the 8 weeks, a salvage value of $3 per unit remaining in inventory is recovered.

The following tables were computed for this problem in the solution to HW#12, Fall '92, with the "finger" indicating the optimal production schedule:

Stage 8:

State	Optimal Values	Optimal Decisions	Resulting State
0	96.00	6	1
1	95.00	5	1
☞ 2	94.00	4	1
3	93.00	3	1
4	92.00	2	1
5	85.00	0	0
6	80.00	0	1
7	81.00	0	2
8	80.00	0	3

Stage 7:

State	Optimal Values	Optimal Decisions	Resulting State
0	80.00	6	5
☞ 1	74.00	0	0
2	74.00	0	1
3	72.00	0	2
4	72.00	0	3
5	68.00	0	4
6	64.00	0	5
7	65.00	0	6
8	66.00	0	7

Stage 6:

State	Optimal Values	Optimal Decisions	Resulting State
☞ 0	73.00	5	1
1	72.00	4	1
2	69.00	6	4
3	68.00	5	4
4	63.00	0	0
5	58.00	0	1
6	58.00	0	2
7	58.00	0	3
8	53.00	0	4

Stage 5:

State	Optimal Values	Optimal Decisions	Resulting State
0	59.00	4	0
☞ 1	53.00	6	3
2	52.00	5	3
3	51.00	4	3
		6	5
4	45.00	0	0
5	45.00	0	1
6	45.00	0	2
7	37.00	0	3
8	38.00	0	4

Stage 4:

State	Optimal Values	Optimal Decisions	Resulting State
0	41.00	5	2
1	40.00	4	2
2	39.00	3	2
		6	5
☞ 3	30.00	0	0
4	30.00	0	1
5	26.00	0	2
6	27.00	0	3
7	28.00	0	4
8	23.00	0	5

Stage 3:

State	Optimal Values	Optimal Decisions	Resulting State
☞ 0	27.00	6	4
1	26.00	5	4
2	21.00	0	0
3	21.00	0	1
4	21.00	0	2
5	15.00	0	3
6	11.00	0	4
7	11.00	0	5
8	11.00	0	6

Stage 2:

State	Optimal Values	Optimal Decisions	Resulting State
0	19.00	4	1
		5	2
		6	3
1	18.00	3	1
		4	2
		5	3
		6	4
2	17.00	2	1
		3	2
		4	3
		5	4
		6	5
3	10.00	0	0
4	5.00	0	1
5	4.00	0	2
6	3.00	0	3
7	2.00	0	4
8	1.00	0	5

Stage 1:

State	Optimal Values	Optimal Decisions	Resulting State
0	7.00	6	5
1	1.00	0	0
2	¯1.00	0	1
3	¯3.00	0	2
4	¯5.00	0	3
5	¯7.00	0	4
6	¯9.00	0	5
7	¯11.00	0	6
8	¯13.00	0	7

a. Suppose that, upon recounting the initial inventory, you find that you had previously overlooked three units, so that you actually have **5** units in stock. What is now your optimal total cost for the 8-week period?

b. If you have 5 units in stock initially, what is now your optimal production schedule? That is, in which weeks should you produce, and how much?

c. Suppose that, in the third week (i.e. stage #6), you have an unexpected cancellation of an order for one unit. Does this change your production schedule for the remaining 6 weeks? If so, what is the new production schedule?

Hint: To answer all of these questions, use the original tables above. (No recomputation of these tables is required.)

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Solutions:

1.

a. Go first.

b. Yes. Choose to go first.

c. Go first and let force the my friend to choose from 26, 21,16,11,6,1, respectively. That is, I take one match first, so the total matches are 26. Then if my friend takes 1, I take 4. If he takes 2, then I take 3 and so on. Thus, the remaining matches will be 21. Therefore I will win. The strategy is to keep (# he takes+# I take =5 per run when I choose one match first).

2.

a. $3800.

Let G(x)=minimum total cost until the end of period 6, given a new car at the end of time x.

Stage 6: G(6)=0

Stage 5:

x	c	c+G

6	-10600	-10600

Thus, G(5)=-10600, x(5)=6

Stage 4:

x	c	c+G
5	4400	-6200
6	-8000	-8000

Thus, G(4)=-8000, x(4)=6

Stage 3:

x	c	c+G
4	4400	-3600
5	7000	-3600
6	-5600	-5600

Thus, G(3)=-5600, x(3)=6

Stage 2:

x	c	c+G
3	4400	-1200
4	7000	-1000
5	9400	-1200
6	-1900	-1900

Thus, G(2)=-1900, x(2)=6

Stage 1:

x	c	c+G
2	4400	2500
3	7000	1400
4	9400	1400
5	13100	2500
6	700	700

Thus, G(1)=700, x(1)=6

Stage 0:

x	c	c+G
1	4400	5100
2	7000	5100
3	9400	3800
4	13100	5100
5	15700	5100
6	3900	3900

Thus, G(0)=3800, x(0)=3

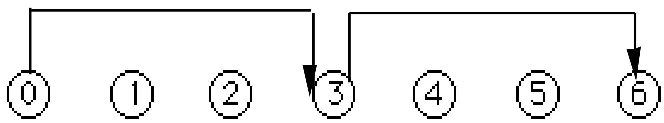
Summary

Year	x	G
0	3	3800
1	6	700
2	6	-1900
3	6	-5600
4	6	-8000
5	6	-10600

6	--	--

Therefore, the minimum total cost is 3800.

b. At what age should I replace my initial car? My second car? Indicate the optimal "path" from "0" to "6" on the diagram below, where, as in the notes, visiting a node "t" means that you replace your car at the end of the t^{th} year.



Replace first car after 3 years, and then use it until end of the period.

c. If x(0)=2, then cost =5100. The additional cost =5100-3800=1300.

3. a. $85.

b.

stage	week	x (# needs to produce)
8	1	0
7	2	6
6	3	0
5	4	6
4	5	0
3	6	6
2	7	0
1	8	0

c. That is D[6] is changed from 4 to 3.

stage	week	x (# needs to produce)
8	1	0
7	2	6
6	3	0
5	4	*5
4	5	0
3	6	6
2	7	0
1	8	0

The units to be produced for week 4 is changed from 6 to 5.

Hint: To answer all of these questions, use the original tables above. (No recomputation of these tables is required.)

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Homework #13

1. *Match Problem.* Suppose that there are 15 matches originally on the table, and you are challenged by your dinner partner to play a variation of the game in last week's homework problem, a variation which makes this a game of chance. Each player must, when it is his/her turn, choose either to
   a) pick up one match

or b) toss a fair coin, and pick up one match if "heads", two matches if "tails", with the player who picks up the last match paying for dinner.
Define P(i) = maximum probability of winning, if i matches remain on the table.

a. If you have the choice, would you choose to go first or second?

b. Assuming that you may choose whether to take the first move or to let your opponent go first, what is your probability of winning the game?

c. Describe your optimal strategy by completing the table below:

i=# matches remaining	Probability of winning if Choice (a)	Choice (b)	P(i)	Optimal Choice
1	0	0	0	a
2	1	0.5	1	a
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				

2. **Optimization of System Reliability:** A system consists of 3 devices, each subject to possible failure, all of which must function in order for the system to function. In order to increase the reliability of the system, redundant units may be included, so that the system continues to function if at least one of the redundant units remains functional. The data are:

Device	Reliability (%)	Weight (kg.)
1	75	1
2	80	2
3	90	3

If we include a single unit of each device, then the system reliability will be the product of the device reliabilities, i.e., (0.75)(0.80)(0.90) = 53.55%. However, by including redunant units of one or more devices, we can substantially increase the reliability. Thus, for example, if 2 redundant units of device #1 were included, the reliability of device #1 will be increased from 75% to $1 - (0.25)^2 = 93.75\%$. That is, the probability that both units fail, assuming independent failures, is $0.25 \times 0.25 = .0625$. Suppose that the system may weigh no more than 10 kg. (Since at least one of each device must be included, a total of 6 kg, this leaves 4 kg available for redundant units.) Assume that no more than 3 units of any type need be considered. We wish to compute the number of units of each device type to be installed in order to maximize the system reliability, subject to the maximum weight restriction.

Assume that the devices are considered in the order: #3, #2, and finally, #1. The optimal value function is defined to be:

$F_n(S)$ = maximum reliability which can be achieved for devices #n, n-1, ... 1, given that the weight used by these devices cannot exceed S (the state variable)

The optimal value for the problem is therefore given by $F_3(10)$. The computation is done in the backward order, i.e., first the optimal value function $F_1(S)$ is computed for each value of the available weight S, then $F_2(S)$, until finally $F_3(10)$ has been computed.

The reliability of each device as a function of the number x of redundant units is $1 - (1-R_i)^x$ where R_i is the reliability of a single unit of device i:

Reliability (%) vs # redundant units

i	1	2	3
1	70	91	97.3
2	85	97.75	99.6625
3	90	99	99.9

The following output is produced during the solution of the problem:

Stage 1

s \ x	1	2	3
1	0.7000		
2	0.7000	0.9100	
3	0.7000	0.9100	0.9730
4	0.7000	0.9100	0.9730
5	0.7000	0.9100	0.9730
6	0.7000	0.9100	0.9730
7	0.7000	0.9100	0.9730
8	0.7000	0.9100	0.9730
9	0.7000	0.9100	0.9730
10	0.7000	0.9100	0.9730

Stage 2

s \ x	1	2	3
3	0.5950		
4	0.7735		
5		0.6843	
6	0.8270	0.8895	
7	0.8270	0.9511	0.6976
8	0.8270	0.9511	0.9069
9	0.8270	0.9511	0.9697
10	0.8270	0.9511	0.9697

Stage 3

s \ x	1	2
6	0.5355	
7	0.6961	
8	0.7443	
9	0.8006	0.5891
10	0.8560	

a. Fill in the two blanks in the tables above.

The tables showing the values of f_3, f_2, and f_1 are:

Stage 1

State	f_1	Optimal Decisions	Resulting State
1	0.7000	1	0
2	0.9100	2	0
3	0.9730	3	0
4	0.9730	3	1
5	0.9730	3	2
6	0.9730	3	3
7	0.9730	3	4
8	0.9730	3	5
9	0.9730	3	6
10	0.9730	3	7

Stage 2

State	f_2	Optimal Decisions	Resulting State
3	0.5950	1	1
4	0.7735	1	2
5	0.8270	1	3
6			
7	0.9511	2	3
8	0.9511	2	4
9	0.9697	3	3
10	0.9697	3	4

Stage 3

State	f_3	Optimal Decisions	Resulting State
6	0.5355	1	3
7	0.6961	1	4
8	0.7443	1	5
9	0.8006	1	6
10	0.8560	1	7

b. Fill in the three blanks in the table above for stage #2.

c. What is the optimal system reliability if 10 kg. is available for the devices ?

d. What is the optimal number of units of each device if 10 kg. is available?

e. What is the optimal system reliability if only 9 kg. were available for the devices? If only 9 kg. were available, how many units of each device should be included in the system?

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