COMPUTATIONAL FINANCE

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Exercise 4

Set-up. Consider a standard European call option in a Black-Scholes model with strike price K and maturity T. Its price $u(S,t)$ satisfies the Black-Scholes PDE

$$\begin{cases} \partial_t u + rS\partial_S u + \frac{1}{2}\sigma^2 S^2 \partial_S^2 u - ru = 0, \\ u(T,S) = (S-K)^+ . \end{cases}$$

Here, the natural spatial domain for u is $[0,\infty[$. As indicated in the lecture, we transform u according to

$$v(x,\tau) = \frac{u(S,t)}{K} \exp\left(\frac{1}{2}(q-1)x + \left(\frac{1}{4}(q-1)^2 + q\right)\tau\right),$$

$$x = \log\left(\frac{S}{K}\right),$$

$$\tau = (T-t)\,\frac{\sigma^2}{2},$$

with the short-hand notation $q = 2r/\sigma^2$. Then one can show that v satisfies the heat equation in the new variables τ (dimension-free time to maturity) and x (log-moneyness):

$$\begin{cases} \partial_\tau v = \partial_x^2 v, \\ v(x,0) = \left(\mathrm{e}^{\frac{x}{2}(q+1)} - \mathrm{e}^{\frac{x}{2}(q-1)}\right)^+ . \end{cases}$$

Here, the natural spatial domain is $\mathbb{R}$. Moreover, we recall from the lecture that we can use the following approximations to the solution far away from $x=0$:

$$v(x,\tau) \approx 0, \quad x \ll -1,$$

$$v(x,\tau) \approx \exp\left(\frac{1}{2}(q+1)x + \frac{1}{4}(q+1)^2\tau\right), \quad x \gg 1.$$

Parameters. We choose the following parameters for the option: $r = 0.1$, $T = 1.0$, $\sigma = 0.3$, $S_0 = 10$, $K = 10$.

Tasks. Implement the implicit finite difference scheme (i.e., backward Euler scheme) and the Crank-Nicolson scheme and compute the option prices in this way. Moreover, study the empirical convergence rates.

You need to choose the following numerical parameters:

- The truncation values for the infinite domain $x_{\min} < 0 < x_{\max}$. To simplify the analysis, you should use the same values for all the runs of the code (i.e., for all the values of $\Delta\tau$ and h to be discussed below). Hence, $x_{\min}$ and $x_{\max}$ must be sufficiently far from 0 so that the observed errors are not significantly influenced by the truncation of the domain.

- The space grid $x_{\min} = x_0 < \cdots < x_{N+1} = x_{\max}$. We suggest to use a uniform grid. In particular, the mesh of the space grid is given by $h = \frac{x_{\max} - x_{\min}}{N+1}$.
- The time-grid $0 = \tau_0 < \cdots < \tau_L = \tau_{\max} = T\frac{\sigma^2}{2}$. Again, we suggest to use a uniform grid with mesh $\Delta\tau = \tau_{\max}/L$.

Hints.

- Use the Black-Scholes formula as reference value to obtain the true errors of your calculation.
- In order to study the convergence rates, choose an appropriate sequence N_m and L_m, $m = 1, \ldots, M$, and compare the corresponding errors e_m with $\Delta\tau_m$ and h_m^2 for the implicit scheme and with $\Delta\tau_m^2$ and h_m^2 for the Crank-Nicolson scheme, respectively.
- It would be wise to choose $\Delta\tau$ proportional to h^2 for the implicit scheme and $\Delta\tau$ proportional to h for the Crank-Nicolson scheme. (Why?)
- Visualize the convergence by our usual log-log plots.
- If the plots and/or the convergence analysis are not convincing, check that M, N_m and L_m are large enough. Moreover, check that $|x_{\min}|$ and $x_{\max}$ are large enough so as not to interfere with the observed errors.
- As indicated in the lecture, this exercise corresponds to identifying the coefficients of a system of linear equations and solving this system.

Submit.

- The source code (in `scilab/matlab/C/...`). The source code should include sufficient documentation.
- A PDF file explaining how the code was developed and discussing the results (preferably written in LaTeX).
- Submit everything per e-mail to papapan@math.tu-berlin.de in a zip file named: `Exercise_4_Surname_Name`.
- Deadline: **August 21, 2015**.