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# ON THE GREATEST PRIME FACTORS OF DECOMPOSABLE FORMS AT INTEGER POINTS

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## 1. Introduction

Let $f\in \boldsymbol{Z}[x, y]$ be a binary form and assume that among the linear factors in the factorization of $f$ at least three are distinct. Mahler [12] proved that $P(f(x, y))\to\infty$ if $X=\max(|x|, |y|)\to\infty$ with $x, y\in \boldsymbol{Z}$, $(x, y)=1$, where $P(n)$ denotes the greatest prime factor of $n$. Mahler's work was generalized by Parry [14]. For irreducible forms $f$ Coates [4] improved Mahler's result by showing that if $\alpha=1/4$, then for any coprime integers $x, y$

$$P(f(x, y)) > c_1(\log\log X)^{\alpha}, \quad X \geqq X_1, \tag{1}$$

where $c_1>0$ and $X_1>0$ depend only on $f$ and can be given explicitly. Sprindžuk [21], [22] established (1) with $\alpha=1$ for all such forms of degree at least 5 and for so-called non-exceptional forms of degree 4. Kotov [11] generalized Sprindžuk's result to binary forms with algebraic integer coefficients. Shorey, van der Poorten, Tijdeman and Schinzel [20] proved that if $f\in \boldsymbol{Z}[x, y]$ has at least three distinct linear factors in its factorization and $\alpha=1$, then (1) holds for any $x, y\in \boldsymbol{Z}$ with $(x, y)=d$, where $d$ is a fixed positive integer.

Schlickewei [17], [18] proved that for a large class of norm forms $F\in \boldsymbol{Z}[x_1, \dots, x_m]$ in $m\geqq 2$ variables and for $\mathbf{x}=(x_1, \dots, x_m)\in \boldsymbol{Z}^m$ with relatively prime components, $P(F(\mathbf{x}))\to\infty$ as $|\mathbf{x}|=\max(|x_1|, \dots, |x_m|)\to\infty$. For index forms $F\in \boldsymbol{Z}[x_1, \dots, x_m]$ Trelina [24] showed that

$$P(F(\mathbf{x})) > c_2(\log\log|\mathbf{x}|\log\log\log|\mathbf{x}|)^{1/2}, \quad |\mathbf{x}| \geqq X_2.$$

Independently, for discriminant forms and index forms $F\in \boldsymbol{Z}[x_1, \dots, x_m]$

$$P(F(\mathbf{x})) > c_3\log\log|\mathbf{x}|, \quad |\mathbf{x}| \geqq X_3, \tag{2}$$

have been established by Papp and the author [8]. Here $\mathbf{x}\in \boldsymbol{Z}^m$ with $(x_1, \dots, x_m)=1$ and $c_2, c_3, X_2, X_3$ are effectively computable positive numbers depending only on $F$. Recently the author [10] proved (2) for a wide class of irreducible norm forms $F(\mathbf{x})$ in $m\geqq 2$ variables (including all binary forms). In [8] and [10] our estimates are established for forms $F(\mathbf{x})\in \boldsymbol{Z}_L[x_1, \dots, x_m]$ at integer points $\mathbf{x}\in \boldsymbol{Z}_L^m$, where $\boldsymbol{Z}_L$ denotes the ring of integers of an arbitrary but fixed algebraic number field $L$.

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In this paper we give a common generalization of our results mentioned above and compute an explicit value of the constant corresponding to $c_3$. Our main result implies the above-quoted theorems of Sprindžuk [21], [22], Kotov [11], Shorey, van der Poorten, Tijdeman and Schinzel [20], Trelina [24], Győry and Papp [8] and Győry [10].

## 2. Results

Before we state our theorem, we establish our notation and introduce some definitions.

A system $\mathcal{L}$ of $n \geqq 2$ linear forms $L_1(\mathbf{x}), \dots, L_n(\mathbf{x})$ in $\mathbf{x}=(x_1, \dots, x_m)$ with algebraic coefficients will be called triangularly connected or, more briefly, *Δ-connected* (cf. [7]) if for any distinct $i, j$ with $1 \leqq i, j \leqq n$ there is a sequence $L_i = L_{i_1}, \dots, L_{i_v} = L_j$ in $\mathcal{L}$ such that for each $u$ with $1 \leqq u \leqq v-1$, $L_{i_u}$, $L_{i_{u+1}}$ have a linear combination with non-zero algebraic coefficients which belongs to $\mathcal{L}$. If in particular $m=2$, then every system $\mathcal{L}$ which contains at least three pairwise non-proportional linear forms is Δ-connected.

Throughout the paper, $L$ will denote a fixed algebraic number field of degree $l \geqq 1$ with ring of integers $\mathbf{Z}_L$, and $U_L$ will be the group of units in $L$. We denote by $\omega(\alpha)$ the number of distinct prime ideal divisors $\mathfrak{p}$ of a non-zero integer $\alpha$ in $L$ and by $\mathcal{P}(\alpha)$ the greatest of the norms $N(\mathfrak{p})$ of these prime ideals. For $\alpha \in U_L$ we take $\mathcal{P}(\alpha)=1$ and $\omega(\alpha)=0$.

If $F(x_1, \dots, x_m) \in \mathbf{Z}_L[x_1, \dots, x_m]$ is a form in $m \geqq 2$ variables, then $F(x_1, \dots, x_m)$ and $F(\varepsilon x_1, \dots, \varepsilon x_m)$ have the same prime ideal decomposition for any $\mathbf{x} = (x_1, \dots, x_m) \in \mathbf{Z}_L^m$ and $\varepsilon \in U_L$. It will be useful to introduce the notation $\overline{|\mathbf{x}|}$ defined by[1])

$$\overline{|\mathbf{x}|} = \min_{\varepsilon \in U_L} \max \left(\overline{|\varepsilon x_1|}, \dots, \overline{|\varepsilon x_m|}\right), \quad m \geqq 2,$$

where $\mathbf{x}=(x_1, \dots, x_m) \in \mathbf{Z}_L^m$. $s_0 \overline{|\mathbf{x}|}$ can be effectively determined and clearly

$$N^{1/l} \leqq \overline{|\mathbf{x}|} \leqq \max \left(\overline{|x_1|}, \dots, \overline{|x_m|}\right) \tag{3}$$

for any $\mathbf{x} \in \mathbf{Z}_L^m$, where $N = \max_{1 \leqq i \leqq m} \left(|N_{L/Q}(x_i)|\right)$. Further, it is clear that in the special case $L=\mathbf{Q}$ $\overline{|\mathbf{x}|}$ coincides with $|\mathbf{x}|$.

Our main result is the following

Theorem. *Let $F(\mathbf{x}) = F(x_1, \dots, x_m) \in \mathbf{Z}_L[x_1, \dots, x_m]$ be a decomposable form of degree $n \geqq 3$ in $m \geqq 2$ variables with splitting field $G$ over $L$, and let $[G: \mathbf{Q}]=g$, $[G: L]=f$. Suppose that the linear factors $L_1(\mathbf{x}), \dots, L_n(\mathbf{x})$ in the factorization of*

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1) $\overline{|\gamma|}$ denotes the maximum absolute value of the conjugates of an algebraic number $\gamma$.

*F form a Δ-connected system and that there is no* $0 \neq \mathbf{x} \in L^m$ *for which* $L_j(\mathbf{x})=0$, $j=1, \ldots, n$. *Let d be a positive integer. Then there exists an effectively computable number* $X_4$ *depending only on F, d and L, such that*

$$(4) \qquad (13f+1)s\log(s+1)+(g+1)\log \mathscr{P} > \log\log \overline{|\mathbf{x}|}$$

*and*

$$(5) \qquad \mathscr{P} > ((13f+1)l)^{-\alpha}(\log\log \overline{|\mathbf{x}|})^{\alpha}$$

*for any* $\mathbf{x} \in \mathbf{Z}_L^m$ *with* $N((x_1, \ldots, x_m)) \leqq d$ *and* $\overline{|\mathbf{x}|} \geqq X_4$, *where* $\mathscr{P}=\mathscr{P}(F(\mathbf{x}))$, $s=\omega(F(\mathbf{x}))$, $\mathscr{P}=P^{\alpha}$ *and P is the maximal rational prime for which* $(F(\mathbf{x}), P) \neq 1$.

It is easily seen that under the conditions and notations of the theorem we have $1 \leqq \alpha \leqq l$,

$$(4') \qquad (13f+1)s\log(s+1)+(g+1)\log \mathscr{P} > \log\log N$$

and

$$(5') \qquad \mathscr{P} > ((13f+1)l)^{-\alpha}(\log\log N)^{\alpha}$$

for any $\mathbf{x} \in \mathbf{Z}_L^m$ with $N((x_1, \ldots, x_m)) \leqq d$ and $N=\max_{1 \leqq i \leqq m}(|N_{L/Q}(x_i)|) \geqq N_1$. For small values of $s$ the estimates (4) and (4′) are obviously much better than (5) and (5′).

Our theorem has several consequences. We first mention an application to diophantine equations. Let $F(\mathbf{x})$ and $d$ be as in the theorem and let $\beta, \pi_1, \ldots, \pi_t$ be fixed non-zero algebraic integers in $L$. Consider the equation

$$(6) \qquad F(\mathbf{x}) = \beta\pi_1^{z_1} \ldots \pi_t^{z_t}$$

in $\mathbf{x} \in \mathbf{Z}_L^m$, $z_1, \ldots, z_t \in \mathbf{Z}$ with $N((x_1, \ldots, x_m)) \leqq d$ and $z_1, \ldots, z_t \geqq 0$. Then (4) gives

$$\max(\overline{|\mathbf{x}|}, e^{\max_k(z_k)}) < C$$

for all solutions $\mathbf{x}, z_1, \ldots, z_t$ of (6), where $C$ is an effectively computable number[2]) depending only on $F$, $d$, $\mathscr{P}(\beta\pi_1 \ldots \pi_t)$, $\omega(\beta\pi_1 \ldots \pi_t)$ and $L$. This result can be regarded as a $\mathfrak{p}$-adic analogue of our Theorem 1 in [7]. (In [7] it is not assumed $F \in \mathbf{Z}_L[\mathbf{x}]$; however, in the applications of Theorem 1 of [7] $F \in \mathbf{Z}_L[\mathbf{x}]$ is always supposed. Thus this is not an essential restriction.)

The following corollary enables us to obtain some information about the arithmetical structure of those algebraic integers of $L$ which can be represented by a decomposable form of the above type.

Corollary 1. *Suppose* $F(x_1, \ldots, x_m)$ *and d are as in the Theorem. Let F be any algebraic integer in L represented by* $F(x_1, \ldots, x_m)$, *where* $x_1, \ldots, x_m \in \mathbf{Z}_L$ *with* $N((x_1, \ldots, x_m)) \leqq d$. *Then*

$$(7) \qquad (13f+1)\omega(F)\log(\omega(F)+1)+(g+1)\log\mathscr{P}(F) > \log\log N$$

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[2]) We could easily obtain an explicit expression for $C$ by computing each constant in the proof of our theorem. *Added in proof:* In my paper „Explicit upper bounds for the solutions of some diophantine equations" (to appear) I explicitly evaluated $C$ in terms of each constant, (generalizing many earlier effective results on norm form, discriminant form and index form equations).

*and*

(8) $$\mathscr{P}(F) > ((13f+1)l)^{-1} \log\log N$$

*if* $N=|N_{L/Q}(F)|\geqq N_2$, *where* $N_2$ *is an effectively computable positive number depending only on* $d$, $L$ *and the form* $F(x_1, \ldots, x_m)$.

Our Corollary 1 generalizes and improves Sprindžuk's theorems [22], [23] concerning rational integers represented by a binary form $f\in \boldsymbol{Z}[x, y]$.

Corollary 2. *Let* $F(\mathbf{x})\in \boldsymbol{Z}_L[x_1, \ldots, x_m]$ *be a decomposable form with the properties specified in the Theorem. Let* $d$ *and* $A$ *be positive numbers with* $d\geqq 1$ *and* $A<1/(g+1)$. *Then there exists an effectively computable number* $X_5$ *depending only on* $F$, $d$, $L$ *and* $A$ *such that if*

$$\mathscr{P}(F(\mathbf{x})) \leqq (\log \overline{|\mathbf{x}|})^A, \quad \mathbf{x}\in \boldsymbol{Z}_L^m, \quad \overline{|\mathbf{x}|} \geqq X_5$$

*and* $N((x_1, \ldots, x_m))\leqq d$, *then*

(9) $$\omega(F(\mathbf{x})) > c_4 \frac{\log\log \overline{|\mathbf{x}|}}{\log\log\log \overline{|\mathbf{x}|}},$$

*where* $c_4=(1-A(g+1))/(13f+1)$.

Let $f\in \boldsymbol{Z}_L[x]$ be a polynomial with at least three distinct roots. Since $\overline{|x|}^{1/l}\leqq \max(\overline{|\varepsilon x|}, \overline{|\varepsilon|})$ for any $x\in \boldsymbol{Z}_L$ and $\varepsilon\in U_L$, our estimates (4), (5), (7), (8) and (9) remain obviously valid for $\mathscr{P}(f(x))$ and $\omega(f(x))$ with $\overline{|x|}$ instead of $\overline{|\mathbf{x}|}$, where $x\in \boldsymbol{Z}_L$ and $\overline{|x|}>X_6$. We remark that for polynomials $f(x)$ with rational integer coefficients Shorey and Tijdeman [19] obtained a much better result than our Corollary 2; they proved $\omega(f(x))\gg(\log\log |x|)/(\log\log\log |x|)$ under the condition $P(f(x))\leqq \exp((\log\log |x|)^A)$, where $A$ is any positive number. As an immediate consequence of this result they derived a good lower bound for $\max_{1\leqq i\leqq y} P(f(x+i))$.

As a consequence of our theorem we obtain the following generalization and improvement, respectively, of the theorems of Coates [4], Sprindžuk [21], [22], Kotov [11] and Shorey, van der Poorten, Tijdeman and Schinzel [20] on the maximal prime factors of binary forms.

Corollary 3. *Let* $f(x, y)\in \boldsymbol{Z}_L[x, y]$ *be a binary form with splitting field* $G$ *over* $L$ *and suppose that among the linear factors in the factorization of* $f$ *at least three are distinct*[3]). *Let* $[G: \boldsymbol{Q}]=g$, $[G: L]=f$ *and* $d\geqq 1$. *Then there exists an effectively computable positive number* $X_7$ *depending only on* $d$, $L$ *and the form* $f(x, y)$ *such that for all pairs* $x, y\in \boldsymbol{Z}_L$ *with* $N((x, y))\leqq d$ *and* $\overline{|\mathbf{x}|}=\min_{\varepsilon\in U_L} \max(\overline{|\varepsilon x|}, \overline{|\varepsilon y|})>X_7$, (4) *and* (5) *hold, where* $\mathscr{P}=\mathscr{P}(f(x, y))$, $s=\omega(f(x, y))$, $\mathscr{P}=P^\alpha$ *and* $P$ *is the maximal rational prime with* $(f(x, y), P)\neq 1$.

[3]) In other words $f$ has at least three pairwise nonproportional linear factors in its factorization.

It follows from (5′) that

$$\mathscr{P}(f(x, y)) > c_5(\log\log N)^{\alpha} \tag{10}$$

for all $x, y \in \mathbf{Z}_L$ with $(x, y)=1$ and $N=\max\left(|N_{L/Q}(x)|, |N_{L/Q}(y)|\right) \geqq N_3$, where $c_5=((13f+1)l)^{-\alpha}$. For irreducible forms $f\in \mathbf{Z}_L[x, y]$ of degree $\geqq 5$ (10) was earlier proved by Kotov [11].

An important special case of Corollary 3 is when $f(x, y)=(x-\alpha_1 y)\dots(x-\alpha_n y)$, where $\alpha_1, \dots, \alpha_n \in \mathbf{Z}_L$ and at least three of them are distinct. This special case of Corollary 3 can be used to obtain an effective result on the diophantine equation $az^q=f(x, y)$ (cf. [20], pp. 63—65).

Corollary 4. *Let $K$ be an extension of degree $n\geqq 3$ of $L$ and let $F(\mathbf{x})= a_0 N_{K/L}(x_1+\alpha_2 x_2+\dots+\alpha_m x_m)\in \mathbf{Z}_L[x_1, \dots, x_m]$ be a norm form in $m\geqq 2$ variables such that $[L(\alpha_i)\colon L]=n_i\geqq 3$, $i=2, \dots, m$, and $n_2\dots n_m=n$. Then with the notations of the Theorem we have* (4) *and* (5).

Corollary 4 implies Corollary 2 of [10] and Theorem 3 of Kotov [11].

Corollary 5. *Let $K$ be as in Corollary 4. Let $\alpha_1, \dots, \alpha_m$ be $m\geqq 2$ algebraic integers in $K$ with $K=L(\alpha_1, \dots, \alpha_m)$ and suppose that $1, \alpha_1, \dots, \alpha_m$ are linearly independent over $L$. Let $F(\mathbf{x})$ denote the discriminant form* $\mathrm{Discr}_{K/L}(\alpha_1 x_1+\dots+\alpha_m x_m)$. *Under the notations of the Theorem, for $F(\mathbf{x})$* (4) *and* (5) *hold.*

Corollary 5 improves Corollary 1 of our paper [8].

Let again $K$ be an extension of degree $n\geqq 3$ of $L$ and let $G$ be the smallest normal extension of $L$ containing $K$. Write $[G\colon \mathbf{Q}]=g$ and $[G\colon L]=f$. Consider an order $O$ of the field extension $K/L$ (i.e. a subring of $\mathbf{Z}_K$ containing $\mathbf{Z}_L$ that has the full dimension $n$ as a $\mathbf{Z}_L$-module) and suppose that $O$ has a relative integral basis $1, \alpha_1, \dots, \alpha_{n-1}$ over $L$. (Such an integral basis exists for a number of orders of $K/L$; see e.g. [2], [13] and [8].) Then we have (cf. [8])

$$\mathrm{Discr}_{K/L}(\alpha_1 x_1+\dots+\alpha_{n-1}x_{n-1}) = [\mathrm{Ind}_{K/L}(\alpha_1 x_1+\dots+\alpha_{n-1}x_{n-1})]^2 D_{K/L}(1, \alpha_1, \dots, \alpha_{n-1}), \tag{11}$$

where $I(x)=\mathrm{Ind}_{K/L}(\alpha_1 x_1+\dots+\alpha_{n-1}x_{n-1})\in \mathbf{Z}_L[x_1, \dots, x_{n-1}]$ is a decomposable form of degree $n(n-1)/2$. It is called the index form of the basis $1, \alpha_1, \dots, \alpha_{n-1}$ of $O$ over $L$.

In the special case $L=\mathbf{Q}$ Trelina [24] obtained lower bounds for $P(I(\mathbf{x}))$. Corollary 1 and Theorem 3 in our paper [8], established independently of Trelina, give lower bounds for $\mathscr{P}(I(\mathbf{x}))$ in the above general case. As a consequence of Corollary 5 we obtain the following generalization and improvement of the estimates of Trelina [24] and Győry and Papp [8].

Corollary 6. *Let $L$, $K$, $d$ and $I(\mathbf{x})$ be defined as above. Then there exists an effectively computable positive number $X_8$ depending only on $I(\mathbf{x}), d, L$ and $D_{K/L}(1, \alpha_1, \dots, \alpha_{n-1})$ such that* (4) *and* (5) *hold for any $\mathbf{x} \in \mathbf{Z}_L^{n-1}$ with $N((x_1, \dots, x_{n-1})) \leqq d$ and $\overline{|\mathbf{x}|} \geqq X_8$, where $\mathscr{P} = \mathscr{P}(I(\mathbf{x}))$, $s = \omega(I(\mathbf{x}) D_{K/L}(1, \alpha_2, \dots, \alpha_{n-1}))$, $\mathscr{P} = P^\alpha$ and $P$ is the maximal rational prime with $(I(\mathbf{x}), P) \neq 1$.*

The proof of our theorem depends on two deep theorems, due to van der Poorten and Loxton [16] and van der Poorten [15], which are essentially sharp inequalities on linear forms in the complex and in the $\mathfrak{p}$-adic case.

## 3. Proof of the Theorem

We first show that we can make certain assumptions without loss of generality. By using a well-known argument we can easily see that there exist algebraic integers $a_2, \dots, a_m$ in $L$ such that $F(1, a_2, \dots, a_m) \neq 0$ (see e.g. [3], p. 77). It suffices to prove the theorem for $F(x_1, a_2 x_1 + x_2, \dots, a_m x_1 + x_m)$, where the coefficient of $x_1^n$ is non-zero. Hence we may suppose that

$$F(\mathbf{x}) = a_0 L_1(\mathbf{x}) \dots L_n(\mathbf{x})$$

with $0 \neq a_0 \in \mathbf{Z}_L$ and

$$L_j(\mathbf{x}) = x_1 + \alpha_{2j} x_2 + \dots + \alpha_{mj} x_m, \quad j = 1, \dots, n,$$

where $\alpha_{ij} \in G$, $2 \leqq i \leqq m$, $1 \leqq j \leqq n$. Writing $\alpha'_{ij} = a_0 \alpha_{ij}$ for $i \geqq 2$ and $\alpha'_{ij} = a_0$ for $i = 1$, we have $\alpha'_{ij} \in \mathbf{Z}_G$ for each $i$ and $j$. We shall prove our theorem for

$$f(\mathbf{x}) = a_0^{n-1} F(\mathbf{x}) = \prod_{j=1}^{n} L'_j(\mathbf{x}),$$

where $L'_j(\mathbf{x}) = \alpha'_{1j} x_1 + \dots + \alpha'_{mj} x_m$. This will imply at once the assertion of the theorem for $F(\mathbf{x})$.

We suppose that there are $r_1$ real and $2r_2$ complex conjugate fields to $G$ and that they are chosen in the usual manner: if $\theta$ is in $G$, then $\theta^{(i)}$ is real for $1 \leqq i \leqq r_1$ and $\theta^{(i+r_2)} = \overline{\theta^{(i)}}$ for $r_1 + 1 \leqq i \leqq r_1 + r_2$. Put $r = r_1 + r_2 - 1$. It is well-known that there exist fundamental units $\eta_1, \dots, \eta_r$ in $G$ and constants $c_6$, $c_7$ such that $|\log |\eta_h^{(i)}|| \leqq c_6$ for $1 \leqq h \leqq r$, $1 \leqq i \leqq g$ and $R_G > c_7$, where $R_G$ denotes the regulator of $G$. Here, and below, $c_6, c_7, \dots$ will denote effectively computable positive numbers which depend only on $F(\mathbf{x})$, $L$ and (some of them) on $d$.

Let $x_1, \dots, x_m$ be any $m$-tuple of algebraic integers in $L$ with $N((x_1, \dots, x_m)) \leqq d$. Put

(12) $$\beta_j = \alpha'_{1j} x_1 + \dots + \alpha'_{mj} x_m, \quad j = 1, \dots, n,$$

and

(13) $$(f(\mathbf{x})) = (\beta_1 \dots \beta_n) = \mathfrak{p}_1^{v_1} \dots \mathfrak{p}_s^{v_s},$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ are distinct prime ideals in $L$. If $X_4$ is sufficiently large and $\overline{|\mathbf{x}|} \geqq X_4$, then Theorem 1 of [7] implies $s>0$ and $P>1$. Let $\mathfrak{P}_1, \dots, \mathfrak{P}_t$ be all distinct prime ideals in $G$ lying above $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. Clearly $t \leqq sf$. Applying now the unique factorization theorem to (13) we get in $\boldsymbol{Z}_G$

$$(14) \qquad (\beta_j) = \mathfrak{P}_1^{U_{1j}} \dots \mathfrak{P}_t^{U_{tj}}, \quad j = 1, \dots, n,$$

where the $U_{kj}$ are non-negative rational integers. Denote by $h_G$ the class number of $G$ and write $U_{kj} = h_G u_{kj} + r_{kj}$ with $0 \leqq r_{kj} < h_G$. We have $\mathfrak{P}_k^{h_G} = (\mu_k)$ with some $\mu_k \in \boldsymbol{Z}_G$. Then from (14) we see that

$$(15) \qquad (\beta_j) = (\chi_j)(\mu_1)^{u_{1j}} \dots (\mu_t)^{u_{tj}},$$

where $(\chi_j) = \mathfrak{P}_1^{r_{1j}} \dots \mathfrak{P}_t^{r_{tj}}$ and

$$|N_{G/Q}(\mu_k)| \leqq P^{gh_G}, \quad |N_{G/Q}(\chi_j)| \leqq P^{gh_G t}.$$

So, following a well-known argument (see e.g. [1], p. 188), we may choose $\mu_k$ and $\chi_j$ such that

$$(16) \qquad \left|\log |\mu_k^{(i)}|\right| \leqq c_8 \log P, \quad \left|\log |\chi_j^{(i)}|\right| \leqq c_8 s \log P, \quad i = 1, \dots, g,$$

and, by (15), we have

$$\beta_j = \varepsilon_j \chi_j \mu_1^{u_{1j}} \dots \mu_t^{u_{tj}}, \quad j = 1, \dots, n,$$

for some unit $\varepsilon_j$ of $G$.

Put $\mathscr{L} = \{L_1', \dots, L_n'\}$. By hypothesis there are two forms in $\mathscr{L}$, say $L_1'$ and $L_2'$, such that $\lambda_1 L_1'(\mathbf{x}) + \lambda_2 L_2'(\mathbf{x}) \in \mathscr{L}$ with non-zero algebraic numbers $\lambda_1, \lambda_2$. Suppose, for convenience, that

$$\lambda_1 L_1'(\mathbf{x}) + \lambda_2 L_2'(\mathbf{x}) + \lambda_3 L_3'(\mathbf{x}) = 0$$

with $\lambda_1 \lambda_2 \lambda_3 \neq 0$. Further, we may assume that $\lambda_1, \lambda_2, \lambda_3 \in \boldsymbol{Z}_G$ and $\max(\overline{|\lambda_1|}, \overline{|\lambda_2|}, \overline{|\lambda_3|}) \leqq c_9$. We obtain now

$$(17) \qquad \lambda_1 \beta_1 + \lambda_2 \beta_2 + \lambda_3 \beta_3 = 0.$$

Put $a_k = \min_q u_{kq}$ and $u_{kq}' = u_{kq} - a_k$ for $q = 1, 2, 3$ and $k = 1, \dots, t$. We may suppose without loss of generality that $U = \max_{k,q} u_{kq}' = u_{11}'$ and $u_{13}' = 0$. Since $\eta_1, \dots, \eta_r$ are fundamental units, we can write

$$\varepsilon_1/\varepsilon_3 = \varrho_1 \eta_1^{w_{11}} \dots \eta_r^{w_{r1}}, \quad \varepsilon_2/\varepsilon_3 = \varrho_2 \eta_1^{w_{12}} \dots \eta_r^{w_{r2}},$$

where $\varrho_1, \varrho_2$ are roots of unity in $G$ and $w_{11}, \dots, w_{r1}, w_{12}, \dots, w_{r2}$ are rational integers. With the notation

$$(18) \qquad \beta_q = \sigma \delta_q, \quad \sigma = \varepsilon_3 \mu_1^{a_1} \dots \mu_t^{a_t}, \quad \delta_q = \chi_q \varrho_q \eta_1^{w_{1q}} \dots \eta_r^{w_{rq}} \mu_1^{u_{1q}'} \dots \mu_t^{u_{tq}'}$$

and $w_{13} = \dots = w_{r3} = 0$, $\varrho_3 = 1$ we get from (17)

$$(19) \qquad \Lambda = -\frac{\lambda_2 \delta_2}{\lambda_3 \delta_3} - 1 = \frac{\lambda_1 \delta_1}{\lambda_3 \delta_3} \neq 0.$$

We are now going to derive an upper bound for $H=\max(U, W)$, where $W=\max_{j,q}|w_{jq}|$. First suppose that $c_{10}s\log P\cdot U>H$ with a sufficiently large $c_{10}$. We may assume that $U\geqq c_{11}s\log P$ with a sufficiently large $c_{11}$, for otherwise (21) immediately follows. We see from (19) that

$$\infty > \operatorname{ord}_{\mathfrak{P}_1}\Lambda \geqq U-c_{12}s\log P \geqq c_{13}U \geqq \frac{c_{14}}{s\log P}H.$$

Further, by (19) we have

$$\Lambda = -\frac{\lambda_2\chi_2\varrho_2}{\lambda_3\chi_3}\eta_1^{w_{12}}\dots\eta_r^{w_{r2}}\mu_1^{u'_{12}-u'_{13}}\dots\mu_t^{u'_{t2}-u'_{t3}}-1. \tag{20}$$

Applying now Theorem 4 of van der Poorten [15] to $\Lambda$, we obtain by (16)

$$H < c_{15}(c_{16}s)^{12(r+sf)+28}P^g(\log P)^{sf+4}. \tag{21}$$

Suppose now that $c_{10}s\log P\cdot U\leqq H$. Assume, for convenience, that $W=|w_{11}|$. From (18) we conclude

$$w_{11}\log|\eta_1^{(i)}|+\dots+w_{r1}\log|\eta_r^{(i)}| = \log|\delta_1^{(i)}|-\log|\chi_1^{(i)}|-\sum_k u'_{k1}\log|\mu_k^{(i)}|$$

for each conjugate with $i=1,\dots,r$. So for some $h$ we must have

$$W\leqq c_{17}\left(\left|\log|\delta_1^{(h)}|\right|+\left|\log|\chi_1^{(h)}|\right|+\sum_k u'_{k1}\left|\log|\mu_k^{(h)}|\right|\right).$$

Thus, by (16) we obtain

$$\left|\log|\delta_1^{(h)}|\right| \geqq c_{18}W-c_{19}s\log P-c_{20}Us\log P\geqq c_{21}H,$$

provided that $c_{10}$ is sufficiently large. Further, by (16) and (18) we have

$$\log|N_{G/Q}(\delta_1)|\leqq \log|N_{G/Q}(\chi_1)|+U\cdot\sum_k\log|N_{G/Q}(\mu_k)|\leqq c_{22}Us\log P.$$

Hence we get for some $m$

$$\log|\delta_1^{(m)}|\leqq -c_{23}H. \tag{22}$$

Formulae (16) and (18) imply

$$\log\left|\frac{\lambda_1^{(m)}}{\lambda_3^{(m)}\delta_3^{(m)}}\right|\leqq c_{24}+(g-1)\log\overline{|\delta_3|}\leqq c_{25}Us\log P<\frac{c_{23}}{2}H. \tag{23}$$

We now omit the superscript $(m)$. It then follows from (22) and (23) that

$$\log|\Lambda|<-\frac{c_{23}}{2}H.$$

Write $\eta_0=-1$. By taking the principal values of the logarithms we obtain from (19) and (18)

$$0<\left|\log\left(-\frac{\lambda_2\delta_2}{\lambda_3\delta_3}\right)\right| \tag{24}$$

$$=\left|\sum_{j=0}^{r}w_{j2}\log\eta_j+\sum_{k=1}^{t}(u'_{k2}-u'_{k3})\log\mu_k-\log\left(-\frac{\lambda_3\chi_3}{\lambda_2\chi_2\varrho_2}\right)\right|<e^{-\delta^*(r+t+1)H},$$

where $\delta^* = (c_{26}(r+t+1))^{-1}$ and $w_{02}$ is a rational integer satisfying

$$|w_{02}| \leqq (r+t+1)H.$$

We can now apply Theorem 3 of van der Poorten and Loxton [16] to (24) and obtain

$$H < c_{27}(c_{28}s)^{10(r+sf)+33}(\log P)^{sf+3}. \tag{25}$$

So (21) and (25) imply

$$H < c_{29}(c_{30}s)^{12(r+sf)+31}P^g(\log P)^{sf+4} \tag{26}$$

and, by (16), (18) and (26), we have

$$\overline{|\delta_q|} < \exp\{c_{31}s\log P + c_{32}H + c_{33}Hs\log P\} < \tag{27}$$
$$< \exp\{c_{34}(c_{30}s)^{12(r+sf)+32}P^g(\log P)^{sf+5}\} = T_1, \quad q = 1, 2, 3.$$

Consider now any $\beta_j$ with $3 \leqq j \leqq n$. By the assumption made on $L'_1, \ldots, L'_n$ there is a sequence $\beta_2 = \beta_{i_1}, \ldots, \beta_{i_v} = \beta_j$ such that for each $u$ with $1 \leqq u \leqq v-1$

$$\lambda_{i_u}\beta_{i_u} + \lambda_{i_{u+1}}\beta_{i_{u+1}} + \lambda_{i_u, u+1}\beta_{i_u, u+1} = 0$$

holds with some non-zero $\lambda_{i_u}, \lambda_{i_{u+1}}, \lambda_{i_u, u+1} \in \boldsymbol{Z}_G$ satisfying $\max(\overline{|\lambda_{i_u}|}, \overline{|\lambda_{i_{u+1}}|}, \overline{|\lambda_{i_u, u+1}|}) \leqq c_{35}$. Further, we may assume $v \leqq n$. We can see in the same way as above that

$$\beta_1 = \sigma\delta_1, \quad \beta_2 = \sigma\delta_2 \tag{28}$$

and

$$\beta_{i_u} = \sigma_u\delta_{u, i_u}, \quad \beta_{i_{u+1}} = \sigma_u\delta_{u, i_{u+1}} \tag{29}$$

for $u = 1, \ldots, v-1$, where $\delta_{u, i_u}, \delta_{u, i_{u+1}} \in \boldsymbol{Z}_G$ with

$$\max_{1 \leqq u \leqq v-1}(\overline{|\delta_{u, i_u}|}, \overline{|\delta_{u, i_{u+1}}|}) < T_1 \tag{30}$$

and $\sigma_u = \vartheta_u\mu_1^{a_{1u}}\ldots\mu_t^{a_{tu}}$ with units $\vartheta_u \in G$ and non-negative rational integers $a_{1u}, \ldots, a_{tu}$. It follows from (28) and (29) that

$$\beta_j = \beta_{i_v} = \sigma\varphi_j/\psi_j \tag{31}$$

with

$$\varphi_j = \delta_2 \prod_{u=1}^{v-1} \delta_{u, i_{u+1}} \quad \text{and} \quad \psi_j = \prod_{u=1}^{v-1} \delta_{u, i_u}.$$

Write $\psi_1 = \psi_2 = 1$ and $\varphi_j = \delta_j$ for $j = 1, 2$. It is clear that

$$\max(\overline{|\varphi_j|}, \overline{|\psi_j|}) < T_1^n, \quad j = 1, \ldots, n. \tag{32}$$

We recall that $\sigma = \varepsilon_3\mu_1^{a_1}\ldots\mu_t^{a_t}$. Denote by $\mu_k^{b_k}$ the highest power of $\mu_k$ with $b_k \leqq a_k$ that divides at least one of the $\psi_1, \ldots, \psi_n$. By taking norms we see that

$$b_k \leqq c_{36}\log T_1, \quad k = 1, \ldots, t.$$

Putting

$$b_k^* = \min(a_k, b_k+1), \quad d_k = a_k - b_k^*, \quad k = 1, \dots, t,$$

and

$$\tau_j = \mu_1^{b_1^*} \dots \mu_t^{b_t^*} \varphi_j / \psi_j,$$

we get

$$\beta_j = \vartheta \mu_1^{d_1} \dots \mu_t^{d_t} \tau_j, \quad j = 1, \dots, n, \tag{33}$$

where $\vartheta = \varepsilon_3$ is a unit and $\tau_j$ are algebraic integers in $G$ satisfying

$$\overline{|\tau_j|} < \exp\{c_{37} s \log P \log T_1\} = T_2. \tag{34}$$

Further, by (13) we have

$$\mathfrak{p}_1^{v_1} \dots \mathfrak{p}_s^{v_s} = (\beta_1 \dots \beta_n) = \big((\vartheta \mu_1^{d_1} \dots \mu_t^{d_t})^n \tau_1 \dots \tau_n\big). \tag{35}$$

Let $k$, $1 \leqq k \leqq s$, be an arbitrary but fixed subscript, and let $\mathfrak{P}$ denote an arbitrary prime ideal in $G$ lying above $\mathfrak{p}_k$. If $\mathfrak{P}^{e_k} \| \mathfrak{p}_k$, $e_k$ does not depend on the choice of $\mathfrak{P}$. Moreover, $\mathfrak{P}$ divides only one of the $\mu_1, \dots, \mu_t$. We shall now follow an argument used in the proof of Theorem 1 of [5] (cf. the deduction (36)⇒(41) of [5]). Let $y_k$ be the greatest rational integer for which

$$\min\left(v_k e_k - \operatorname{ord}_{\mathfrak{P}}\left(\prod_{j=1}^{n} \tau_j\right), v_k e_k\right) \geqq n h_L y_k e_k \tag{36}$$

holds for each $\mathfrak{P}$ with $\mathfrak{P} | \mathfrak{p}_k$, where $h_L$ denotes the class number of $L$. From (35) it follows that $y_k \geqq 0$. By the definition of the $y_k$ there is a $\mathfrak{P}$, lying above $\mathfrak{p}_k$, such that

$$n h_L (y_k + 1) e_k > \min\left(v_k e_k - \operatorname{ord}_{\mathfrak{P}}\left(\prod_{j=1}^{n} \tau_j\right), v_k e_k\right). \tag{37}$$

Since (34) implies

$$\operatorname{ord}_{\mathfrak{P}}\left(\prod_{j=1}^{n} \tau_j\right) \leqq c_{38} \log T_2,$$

we get from (36) and (37)

$$0 \leqq v_k e_k - n h_L y_k e_k \leqq c_{39} \log T_2. \tag{38}$$

If now $\mathfrak{P}$ is an arbitrary prime ideal in $G$ lying above $\mathfrak{p}_k$ and $\mathfrak{P} | (\mu_p)$, then (35), (36) and (38) give

$$0 \leqq d_p \operatorname{ord}_{\mathfrak{P}} \mu_p - h_L y_k e_k \leqq c_{40} \log T_2. \tag{39}$$

Let now $\mathfrak{p}_1^{h_L y_1} \dots \mathfrak{p}_s^{h_L y_s} = (\varkappa)$, where $\varkappa \in \mathbf{Z}_L$, and choose $\xi$ in such a way that

$$\mu_1^{d_1} \dots \mu_t^{d_t} = \varkappa \xi. \tag{40}$$

In view of (39) $\xi$ is an algebraic integer in $G$ and

$$|N_{G/Q}(\xi)| \leqq \exp\{c_{41} s \log P \log T_2\}. \tag{41}$$

It follows from (33) and (40) that

$$\omega = \vartheta^n \xi^n \tau_1 \dots \tau_n \in \mathbf{Z}_L.$$

Further, Lemma 3 of [6] together with (34) and (41) imply that there is a unit $\theta_1 \in L$ and an $\omega' \in \mathbf{Z}_L$ such that

$$\omega = \theta_1^n \omega'$$

and

(42) $$\overline{|\omega'|} < \exp\{c_{42} s \log P \log T_2\}.$$

Thus by (34) and (42) we have

(43) $$\overline{|\theta_1^{-1} \vartheta \xi|} < \exp\{c_{43} s \log P \log T_2\}.$$

Finally, writing $\xi_j = \theta_1^{-1} \vartheta \xi \tau_j$ we get

(44) $$\beta_j = \theta_1 \varkappa \xi_j, \quad j = 1, \dots, n,$$

and, by (34) and (43),

(45) $$\overline{|\xi_j|} < \exp\{c_{44}\, s \log P \log T_2\} = T_3.$$

By hypothesis there is no $0 \neq \mathbf{x} \in L^m$ for which $L_j'(\mathbf{x}) = 0$, $j = 1, \dots, n$. Consequently, the only solution in $L$ of the system of equations

(46) $$L_j'(\mathbf{x}) = \beta_j, \quad j = 1, \dots, n,$$

is the $\mathbf{x} = (x_1, \dots, x_m)$ considered above. Since $f(\mathbf{x})/a_0^n$ is a product of irreducible norm forms over $L$, (46) contains all conjugates of each equation over $L$. Following now an argument of the proof of Lemma 2 of [7], we can easily see that (46) has no other solutions in the complex field. So $m \leqq nf$, and by Cramer's rule we have

(47) $$x_i = \theta_1 \varkappa v_i / v, \quad i = 1, \dots, m,$$

where $v, v_i \in \mathbf{Z}_G$, $v_1, \dots, v_m$ are not all zero,

(48) $$\overline{|v|} \leqq c_{45}$$

and, by (45),

(49) $$\overline{|v_i|} \leqq c_{46} T_3, \quad i = 1, \dots, m.$$

In view of (47) we obtain in $\mathbf{Z}_G$

$$|N_{G/Q}(\varkappa)|\, N((v_1, \dots, v_m)) = |N_{G/Q}(v)|\, N((x_1, \dots, x_m)).$$

Hence, by (48),

(50) $$|N_{L/Q}(\varkappa)| \leqq |N_{G/Q}(v)|^{1/f} d \leqq c_{47}.$$

Thus we can write $\theta_1 \varkappa = \theta_2^{-1} \varkappa'$ with a unit $\theta_2 \in L$ and an algebraic integer $\varkappa' \in L$ satisfying

(51) $$\overline{|\varkappa'|} \leqq c_{48}.$$

It follows now from (47) that

$$x_i' = \theta_2 x_i = \varkappa' v_i/v, \quad i = 1, \ldots, m,$$

and this implies

$$x_i'^f = N_{G/L}(x_i') = N_{G/L}(\varkappa' v_i)/N_{G/L}(v), \quad i = 1, \ldots, m.$$

By the inequality (24) of [7] we have

$$\overline{|x_i'|}^f \leqq \overline{|N_{G/L}(\varkappa' v_i)|}\, \overline{|N_{G/L}(v)|}^{l-1} \leqq \overline{|\varkappa' v_i|}^f\, \overline{|v|}^{(l-1)f},$$

whence, by (48), (49), (51), (45), (34) and (27) we obtain

$$\max_{1 \leqq i \leqq m} \overline{|x_i'|} < c_{49} T_3 \leqq \exp\{c_{50}(c_{51}s)^{12(r+sf)+34} P^g (\log P)^{sf+7}\}. \tag{52}$$

From (52) we deduce

$$\log\log \overline{|\mathbf{x}|} < \log c_{50} + \big(12(r+sf)+34\big)\log(c_{51}s) + g\log P + (sf+7)\log\log P. \tag{53}$$

If $X_4$ is sufficiently large, then $P$ is also sufficiently large and $s > (\log P)^{3f/(3f+1)}$ implies

$$\log c_{50} + \big(12(r+sf)+34\big)\log c_{51} + (12r+34)\log(s+1) + (sf+7)\log\log P$$

$$\leqq \left(f + \frac{1}{2}\right) s \log(s+1).$$

On the other hand, for $s \leqq (\log P)^{3f/(3f+1)}$ we have

$$\log c_{50} + \big(12(r+sf)+34\big)\log c_{51} + (12r+34)\log(s+1) + (sf+7)\log\log P \leqq \log P.$$

Hence (53) gives

$$\log\log \overline{|\mathbf{x}|} < \left(13f + \frac{1}{2}\right) s\log(s+1) + (g+1)\log P, \tag{54}$$

whence (4) follows.

By prime number theory we can choose $X_4$ such that even $\pi(P) \leqq (1+\delta)P/\log P$ holds with $\delta = 1/\big(2(26f+1)\big)$. Then $s \leqq l\pi(P) \leqq (1+\delta)lP/\log P$ and thus

$$\left(13f + \frac{1}{2}\right) s\log(s+1) + (g+1)\log P \leqq (13f+1)lP. \tag{55}$$

Finally, in consequence of (54), (55) and $\mathscr{P} = P^\alpha$ we obtain (5).

In order to prove (4′) and (5′) it suffices to observe that (53) and (3) imply

$$\log\log N < \log(lc_{50}) + \big(12(r+sf)+34\big)\log(c_{51}s) + g\log P + (sf+7)\log\log P.$$

If $N$ is sufficiently large, we get (4′) and (5′) in the same way as we deduced (4) and (5) from (53).

## 4. Proofs of the Corollaries

*Proof of Corollary* 1. Let $\varepsilon$ be a unit in $L$ such that $\overline{|\mathbf{x}|}=\max(\overline{|\varepsilon x_1|}, \ldots, \overline{|\varepsilon x_m|})$. Then

$$N = |N_{L/Q}(F)| = \left|N_{L/Q}\big(F(\varepsilon\mathbf{x})\big)\right| \leqq c_{52}\overline{|\mathbf{x}|}^{nl}. \tag{56}$$

Therefore, for sufficiently large $N$, (4) implies (7), but only with $\log\log N-\log(2ln)$ in place of $\log\log N$. Following the argument applied at the end of the above proof, we obtain (7) and (8) from (53) and (56).

*Proof of Corollary* 2. Suppose

$$\omega\big(F(\mathbf{x})\big) \leqq c_4\frac{\log\log\overline{|\mathbf{x}|}}{\log\log\log\overline{|\mathbf{x}|}}$$

for some $\mathbf{x}\in \boldsymbol{Z}_L^m$ with $\overline{|\mathbf{x}|}\geqq X_5$ and $N\big((x_1, \ldots, x_m)\big)\leqq d$. Then by our theorem we have

$$\log\log\overline{|\mathbf{x}|} < (13f+1)\,\omega\big(F(\mathbf{x})\big)\log\big(\omega(F(\mathbf{x}))+1\big)+(g+1)\log\mathscr{P}\big(F(\mathbf{x})\big)$$
$$\leqq (13f+1)\,c_4\log\log\overline{|\mathbf{x}|}+A(g+1)\log\log\overline{|\mathbf{x}|},$$

provided that $X_5$ is sufficiently large. Since $(13f+1)c_4+A(g+1)=1$, we have arrived at a contradiction and thus (9) is proved.

*Proof of Corollary* 3. By assumption there are at least three pairwise non-proportional linear factors in the factorization

$$f(x, y) = \prod_{i=1}^{n}(\alpha_{i1}x_1+\alpha_{i2}y).$$

Consequently, the linear factors $\alpha_{i1}x+\alpha_{i2}y$, $i=1, \ldots, n$, form a $\Delta$-connected system and the system of equations

$$\alpha_{i1}x+\alpha_{i2}y = 0, \quad i = 1, \ldots, n,$$

has no non-trivial solution $x, y$ in $L$. So the assertion of Corollary 3 follows at once from our theorem.

*Proof of Corollary* 4. $F(\mathbf{x})$ can be written in the form

$$F(\mathbf{x}) = a_0\prod_{i=1}^{n}(x_1+\alpha_2^{(i)}x_2+\ldots+\alpha_m^{(i)}x_m),$$

where $\alpha_j^{(1)}, \ldots, \alpha_j^{(n)}$ denote the conjugates of $\alpha_j$ over $L$. As we showed in [7] (see also [9]), the conjugates $x_1+\alpha_2^{(i)}x_2+\ldots+\alpha_m^{(i)}x_m$ of $x_1+\alpha_2x_2+\ldots+\alpha_mx_m$ over $L$ form a $\Delta$-connected system. Further, by virtue of the assumption $[L(\alpha_1)\colon L]\ldots[L(\alpha_m)\colon L]=n$, the only solution of the system of equations

$$x_1+\alpha_2^{(i)}x_2+\ldots+\alpha_m^{(i)}x_m = 0, \quad i = 1, \ldots, n,$$

in $L$ is $x_1=\ldots=x_m=0$. So our theorem implies the required assertion.

*Proof of Corollary* 5. Let $L(\mathbf{x})=\alpha_1 x_1+\ldots+\alpha_m x_m$ and let $L^{(1)}(\mathbf{x}), \ldots, L^{(n)}(\mathbf{x})$ be the conjugates of $L(\mathbf{x})$ over $L$. Put

$$l_{ij}(\mathbf{x}) = L^{(i)}(\mathbf{x})-L^{(j)}(\mathbf{x}).$$

In proving Theorem 4 in [7] we showed that

$$F(\mathbf{x}) = \mathrm{Discr}_{K/L}(\alpha_1 x_1+\ldots+\alpha_m x_m) = (-1)^{n(n-1)/2} \prod_{\substack{i,j=1\\ i\neq j}}^{n} l_{ij}(\mathbf{x})$$

satisfies all conditions made in our theorem. Thus (4) and (5) clearly follow.

*Proof of Corollary* 6. If $X_8$ is sufficiently large and $\overline{|\mathbf{x}|}\geqq X_8$, by Corollary 5 and (11) we have $\mathscr{P}(D(\mathbf{x}))=\mathscr{P}(I(\mathbf{x}))$, where $D(\mathbf{x})=\mathrm{Discr}_{K/L}(\alpha_1 x_1+\ldots+\alpha_{n-1}x_{n-1})$. Thus Corollary 5 proves the assertion of Corollary 6.

## References

[1] BAKER, A.: Contributions to the theory of Diophantine equations. I and II. - Philos. Trans. Roy. Soc. London Ser. A 263, 1967/68, 173—208.

[2] BERWICK, W. E. H.: Integral bases. - Reprinted by Stechert - Hafner Service Agency, New York—London, 1964.

[3] BOREVICH, Z. I., and I. R. SHAFAREVICH: Number theory. - Academic Press, New York—London, 1967.

[4] COATES, J.: An effective $p$-adic analogue of a theorem of Thue. II. The greatest prime factor of a binary form. - Acta Arith. 16, 1969/70, 399—412.

[5] GYŐRY, K.: On polynomials with integer coefficients and given discriminant, V. $\mathfrak{p}$-adic generalizations. - Acta Math. Acad. Sci. Hungar. 32, 1978, 175—190.

[6] GYŐRY, K.: On the solutions of linear diophantine equations in algebraic integers of bounded norm. - Ann. Univ. Sci. Budapest. Eötvös Sect. Math. (to appear).

[7] GYŐRY, K., and Z. Z. PAPP: Effective estimates for the integer solutions of norm form and discriminant form equations. - Publ. Math. Debrecen 25, 1978, 311—325.

[8] GYŐRY, K., and Z. Z. PAPP: On discriminant form and index form equations. - Studia Sci. Math. Hungar. (to appear).

[9] GYŐRY, K., and Z. Z. PAPP: Norm form equations and explicit lower bounds for linear forms with algebraic coefficients. - To appear.

[10] GYŐRY, K.: Effective upper bounds for the solutions of norm form equations in the $\mathfrak{p}$-adic case. - Preprint, 1978.

[11] KOTOV, S. V.: The Thue—Mahler equation in relative fields. - Collection of articles in memory of Juriĭ Vladimirovič Linnik, Acta Arith. 27, 1975, 293—315 (Russian).

[12] MAHLER, K.: Zur Approximation algebraischer Zahlen, I. Über den grössten Primteiler binärer Formen. - Math. Ann. 107, 1933, 691—730.

[13] NARKIEWICZ, W.: Elementary and analytic theory of algebraic numbers. - PWN—Polish Scientific Publishers, Warszawa, 1974.

[14] PARRY, C. J.: The $\mathfrak{p}$-adic generalization of the Thue—Siegel theorem. - Acta Math. 83, 1950, 1—100.

[15] van der Poorten, A. J.: Linear forms in logarithms in the $p$-adic case. - Transcendence Theory: Advances and Applications, edited by A. Baker and D. W. Masser, Academic Press, London—New York—San Francisco, 1977, 29—57.

[16] van der Poorten, A. J., and J. H. Loxton: Multiplicative relations in number fields. - Bull. Austral. Math. Soc. 16, 1977, 83—98 and 17, 1977, 151—156.

[17] Schlickewei, H. P.: On norm form equations. - J. Number Theory 9, 1977, 370—380.

[18] Schlickewei, H. P.: On linear forms with algebraic coefficients and Diophantine equations. - J. Number Theory 9, 1977, 381—392.

[19] Shorey, T. N., and R. Tijdeman: On the greatest prime factors of polynomials at integer points. - Compositio Math. 33, 1976, 187—195.

[20] Shorey, T. N., A. J. van der Poorten, R. Tijdeman, and A. Schinzel: Applications of the Gel'fond—Baker method to diophantine equations. - Transcendence Theory: Advances and Applications, edited by A. Baker and D. W. Masser, Academic Press, London—New York—San Francisco, 1977, 59—77.

[21] Sprindžuk, V. G.: The greatest prime divisor of a binary form. - Dokl. Akad. Nauk BSSR 15, 1971, 389—391 (Russian).

[22] Sprindžuk, V. G.: The structure of numbers representable by binary forms. - Dokl. Akad. Nauk BSSR 17, 1973, 685—688, 775 (Russian).

[23] Sprindžuk, V. G.: An effective analysis of the Thue and Thue—Mahler equations. - Current problems of analytic number theory, Proc. Summer School Analytic Number Theory, Minsk, 1972, pp. 199—222, 272, Izdat. "Nauka i Tehnika", Minsk, 1974 (Russian).

[24] Trelina, L. A.: On the greatest prime factor of an index form. - Dokl. Akad. Nauk BSSR 21, 1977, 975—976 (Russian).

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