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**REVERSAL OF MISFORTUNE:  
PARADOX IN OPTIMIZATION ACROSS CONTINGENCIES**

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**I: INTRODUCTION**

Often an economic agent can be dissatisfied with an endowed distribution of outcomes, events, or utilities and desire to improve this distribution by transferring income or resources across individuals or states of the world. Some kinds of utility generating resources may be easy to transfer, while others are difficult -- maybe impossible -- to reallocate. For example, perfect lump-sum transfers of some resources such as earned income may be possible but other resources, such as the time endowments of individuals, may be completely non-transferable. This multi-state optimization theme recurs in a wide variety of economic contexts, ranging across taxation and income distribution, international trade and market disruption, labor contracts and unemployment insurance, and many others.

Jack Hirshleifer's influence on our understanding of such inter-contingency allocations spans many decades of his professional life. Well before the Quarterly Journal of Economics articles that established his authority on the subject (Hirshleifer, 1965, 1966) Jack had cast issues of preparation for and survival from nuclear war in those terms (Hirshleifer, 1953, 1956). In fact, a preoccupation with how to manage uncertain adversity continued to inform his entire life's work, including his foundational contributions of his later years in creating the economics of conflict as a field of study (Hirshleifer, 1987, 2001).

Yet because economic analyses of such topics, including those of Hirshleifer, have frequently been so context driven, the generality of this theme across cases and its structure seem to have gone unnoticed, and the generality of a

particular paradoxical result (which we will denote a "reversal theorem") unappreciated. Nor does it appear that the result has been formally derived with the required assumptions explicitly identified. For example, in the public finance literature, a paradoxical implication of lump-sum distribution in a first best environment has been reported, namely that such optimal first best redistribution will reverse the preference rankings of the endowed distribution of utilities. That is when lump sum transfers are available, an optimizing resource manager will necessarily redistribute so much that originally "bad" outcomes become preferred to originally better ones. (Atkinson and Stiglitz 1980 p. 351)

Or as another example the literature on unemployment insurance and efficient labor contracts clearly implies that the same principle readily extends to the inter-contingency allocations of an expected utility maximizing individual. Specifically, if fair insurance is available, the rational resource owner will necessarily over insure, buying so much insurance that the otherwise "bad" contingency becomes preferred. (Green and Kahn 1983, Topel and Welch 1986, Milgrom 1988) In both cases, first best inter-personal or inter-contingency transfers entail a complete reversal in the utility rankings of different individuals/outcomes. As we show presently this "reversal theorem" requires that welfare be an additive function of equally weighted, identical, and individually risk neutral (or risk averse) utility functions. Provided such conditions are met the theorem obtains in many situations; we list several of the more important.

**Optimal Income Tax.** The optimal income tax literature has generally avoided lump-sum taxation because of assumed imperfect knowledge about the characteristics of individuals. Where lump-sum taxation is available, however, the reversal theorem will ordinarily apply; individuals usually enjoy non-transferable time endowments, and additive social welfare functions are the most common. As a result, the rank order of individual utilities after optimal redistribution will be precisely the opposite of the initial rank order.

**Resource Unemployment Insurance and Compensation.** Frequently, a resource owner can sell some or all of his endowment in good times, but be cut off from his preferred market in bad times and be obliged therefore to consume his entire endowment. For example, a worker may choose his hours between leisure and labor at a fixed wage in good times, but be thrown out of work altogether in bad and compelled to consume his entire endowment as leisure (or be forced to work at a lesser wage). Clearly the worker's time endowment cannot be transferred across contingencies. If his welfare is the expectation of one, risk neutral, state independent utility, and this in turn is a function of earnings and leisure, then fair insurance purchase allows him to lump-sum transfer income across contingencies, and our theorem applies. He will buy so much insurance that he is better off ex post if unemployed. Note that the conclusion applies equally when the resource owner can self-insure through savings/borrowing at a probabilistically fair price.

**Provision Against Trade Interruption.** A small country which can sell its exportable at the world price in normal times, may be cut off entirely from imports in time of war or emergency and obliged to consume all the goods it had planned to export (at least until it can adjust production). If it has a risk neutral expected utility function, it should wish to insure at fair odds. It, therefore, would conform to our theorem as well, purchasing so much insurance that it is better off if embargoed and actually prefers this "bad" outcome to continuing trade (and paying insurance premiums).<sup>1</sup>

**Rawlsian Convocations.** Next consider a constituent in Rawls's celestial convocation. From behind his veil of ignorance he identifies a life of opportunity vs. one truncated by disability. If he can lump sum transfer between outcomes ex ante at fair odds, his behavior will conform to our theorem. He will arrange the distribution among outcomes such that ex post the disabled are better off than the lucky.

**Provision for Retirement.** Lastly, a worker might be guaranteed  $n$  years of employment at a known wage followed by  $m$  certain years of unemployment/retirement. The structure of this example resembles the others closely. Anticipating such prospects the worker will want to save for retirement. With an additive inter-temporal welfare function (and compound interest earned at the same rate as his pure time preference or inter-temporal utility discount) the earner will definitely save enough that he reaches a higher utility level when retired than he did when working.

Although this "reversal paradox" as stated above is recognized in several niches in the literature, the paradigm itself and its structure seem not to have been given adequate explicit focus. This is true notwithstanding that questions of income redistribution, of unemployment insurance and labor contract, or protection against market disruption and so on have been intensively explored. This paper examines the underlying phenomenon common to each of these contexts, its structure, and the conditions under which the "reversal theorem" will and not obtain.

Several features identify the phenomenon this paper examines. The problem involves incomplete markets in that the bad outcome derives from the loss or curtailment of a market for one's resource, which in turn cannot be transferred between individuals or contingencies directly. Second, some other good(s) is lump-sum transferable. Third, the resource which the lapse of a market makes non-tradable has a residual or reservation value for the subject. These three characteristics apply to the poor or unemployed individual whose time cannot be transferred to other people or across contingencies; only money or the numeraire commodity his time buys can be traded. They apply to a

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<sup>1</sup> Of course a country anticipating cut off from foreign markets might contemplate other measures than taking out insurance (via agreements with allies supposedly). For instance it might, if feasible, stockpile the import at risk. Similarly, the individual at risk of involuntary unemployment, might prepare an alternative individually productive use for his surplus leisure. Alternatives like these may approximate insurance purchase under some circumstances (McGuire, 1990).

country faced with the prospect of trade cut-off under which it cannot export or import, a country which might secure a replacement for needed imports by storage or stockpiling. In the same vein, they apply to the constituent in Rawls's celestial convocation who from behind a veil of ignorance may desire compensation under outcomes with no earnings. In all these cases leisure time, embargoed exports, or years of life, have a residual value to the subject.

To emphasize the essential structure of the problem, our formal analysis begins with the lump-sum interpersonal transfer case, making the simplest assumptions possible: (a) two goods, income and leisure; (b) two individuals; (c) zero costs of resource transfer; and (d) risk neutral, state independent, linear homogeneous utility functions to name the more important. Section II will derive the "reversal theorem" for optimal income taxation. Section III will extend this theme to intercontingency optimization and resource transfer via insurance. Once the simplest model is understood, extensions to more complex and realistic assumptions will be addressed in later sections.

## II: OPTIMAL INCOME DISTRIBUTION: LINEAR HOMOGENEOUS RISK NEUTRAL UTILITY

### II.A. Assumptions

- a. Consider a society of two individuals ( $j = 0, 1$ ), with identical utility functions  $U$ .
- b. These utility functions are a state independent, and strictly quasi-concave in two arguments, viz. consumption of  $x^1$  if "lucky"  $x^0$  if not) and of  $y$  ( $y^1$  if lucky and  $y^0$  if not). Utility,  $V(x^j, y^j)$ , is linear homogeneous in the arguments  $(x, y)$ , or is any constant ratio transformation  $U = f(V) = BV$  ( $B$  a constant) of the same with constant positive first derivatives  $f'(V) = B^{-1}$ . Each indifference curve, by strict quasi-concavity, has strictly a diminishing marginal rate of substitution (MRS).
- c. Assume both individuals have equal endowments of  $x = \hat{x}$ ;  $x$  might be thought of as leisure, and  $\hat{x}$  as the endowed twenty four hours/day. One individual, say Mr. 1, the "lucky" one can trade his  $x$  for  $y$ , at a constant wage,  $w$ . To maximize his utility he gives up, say  $z^1$  of  $\hat{x}$ , so that absent redistribution  $y^1 = wz^1$  and  $x^1 = (\hat{x} - z^1)$ , the initial utility for Mr. 1 is

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<sup>2</sup> The cases representing risk aversion or risk preferring utility in which  $f(V)$  is any positive smooth continuous concave or convex transformation with continuously increasing or decreasing first derivatives will be dealt with presently in Section V.

$$U^1(i) = U[(\hat{x} - z^i), wz^i]$$

The other person, Mr. o is "unlucky." He cannot work or trade at all. Absent redistribution, his initial utility is

$$U^0(i) = U[\hat{x}, 0]$$

Assuming diminishing marginal rates of substitution,  $U^0(i) < U^1(i)$ , where "i" stands for "initial" pre-transfer values.

Now suppose lump-sum transfers, T, are allowed between Mr. 1 and Mr. o and the society desires to maximize the equally weighted sum of utilities.

$$W = U^0 + U^1$$

## II.B. The Optimal Lump-Sum Transfer

**Theorem:** At the maximum of  $W = W^*$ ,  $U^0(*) > U^1(*)$ , (where "\*" indicates solution values), provided  $f'' = 0$  throughout. In words, to maximize aggregate utility, society will lump-sum redistribute so much income that Mr. o is better off after receiving the transfer than Mr. 1 is after paying it.

**Proof:** The maximand for this problem is

$$(1) \quad W = U^0[\hat{x}, T] + U^1[(\hat{x} - z), (wz - T)]$$

Because Mr. 1 is assumed to freely exchange z for y at the rate w after the transfer no less than before, the variables of choice are both z and T. First order conditions with respect to these two variables are:

$$(2) \quad -U_x^1 + wU_y^1 = 0$$

$$(3) \quad -U_y^1 + U_y^0 = 0$$

where  $U_h^k$  indicates the marginal utility of good h to person k.

The risk neutral case with U linear homogeneous,  $[f(V) = U = BV]$  is pivotal for extensions to risk averse and risk preferring utility functions will hinge; therefore, we will develop it in detail. By eq. (3) the marginal utility of good y after the optimal transfer is the same across individuals; thus the ratio of x to y consumed and the marginal utility of x must also be the same across individuals. This follows from the homogeneity assumption. It follows from  $x^0(*) = \hat{x}$  and, therefore, from Mr. 1's budget constraint that  $x^0(*) > x^1(*)$ . With  $[x^0(*)/y^0(*)] = [x^1(*)/y^1(*)]$ , it then also follows that  $y^0(*) > y^1(*)$ . This states that consumption of both commodities is higher after the transfer for the

initially unlucky Mr. 0 than for the initially lucky Mr. 1; thus utility is higher for the former. QED.

### II.C. Optimal Transfers Induce The Poor to Choose Zero Earnings

As a matter of fact the social welfare maximizing society under the above assumptions will transfer so much that after the transfer the recipient would choose not to sell his leisure or otherwise trade his resource even if he could (which he can not do of course). Thus the optimal compensation for adversity in the form of loss of employment is to provide so much that one would **choose not to work**, i.e. so much that the state forced on the unemployed resource owner is one he would voluntarily choose.

The intuitive explanation for this finding is straightforward. Individuals with zero or low wages<sup>3</sup> are more efficient consumers of leisure than are individuals with positive or high wages. Hence society can improve the efficiency of the total allocation of time between the market and leisure sectors by inducing low or zero wage individuals **to choose** to spend relatively large amounts of their time at leisure, and high wage individuals **choose** to spend relatively large amounts of their time at work. Redistributing income away from high wage individuals induces them to choose greater market activity while inducing low wage individuals to choose less.

### II.D. Optimal Solution is Unique

Fig. 1 assists in understanding this result. To the left of  $\hat{x}$  is the opportunity set during employment and the effects on utility-when-employed of endowment sales. The  $\hat{x}$ -endowment may be sold for two purposes: to obtain  $y$  for consumption when employed, or to finance transfers to the disadvantaged for benefits as unemployed. Being employed the lucky Mr. 1 would exchange  $z$  of  $\hat{x}$  at price  $w$  along price line  $q^0$  up to a point of tangency at  $U^1(i)$  --  $i$  being initial utility when lucky and employed. If no redistribution actually were provided unemployment would give consumption point  $(\hat{x}, 0)$  and utility  $x^0(i)$  for the unlucky person. However, by transferring  $T(k)$  the when-employed budget line shifts in to  $q^k$ , utility for those employed drops to  $U^1(k)$  and utility of those unemployed rises to  $U^0(k)$ . As  $T$  increases the consumption point when employed retreats along the income expansion path,  $S$ , indicating that the

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<sup>3</sup> Section IV of this paper extends the above analysis of redistribution between individuals with positive and zero wages to redistribution between individuals with low and high wage rates. Our characterization of the intuition of our result anticipates this extension.

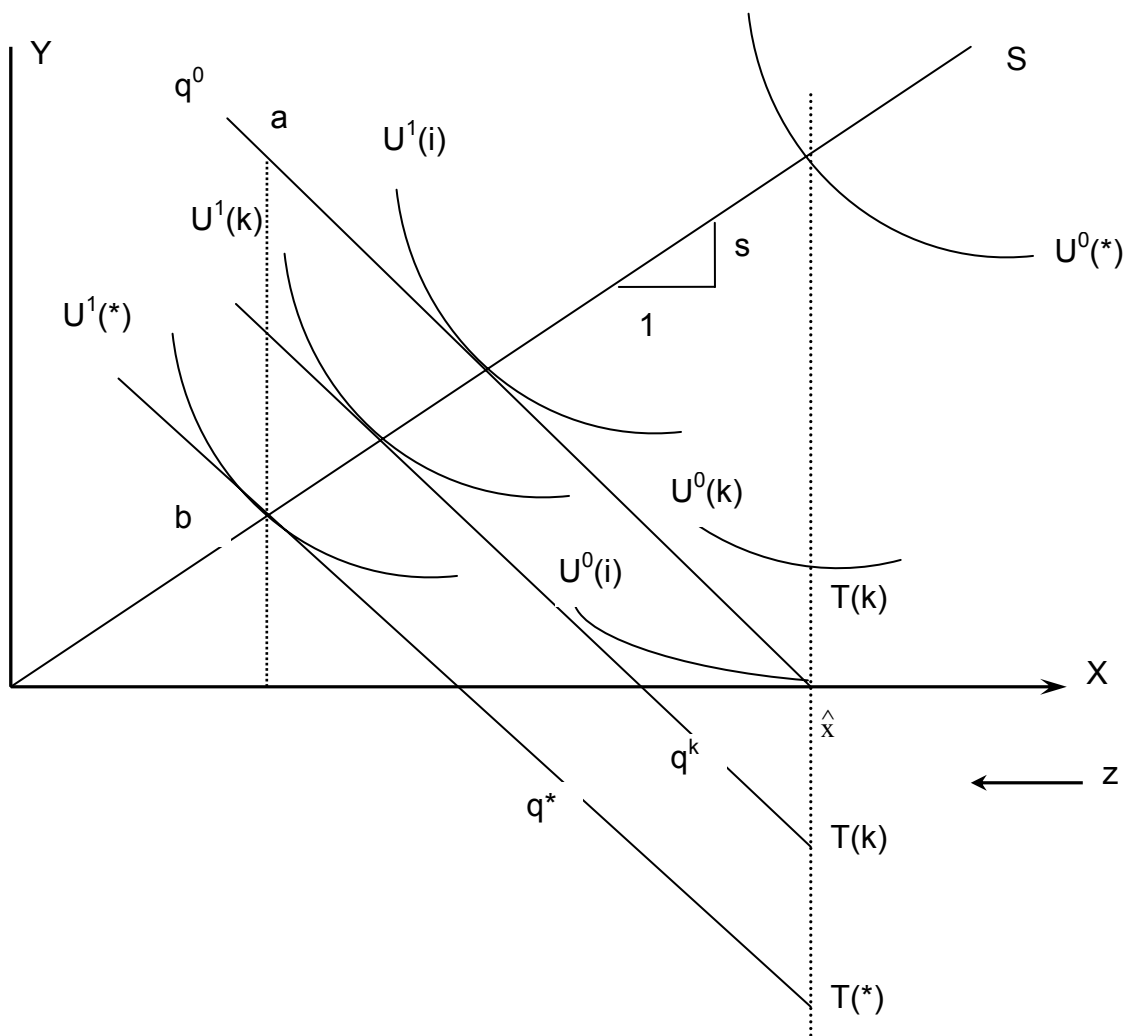


Figure 1

Optimal First Best Lump-Sum Income Redistribution

lucky Mr. 1 works more and more as he is lump-sum taxed to provide increasing compensation to the unlucky Mr. 0. (This expansion path has slope "s" indicating the if-employed consumption ratio  $y^1/x^1$ ). Correspondingly, the consumption point if unemployed climbs along the vertical through  $\hat{x}$ . The optimum in social welfare i.e. the aggregate of unweighted utility is reached at the unique intersection of these two curves giving  $y^0(*) = T(*)$  and unemployment utility of  $U^0(*)$ . It is also obvious from inspection of Fig. 1 that, in this identical two person case, the optimal lump-sum tax will induce the worker to exactly double his work effort over what it would be in the absence of

the transfer, to double  $z$  so as to provide sufficient income for transfers and (at his after transfer budget) to maximize his utility.<sup>4</sup>

### II.E. Corner Solutions

Note that this model with risk neutrality implies that to maximize the aggregate of unweighted utilities the "lucky" worker might be lump-sum taxed all the way back to the origin, where he allocates his entire endowment to work so that  $z = \hat{x}$  and his entire earnings are transferred away so that  $T = w\hat{x}$ . This in turn implies that the lucky worker may run out of earnings altogether before marginal utilities can be equalized. In this case a corner solution is reached with Mr. 1 at the origin after transfers and Mr. 0 on the vertical through  $\hat{x}$  but short of the point of its intersection with the income expansion path/ray "S". When there are equal numbers of lucky and unlucky individuals, such a corner could only be reached if  $w < s$ . As is easily shown  $s > w$  if and only if the supply of labor  $z(w) > (\hat{x})/2$ ; in other words if and only if more than 50% of the endowment is offered for sale. Such a high proportion may be implausible for individual labor supply though less so for small countries supplying an export product. However, this crucial or cutoff proportion  $z(w)/(\hat{x})$  -- which determines whether optimal distribution will press the lucky, employed individuals to a corner -- declines as more are unemployed in proportion to the "lucky" employed. For example, if there were two identical unlucky individuals of "type-0" and one lucky one of "type-1" then the necessary condition for optimal redistribution to go to a corner is that  $s > w/2$ , and this in turn requires  $z(w) > (\hat{x})/3$ . The condition generalizes to  $nw < ms$  where  $n$  indicates the number employed, and  $m$  the number unemployed; correspondingly, a corner solution therefore requires  $z(w) > \hat{x}[n/(n+m)]$ , where  $n/m$  represents the ratio of lucky to unlucky individuals. Thus, the higher the ratio  $n/m$  the higher must be  $z(w)/\hat{x}$  for optimal distribution to push the lucky to a corner solution.

### II.F. Welfare Effects of a Wage Increase

This in turn raises the question of the effects of an increase in wage on the post transfer welfare of the lucky worker, who increases his work effort as a result of the lump-sum tax -- actually doubles his work effort as we have seen if he must support one entire unemployed individual (i.e. if  $n/m = 1$ ). Obviously, the unlucky Mr. 0 benefits unambiguously from an increase in  $w$  for Mr. 1. The question is whether the lucky individuals benefit from an

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<sup>4</sup> In Fig. 1 the distance  $ab$  must equal distance  $\hat{x}T(*)$  to transfer the amount  $T(*)$ . Therefore, the triangles with bases  $ab$  and  $\hat{x}T(*)$  are congruent and of equal height.



increase in their wage after the optimal transfer of income. Insight into an answer may accompany the observation that if an increase in wage caused no change in pre-tax labor supply, it would cause no change in post-tax labor supply (which if  $n/m = 1$  is double pre-tax supply) and therefore would increase the lucky person's after tax measured income without increasing work effort. Thus if the wage-price elasticity of supply of labor is zero, then labor of "type-1" will benefit from a wage increase even after paying the lump-sum tax. This proposition obviously obtains *a fortiori* if the pre-tax labor supply is backward bending at the point of a wage increase, since after-tax consumption of leisure and after-tax measured income both will increase with an increase in wage rate. However, if the supply of labor is sufficiently elastic, might not the pre-tax labor supply increase so much that after doubling his work effort so as to provide sufficient lump-sum taxes to the unlucky, the "lucky" worker actually finds he is worse off as a result of the wage increase combined with the transfer? We now turn to this question.

The precise condition for an increase in wage rate to harm the lucky workers follows from the first order conditions (equations 2 and 3), and linear homogeneity of the utility function. To derive this condition we introduce additional notation as follows:

$n$	$(m)$	=	number of employed (unemployed)
$s(w)$		=	optimal consumption ratio $y/x$ ;
$s(w)\hat{x}$		=	transfer of $y$ received by each unemployed or unlucky individual required to effect social optimum assuming an interior maximum
$s_w$		=	$\partial s(w)/\partial w = (s/w) \sigma$
$\sigma$		=	elasticity of substitution in consumption;
$z(w)$		=	optimal labor supply as a function of $w$ if <i>no lump sum transfers</i> across individuals or contingencies were made;
$z(w)(n+m)/n$		=	optimal labor supply of each lucky individual -- allowance made for work required to pay for optimal transfer to unlucky people;
$wz(w)(n+m)/n$		=	gross earnings of lucky people before their payments of transfers;
$(m/n)s(w)\hat{x}$		=	transfer payments <b>from</b> each lucky working person needed to transfer $s(w)\hat{x}$ to each unlucky unemployed person;
$z_w$		=	$\partial z(w)/\partial w$

With " $n$ " lucky-employed individuals and " $m$ " unlucky-unemployed individuals then for each dollar collected from the  $n$  lucky  $n/m$  dollars are received by each unlucky person. Therefore, the welfare of the lucky, after the optimal

transfer has been paid and assuming an interior solution becomes:

$$(4) \quad U^1 = U^1[[\hat{x} - (n+m)/n]z(w)], [((n+m)/n)wz(w) - (m/n)s(w)\hat{x}]]$$

The effect of an increase in wage on post transfer utility of the lucky working person is found from differentiating equation (4) with respect to wage. This yields:

$$(5) \quad dU/dw = -[(n+m)/n]U_X^1 z_w + U_Y^1 [\{(n+m)/n\}[z(w) + wz_w]] - (m/n)\hat{x}s_w$$

Then substituting  $w = U_X^1 / U_Y^1$  from the first order conditions;

$$(6) \quad dU/dw = U_Y^1 [-(n+m)/n]wz_w + [\{(n+m)/n\}[z(w) + wz_w]] - (m/n)\hat{x}s_w$$

As in the usual case, the value of income earned from the increase in labor supplied just offsets the value of leisure foregone, so that (6) simplifies to:

$$(7) \quad dU/dw = U_Y^1 [(n+m)/n]z(w) - (m/n)\hat{x}s_w$$

Thus, the welfare effect boils down to whether the increase in earnings on infra-marginal labor supply is sufficient to pay the extra transfer required. From equation (7) it might appear that a large labor supply -- by making the increment

to infra-marginal earnings due to a wage increase correspondingly large -- would indicate a greater net gain to the lucky from such an increase in wage. This would be mistaken however. For substituting  $s_w = \partial s(w)/\partial w = (s/w)\sigma$  and  $s/w = [wz/[w(\hat{x} - z)]]$  gives:

$$(8) \quad dU/dw = U_Y^1 [(n+m)/n]z/\hat{x} - [(m/n)z/(\hat{x} - z)]\sigma \hat{x}$$

Thus, other things equal (given values of m, n, and  $\sigma$ ) a large labor supply increases both infra-marginal earnings for the employed and it also increases the required transfer payment. For a given substitution elasticity the negative effect on the required increase in transfer outweighs the positive effect on infra-marginal earnings. Thus the condition for an increase in wage to cause a benefit versus a loss to the people who receive it can be written

$$(9) \quad dU/dw \begin{matrix} \geq \\ < \end{matrix} 0 \leftrightarrow [(\hat{x} - z(w))/\hat{x}] \begin{matrix} \geq \\ < \end{matrix} \{m/(n+m)\} \sigma$$

Ceteris paribus a lower substitution elasticity means a lesser increment in the transfer required and therefore a higher gain (lesser loss) for the lucky-employed, while a lower supply of labor also reduces the required transfer proportionately more than it reduces the inframarginal gain from which such transfers will be made. This bizarre effect states

that a wage increase may make an unemployed person so much more efficient a consumer of goods that those employed must work very much more to provide sufficient income to the unemployed -- so much more that those employed actually suffer from a wage increase.<sup>5</sup>

The above analysis is automatically generalizable to n different goods with given relative prices, as long as these relative prices are unaffected by lump-sum redistributions. For then Hicks's Composite Good Theorem implies that we can aggregate all such goods into a single composite.

### III: RESOURCE UNEMPLOYMENT AND OPTIMAL Insurance

One important generalization of the foregoing analysis extends to the optimal distribution of income among contingencies! Just as first best lump-sum redistributions across people will reverse the order of utilities, the same is true across contingencies for the same entity (or for different ones). Many instance of the reversal theorem fall into this category; the resource owner who can sell some or all of his endowment at in good times, but be cut off from his market entirely in bad; the worker facing a risk of unemployment; the country facing a risk of trade disruption; the Rawlsian citizen confronting a risk of disability. As in the optimal tax problem the reversal paradox is acknowledged in the insurance and particularly the labor contract literature,<sup>6</sup> but has not been analyzed for itself.<sup>7</sup>

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<sup>5</sup> Another way to look at the same effect is to compare the shape of the indifference curve which passes through the post-transfer consumption point of the lucky-employed person, with the opportunity curve which passes through the same point and describes his consumption path as wage varies. Let  $\mathcal{L}[x(w),y(w)]$  indicate that opportunity set where  $x = [\hat{x} - \{(n+m)/n\}z(w)]$ , and  $y = [\{(n+m)/n\}wz(w) - (m/n)s(w)\hat{x}]$ . The slope of this opportunity curve is

$$(n1) \quad \frac{\partial y / \partial w}{\partial x / \partial w} = - \frac{[\{(n+m)/n\}[z(w) + wz_w] - (m/n)\hat{x}s_w]}{[(n+m)/n]z_w}$$

and the slope of the indifference curved is  $-w$ . Therefore utility declines if the opportunity curve has a smaller negative value than  $-w$ , i.e. if

$$(n2) \quad - \frac{[\{(n+m)/n\}[z(w) + wz_w] - (m/n)\hat{x}s_w]}{[(n+m)/n]z_w} > -w \quad \text{which occurs if}$$

$$(n3) \quad x/\hat{x} < [m/(n+m)]\sigma$$

which is the same result as (9) above.

<sup>6</sup> Such would include analyses of employment contracts (such as Milgrom, 1988), unemployment insurance (e.g.

### III.A. The Equivalence Between Lump-sum Transfers and Fair Insurance

The essential equivalence of these two transfer problems can be illustrated if we ask how much consumption is it rational to transfer between contingencies via insurance purchase. One benchmark exchange rate between contingencies is the "fair price," the price at which expected benefits equal expected costs, or expected monetary gain/loss is nil. An ability to transfer income between contingencies at fair prices is equivalent to an ability to lump-sum transfer income. And when fair insurance is available, the rational resource owner if he is risk neutral (or risk averse<sup>8</sup>) will necessarily over insure, so much that the otherwise "bad" contingency becomes preferred.<sup>9</sup> Using the same innocuous assumptions of (a) two goods, (b) two contingencies, (c) fair insurance, and (d) risk neutral, state independent linear homogeneous utility, this result follows from the necessity that when markets are incomplete an optimizing subject must trade off the benefits from an optimal commodity mix under alternative contingencies against unequal levels of utility in the alternative states of the world.<sup>10</sup> It depends in no way on arguments from moral hazard (Pauly 1969)<sup>11</sup>. Nor does it involve any presumed ability to protect against unemployment by probability improving "self-protection" measures (Erlach and Becker, 1972), nor interactions between these and risk aversion (McGuire, Pratt and, Zeckhauser, 1991). On the contrary, in the case to be studied here the probability of trade cutoff or unemployment is completely fixed.

### III.B. Assumptions

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Baily, 1974; Feldstein, 1976; Shavell and Weiss, 1979), or stockpiling of imports subject to embargo (e.g. Tolley and Wilman, 1978; McGuire, 1990) and the immense literature on provision for retirement. We emphasize that buying insurance may be only one of many possible protective measures against risk of unemployment. Others might include stockpiling, limited autarky, efforts to change the probabilities of bad outcomes or the analysis of the "perverse" incentives afforded by unemployment insurance (Feldstein, 1974, 1976).

<sup>7</sup> The closest relative to this paradigm we have found in the literature is the case of insurance against the loss of an "irreplaceable asset" as analyzed by Cook and Graham (1977). The phenomenon considered here although kin to that of Cook and Graham differs from them in that we are considering the case of being deprived of all replaceable assets and thrown back on one inalienable asset -- e.g. time, domestic resources, seat at the celestial convocation (i.e. claim to existence in some incarnation or other) etc. -- almost the exact converse of Cook and Graham.

<sup>8</sup> The proof as it relates to risk averse utility will be reserved for section IV.

<sup>9</sup> This conclusion contrasts with that of Cook and Graham (1977) who conclude that only if the "irreplaceable asset" is inferior will the expected utility maximizing decision maker over insure.

<sup>10</sup> We thank Mark Pauly for pointing out that our result reflects the Arrow-Debreu (1963, 1959) theorem that for Pareto efficiency the number of markets must equal or exceed the number of contingencies plus goods.

<sup>11</sup> Those arguments hinge on the incentive insurance may provide a subject to take less care thereby raising the chance of the bad outcome, which he has insured himself against, but benefiting from the savings in effort on self protection.

The assumptions for this intercontingency analysis strictly parallel those for optimal income distribution.<sup>12</sup> In this version of the problem the agent acts so as to maximize the expected value of a Von Neumann-Morgenstern expected composite utility

$$(10) \quad W = pU^1(\cdot) + (1-p)U^0(\cdot)$$

where  $p$ , probability of employment and  $(1-p)$  that of unemployment are known and fixed. He can protect himself against loss of  $y$ -consumption if unemployed by paying out a premium which costs  $t = (1-p)b/p$  when (or if) employed. In return he receives insurance benefit  $b$  if unemployed. Since  $(1-p)b - pt = 0$  this insurance is actuarially fair.

### III.C. The Optimal Insurance Purchase

Theorem: As with interpersonal transfers, at the maximum of  $W = W(*)$  -- where "\*" indicates solution values --  $U^0(*) > U^1(*)$ , provided  $f'' = 0$  throughout. To maximize his expected utility, this agent will purchase so much fair insurance that he is better off if the "bad" event is realized and he collects his insurance benefit. This follows from the first order conditions for maximization of

$$(11) \quad W = pU^1[(\hat{x} - z), (wz - \{(1-p)b/p\})] + (1-p)U^0[\hat{x}, b]$$

with respect to  $z$  and  $b$ . The FOC's are the same as those for the optimal redistribution problem:

$$(2 \text{ repeated}) \quad -U_x^1 + wU_y^1 = 0$$

$$(3 \text{ repeated}) \quad -U_y^1 + U_y^0 = 0$$

Where  $U_h^k$  indicates the marginal utility of good  $h$  in state  $k$ .<sup>13</sup>

For risk neutral linear homogeneous utility [ $f(V) = U = B V$ ] equalization of the marginal utility of good  $y$  across contingent states means again that the marginal utility of  $x$  must also be the same across contingencies, and therefore following the same argument as made in the interpersonal redistribution case  $x^0(*) > x^1(*)$  and  $y^0(*) > y^1(*)$ . With consumption of both commodities higher under unemployment than employment, utility must be higher in the

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<sup>12</sup> The agent also might be assumed to possess a positive and inalienable but indefinitely small endowment of good  $y$ . This assumption is made to insure that the agent's utility is still defined even after all opportunity to sell (part of) his  $\hat{x}$ -endowment is prevented.

<sup>13</sup> The reason these conditions are identical is that the opportunity to purchase fair insurance produces a price for intercontingency transfer which cancels out the probability weights in the expected utility maximand. Or if we took values of  $p = 1/2$  the maximands of equations (1) and (4) would be precisely the same.

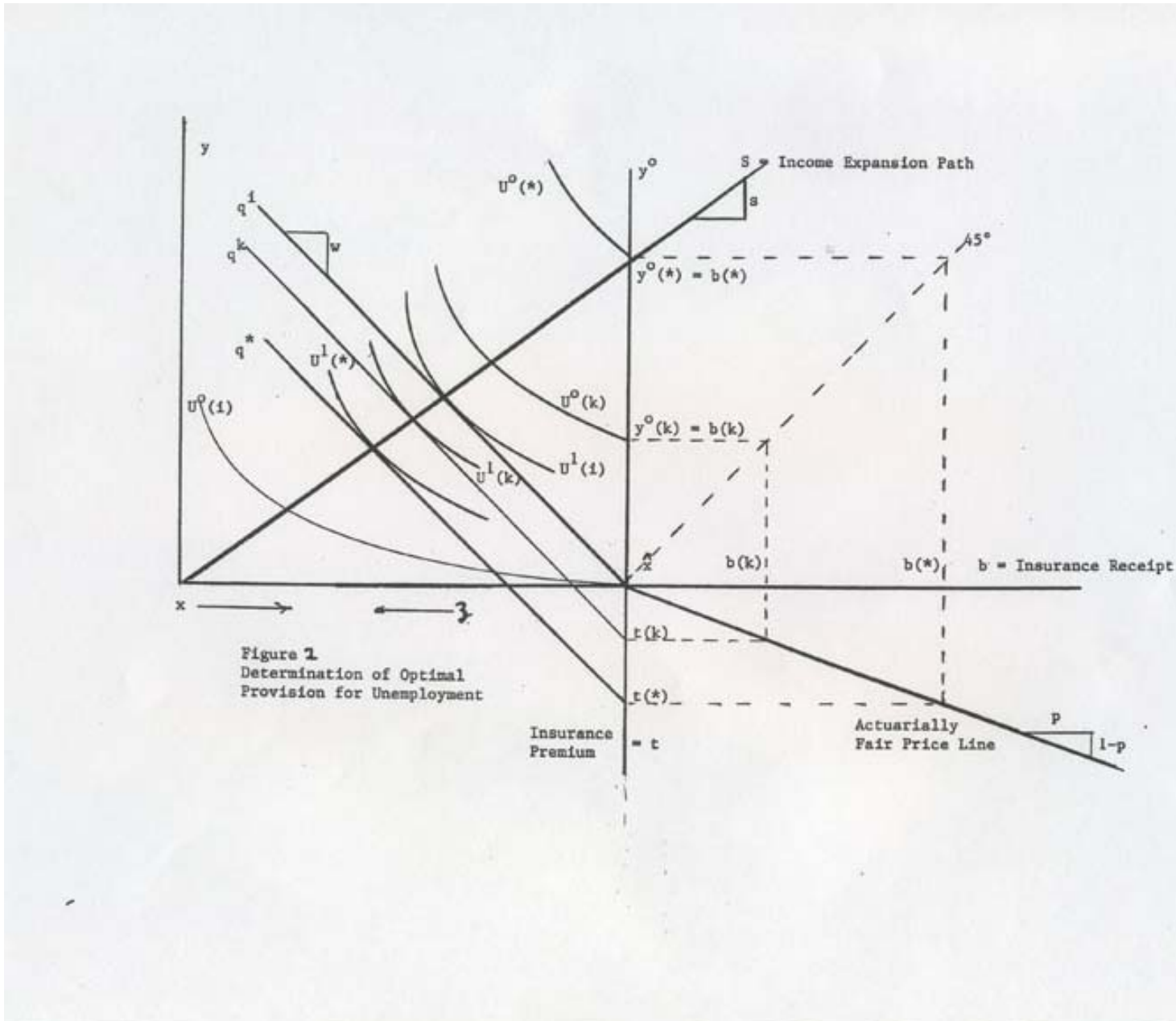
14  
former state .

Again the rational resource owner will insure or save so much that in the bad state of the world he would choose not to trade his resource even if he could (which he can not of course). As in the interpersonal transfer case, the optimal preparation for unemployment is to provide so much that in the event one would choose not to work. This result is pictured in Fig. 2 --- similar to Fig. 1 with an addition to the right of the vertical through  $\hat{x}$  . Now the  $\hat{x}$ -endowment may be sold for two purposes: to obtain  $y$  for consumption when employed, or to pay insurance

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<sup>14</sup> The direct analogy between this model and that of saving for retirement in the absence of bequest motive should be mentioned. Saving from earned income during  $n$  working years so as to enjoy one's entire endowment of leisure during  $m$  retirement years has exactly the same structure as the insurance problem with  $p = n/(n+m)$ . With state independent, linear homogeneous utility, together with the other assumptions specified in the foregoing analyses (plus zero bequest motives), the rational planner will save so much that his income/leisure ratio in retirement is the same as during his working life. As a result consumable commodity income is much greater during retirement when the entire endowment of leisure is consumed than it is during years of work.

This result is easily established for the individual with  $n$  certain years of working life followed by  $m$  certain years of retirement. During each working year his utility is  $U[(\hat{x} - z), (wz - t)]$  ; while during each retirement year is utility is  $U[\hat{x}, b]$ . Assume a pure year-to-year rate of time preference of  $i$  and a marginal rate of productivity of  $i$ . The individual's utility maximizing consumption profile then calls for equalizing marginal utilities of  $x$  and of  $y$  across outcomes. With CRS utility this can only result where  $y/x$  is the same when working and retired.



premiums (or savings) for benefits if unemployed. To the lower right of  $\hat{x}$  the fourth quadrant of the diagram shows the intercontingency transformation of premiums into insurance reimbursements at fair-odds prices.<sup>15</sup> If no insurance were purchased under conditions of employment, exchange of  $\hat{x}$  at price  $w$  would be pursued along price line  $q^0$  up to

<sup>15</sup> This "intercontingency transformation line" could also be useful to analyze the optimal income redistribution among different people when the numbers of transferors and transferees are different. For  $n$  "lucky" payors and  $m$  "unlucky" payees a slope of  $n/m$  would show the tradeoff between transfers per giver and receipt per receiver. Similarly for  $n$  years of work and  $m$  of retirement, a slope of  $n/m$  would show the ratio of annual savings to receipts, ignoring interest accumulations.

a point of tangency at  $U^1(i)$  --  $i$  being initial utility when employed, and unemployment would give consumption point  $(\hat{x}, 0)$  with utility  $U^0(i)$ . However, by spending  $t$  to purchase  $b$  of insurance at the fair price of  $t = [(1-p)/p]b$  the employment line under the good contingency shifts in, utility if employed drops and utility if unemployed rises. As  $t$  and  $b$  increase the consumption point when employed retreats as before along the income expansion path,  $S$ ; and the consumption point if unemployed climbs along the vertical through  $\hat{x}$ . The optimum in expected utility is reached at the unique intersection of these two curves giving  $y^0(*) = b(*)$  and unemployment utility of  $U^0(*)$ .

### III.D. Benefits and Costs of Insurance

A complementary way to summarize this result is to calculate the costs (measured in expected utility terms) **of insurance premiums expended** as the reduction in  $V^1 = U^1$  weighted by  $p$ , while the benefit (again in expected utility terms) **from insurance benefits received** is calculated as the gain in  $V^0$  or  $U^0$  weighted by  $(1-p)$ . With  $U = B$   $V$ , and  $V$  a CRS function, the weighted cost is linear as consumption-when-employed retreats along ray  $S$ , and therefore expected marginal utility cost is constant. On the other hand, the probability weighted utility gain is diminishing as consumption-when-unemployed climbs the vertical through  $\hat{x}$ <sup>16</sup>. Thus expected marginal benefit is uniformly decreasing and the optimal insurance purchase is uniquely determined.

### III.E. Comparisons of Income and Consumption Across Contingencies

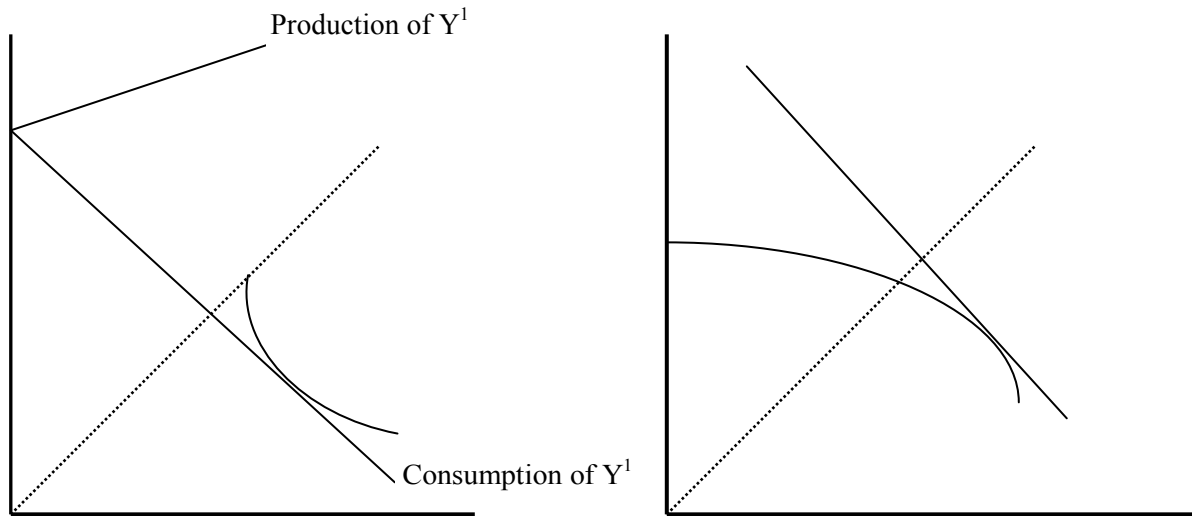
Figures 3a. and 3b. show the same effect with the inter-contingency consumption opportunity frontier, and utility possibility curve. As the risk neutral subject buys insurance at fair odds to transfer goods-consumption from the good to bad outcome his he gains  $[p/(1-p)][(s+w)/s]$  in  $y^0$  for each  $t = 1$  unit of  $y$  he gives up. Note that the resource owner offsets some of the loss in  $y^1$  consumption (i.e. the costs of insurance premiums) by working more (or increasing his production of  $y^1$ ). Therefore the trade-off between goods-consumption in the two contingencies is superior to a fair odds trade off, and the slope of the inter-contingency goods-transformation curve in Fig. 3a is flatter than and superior to fair odds. As the subject buys small amounts of insurance, first he will achieve equal utility

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<sup>16</sup> If the elasticity of substitution  $r$  were (a) infinite, or (b) zero and indifference curves, therefore, were (a) straight lines or (b) sharp cornered, the benefit curve would (a) also be linear or (b) be linear and have a kink at the



across contingencies. The utility gain from the insurance benefit is great because it is combined with the subject's resource (all of which is available when the market is lost). This allocation falls far short of maximization of expected utilities since at such an iso-utility allocation, the marginal utility of goods-consumption is greater in the bad than in the good



state of the world. After this iso-utility allocation is reached, further insurance purchases yield an iso-commodities-consumption allocation, where the amount of measured y-income is the same across contingencies. Expected utility is still not maximized since with  $y^1 = y^0$  and  $x^0 > x^1$  the marginal utility of y is still greater in state "o" than in state "1". Accordingly, Fig. 3a and 3b show the tangency maximum in expected utility to the right of both the iso-utility and the iso-goods-consumption points.

### III.F. Corner Solution Limits on Insurance

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benefit limit where the income expansion path and vertical through  $\hat{x}$  intersect.

It follows from the construction of Fig. 2 that under risk neutrality, the optimized (insurance protected) unemployment utility is unaffected by the probability of unemployment provided resource allocations to insurance do not force a corner solution. For lower values of  $p$  -- i.e. lesser likelihood of uninterrupted trade, or employment -- the optimized value  $U^0(*)$  remains unchanged with the entire utility deficit absorbed by lower and lower  $U^1(*)$ .<sup>17</sup>

**Proof:** From homogeneity with  $V^1_Y(*) = V^0_Y(*)$  the unemployment and employment consumption bundles lie on the same ray through the origin; but the unemployment consumption bundle must lie on the vertical through  $\hat{x}$ . The intersection of these two lines is unique and independent of  $p$ .

**Corollary:** As the probability of unemployment<sup>18</sup> increases from zero, gross earnings when employed (inclusive of insurance premiums) rise monotonically; insurance premium payments rise monotonically at a rate faster than the rise in earnings; and the proportion of gross earnings replaced declines monotonically from  $[(s+w)/sw]$  to  $[s/w]$  as  $p$  declines from  $p = 1$  to  $p = \hat{p}$  where  $w =$  wage rate, or sales price of exports, and  $s =$  slope of income expansion path at wage-price  $w$ , and  $\hat{p}$  is as defined next.

**Corollary:** For every value of  $s$  there is a value of  $p = \hat{p}$  which induces the risk neutral consumer insuring at fair odds, to allocate his entire earnings-if-employed to insurance premiums. Similarly, for every  $p$  there is an  $s = \hat{s}$  which induces all earnings to be allocated to insurance. Moreover, such an insurer will give up his entire endowment of  $\hat{x}$  to earn  $y$  to buy premiums to cover his unemployment.<sup>19</sup>

**Proof:** By inspection of Fig. 2, for any  $p$ , or fair odds line, if  $s$  is great enough,  $y^0(*) = b(*)$  will be great enough in turn to require a  $t(*)$  which absorbs  $\hat{x}$  in its entirety. It also follows that for the same wage line, the more industrious the worker, i.e. one with a higher slope  $s = y/x$  of income expansion path  $S$  will reach this corner solution

<sup>17</sup> The analog of this result in the interpersonal transfer case is that as more unlucky poor are added to the population and the ratio of lucky to unlucky declines, the post-transfer utility of the unlucky is unaffected until the rich are driven to a corner where they work full time and transfer their entire earnings to the poor. Only after this corner is reached will the poor's post transfer utility decline with increases in their numbers.

<sup>18</sup> Analysis of the effects of changes in probability of unemployment on optimal insurance purchase, and on the utility distribution between the "lucky" and "unlucky" states has an analog in the optimal income distribution problem as between different individuals. Changes (increases) in probability of unemployment correspond to changes (increases) in the ratio of "unlucky" to "lucky" individuals.

<sup>19</sup> For example with a Cobb Douglas utility function i.e. ---  $V = x^a y^{1-a}$  --- the critical value of  $\hat{p} = (1-a)$ ; for  $p < (1-a)$  the consumer cannot reach an optimum and is confined to the corner where  $x^1 = 0$  and  $y^0 < y^0(*)$ . We thank Leonard Mirman for pointing this out and for the example.

at lower risks of unemployment (1-p) than will a less industrious worker. Another way of stating the same effect is to say the higher the wage, at any given risk of unemployment, the greater will be the proportion of gross earnings allocated to insurance.

### III.G. Welfare Effects of a Wage or an Export Price Increase

The effect a rise in the price of the resource, as in the case of optimal income distribution, may increase or decrease the supply of the resource (labor or export good for example) depending on its supply price elasticity. However the amount of insurance purchased and therefore the utility in the bad or unlucky state of the world will unambiguously increase with an increase in price. Moreover, with sufficiently elastic supply of the resource, an increase in price will require an increase in supply so great as to lower net-of-insurance-premium welfare in the good or lucky state of the world.

## IV: FURTHER PROPERTIES OF OPTIMAL ALLOCATIONS UNDER RISK NEUTRAL UTILITY

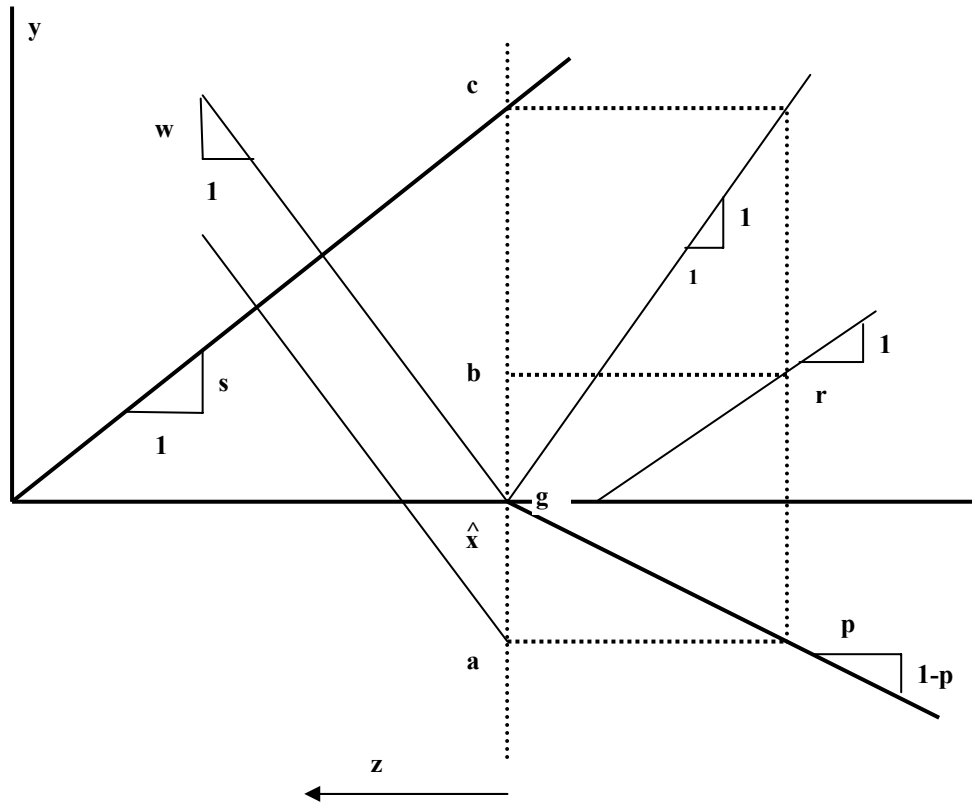
### IV.A. Extension to "Unfair" Insurance: Insurance Loading

Fair insurance is rare as is lump-sum income transfer. Insurance providers usually add a "loading factor" to recover administrative costs, to yield a profit, or anticipate moral hazard or adverse selection. To represent this effect, let the unit price of insurance be not  $(1-p)/p$  which is the "fair" price, but instead  $[(1-p)/p]r + g$ . This adjusts the fair price by a fixed unit charge "g" and a proportional risk inflator, "r." With this adjustment the necessary conditions for a expected utility maximum become

$$(12) \quad -U_X^1 + wU_Y^1 = 0$$

$$(13) \quad U_Y^0 - p[(1-p)/p]r + g/(1-p) = 0$$

If "g" in eq. (13) is zero and only the risk-inflation factor enters, the change compared to fair insurance is slight. (a) The optimal purchase (receipt) of insurance remains fixed independent of p, though at a lower amount than with fair insurance. (b) It no longer seems possible a priori to assert that welfare when unemployed is necessarily greater than when employed; the outcome can tip either way depending on the ordinal utility function. However, if earnings net of the insurance premium are fully replaced or more it is clear welfare is higher in the unemployed state; and it is clear moreover that the higher the likelihood of unemployment the more probable will this be true, since earnings in the good state decline with greater (1-p) while ex post consumption in the bad state is unaffected.. (c)



$a \hat{x}$  = Insurance Payment.  
 $\hat{x} c$  = Receipt under fair pricing.  
 $\hat{x} b$  = Receipt under loaded pricing.

**Figure 3c**  
**Optimal Insurance with Non-Actuarial Insurance Rates**  
 $p[(1-p)/p)r + g]/(1-p)$

Similarly, as in the no-load fair insurance case, when the probability of unemployment  $(1-p)$  rises, unemployment utility is unaffected (though it reaches a lower level than in the fair insurance case) and all the burden is allocated to the state when employed. (d) Likewise more industrious individuals will allocate a greater share of their earnings to insurance premiums than will less industrious.

A positive value for loading factor "g" changes the analysis lending another sort of predictability. For now as the probability of unemployment increases and  $p$  declines, the actuarial cost of the fixed load  $g$  is recovered with increasing likelihood (spread over a larger set of contingencies) so that the amount

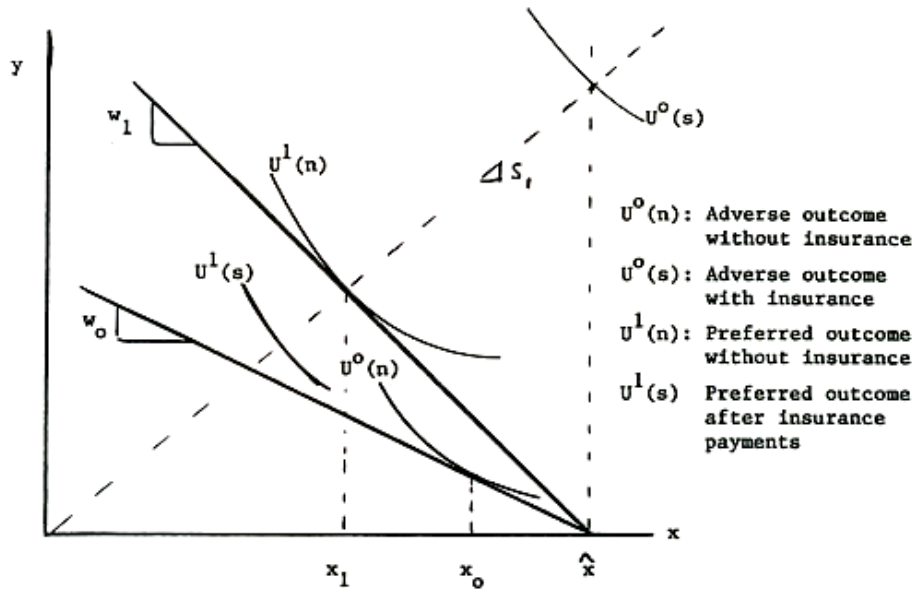
of insurance purchased increases as  $p$  declines. Pictorially, with  $g > 0$ , as  $p$  declines the optimal solution point ascends the vertical through  $\hat{x}$ , approaching the solution with  $g = 0$

Now the seller of insurance can counteract this tendency driven by the rationality of greater insurance purchase with greater risk, by making/causing  $r$  (the inflation factor) or  $g$  (the fixed load factor) or both of these to change with risk. But analysis of such a game between buyers and sellers would take us beyond our focus here.

#### **IV.B. Insurance Against A Demotion, Terms of Trade Reversal, or Partial Disability**

We have been concerned with an individual who earns nothing, a country which is totally embargoed, a Rawlsian citizen born completely disabled, or retirement which allows no gainful employment whatsoever. What of the case where the "bad" outcome is less extreme? Rather than total unemployment, the worker is demoted to a lower wage job; rather than an embargo a country risks adverse developments in its terms of trade; rather than total disability the celestial citizen contemplates diminished opportunity; and the retiree can pick up odd jobs at a lessened wage? How does enriching and softening the scenario for the "bad" outcome, alter the result? The answer is that we can generalize the foregoing analysis not only to differences in wage rates caused by differences in ability, but also differences in the effective prices paid for different goods by different individuals. Perhaps some individuals are more efficient searchers than others, or have more efficient household production functions. Individuals who are more efficient at producing a particular good, and therefore experience a lower effective price for that good would have a higher level of initial utility than less efficient individuals. As long as the good has a positive income elasticity, marginal utility of income is lower to the more efficient individuals when the level of utility is the same for all. Consequently, a lump-sum transfer or fair insurance purchase would also reduce the utility level of the more efficient below that of the less efficient in this case as well.

Figure 4



To be more specific suppose the bad outcome is a reduction in wage from  $w_1$  in good times to  $w_0$  in bad times.

This situation is pictured in Fig. 4. Now the expected utility maximand becomes

$$(14) \quad pU[(\hat{x} - z_1), (w_1 z_1 - \{1-p\}b)] + (1-p)U[(\hat{x} - z_0), (w_0 z_0 + pb)]$$

with variables of choice now work effort in each of the two states  $z_1, z_0$ , and insurance  $b$ . For interior solutions the necessary conditions become

$$(15) \quad U_x^0(*)/U_y^0(*) = w_0$$

$$(16) \quad U_x^1(*)/U_y^1(*) = w_1$$

$$(17) \quad U_y^0(*) = U_y^1(*)$$

However, all three necessary conditions cannot be satisfied at once unless  $w_1 = w_0$ ; but the essence of the problem lies in the assumption that the two wages differ. The last condition requires equal marginal utilities of good  $y$  (and by inference under CRS utility of good  $x$  as well) across contingencies. But this is inconsistent with unequal marginal rates of substitution as required by the first two conditions when  $w_1 \neq w_0$ . Accordingly, the expected utility maximizing individual **either** can purchase optimal insurance with an outcome at the intersection of the vertical

through  $\hat{x}$  and the consumption ray with slope  $s_1$ , **or** must settle for sub-optimal insurance due to his resource limits. If optimal insurance is available, the rational individual will refrain from working at all at the lower wage  $w_0$ . If the only insurance available is inadequate, however, then the individual may have to depend in part on the safety net provided by work at the lower wage.

## V: EXTENSION TO RISK AVERSE AND RISK PREFERRING UTILITY

### V.A. Risk aversion: $f'' > 0, f''' < 0$

To address the effect of risk aversion we will employ an extension to multi-commodity situations of the Arrow-Pratt (1963, 1964) definition of risk aversion as proposed by Kihlstrom and Mirman (1981) viz. a concave transformation of the common utility function  $V(x^1, y^1)$  representing the same ordinal preferences across individuals or contingencies. The "underlying" linear homogeneous function,  $V$ , is not altered; the same ordinal rankings and indifference curves as in Figs. 1 or 2 apply, except now these are renumbered. Each indifference curve, by strict quasi-concavity, has strictly a diminishing marginal rate of substitution (MRS). The necessary condition for a maximum to replace eq. (3) can be written

$$(18) \quad -[f'(V^1)][V_y^1(s^1)] + [f'(V^0)][V_y^0(s^0)] = 0$$

With  $V$ , first degree homogeneous, its first partial derivatives are functions of the consumption ratios,  $y^k/x^k = s^k$  only. (Since this is important in the argument to follow, the dependence is shown explicitly).

To show that  $U^0(*) > U^1(*)$  after optimal insurance has been purchased at fair odds, now assume  $U^0(*) < U^1(*)$  contrary to what is to be proved. Note that the indifference curve through  $[x^1(*), y^1(*)]$  --- being convex with negative slope -- intersects the vertical through  $\hat{x}$ , at a lesser value of  $s$  than  $s^1$ . Therefore, the value of  $s^0$  is still lower for a lower indifference curve  $V^0 < V^1$ . Therefore, if  $V^0 < V^1$  it follows that  $s^1(*) > s^0(*)$ . In turn this entails  $V_y^1(*) < V_y^0(*)$ . Whence to maintain eq (18),  $f^{19}[V^1(*)] > f^{19}[V^0(*)]$ ; this in turn entails  $V^0(*) > V^1(*)$ , which contradicts  $U^1(*) > U^0(*)$ . Thus if  $f'' > 0$  and  $f''' < 0$ ,  $U^0(*)$  is strictly greater than  $U^1(*)$ . QED.

Intuitive support for this conclusion might begin by considering an allocation which yields the same value of  $V$  equally or irrespective of which state of the world occurs, i.e. the value of  $V = V^1 = V^0$  at this allocation the marginal utility of benefits vs. the marginal cost of further incremental insurance is determined strictly by the consumption mix,  $y^0/x^0 < y^1/x^1$ . With  $V^0 = V^1$ , then at this trial allocation  $f^{19}[V^0] = f^{19}[V^1]$ . But the consumption

mix strictly favors adding  $y$  to state "o"; therefore, an iso-utility allocation cannot be maximal. However, for further insurance beyond this trial value where  $U^0 = U^1$  the concave transformation of  $V$  changes the probability weighted marginal utility cost of insurance from a constant to an increasing function of the amount of insurance provided, and at the same time accelerates the decline of the marginal benefit function. The upshot of these two effects is that risk aversion reduces the equilibrium insurance purchase below the risk neutral level but not so much that unemployment becomes less desirable than employment.<sup>20</sup>

**V.B. Risk preference;  $f'' > 0, f''' > 0$**

The potential for multiple optima inherent in the non-convexities introduced by risk preference shows up pointedly in this analysis as well. Compared with the risk neutral outcome, a positive convex transformation  $f(V)$  can lead to two types of alternative solution. Imagine starting from the risk-neutral optimum (with  $U^0(*) > U^1(*)$ ); then increase  $f''$  slightly from zero. One possible effect is for more insurance to be purchased and the superiority of "bad" over "good" luck to increase. A second is that risk preference requires purchase of less fair insurance-- so much less that unemployment is not fully insured and remains the less desirable outcome.

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<sup>20</sup> The log utility function  $U = [a\text{Ln}(x) + (1-a)\text{Ln}(y)]$  gives a good example of risk averse utility in a bi-commodity world. In this case maximization of

$$p[a\text{Ln}(\hat{x}-z) + (1-a)\text{Ln}(wz\{(1-p)/p\}b)] + (1-p)[\{ \text{Ln}(\hat{x}) \} + \{(1-a)\text{Ln}(b)\}]$$

with respect to  $z$  and  $b$  yields

$$\begin{aligned} y^1(*) &= y^0(*) = pwz(*) \\ x^1(*) &= \{\hat{x} / [1 + (1/ap) - 1/p]\} \\ x^0(*) &= \hat{x} > x^1(*) \end{aligned}$$

Therefore  $U^0(*) > U^1(*)$ . Bill Evans suggested this example, similar to Atkinson-Stiglitz (1978, p. 344).



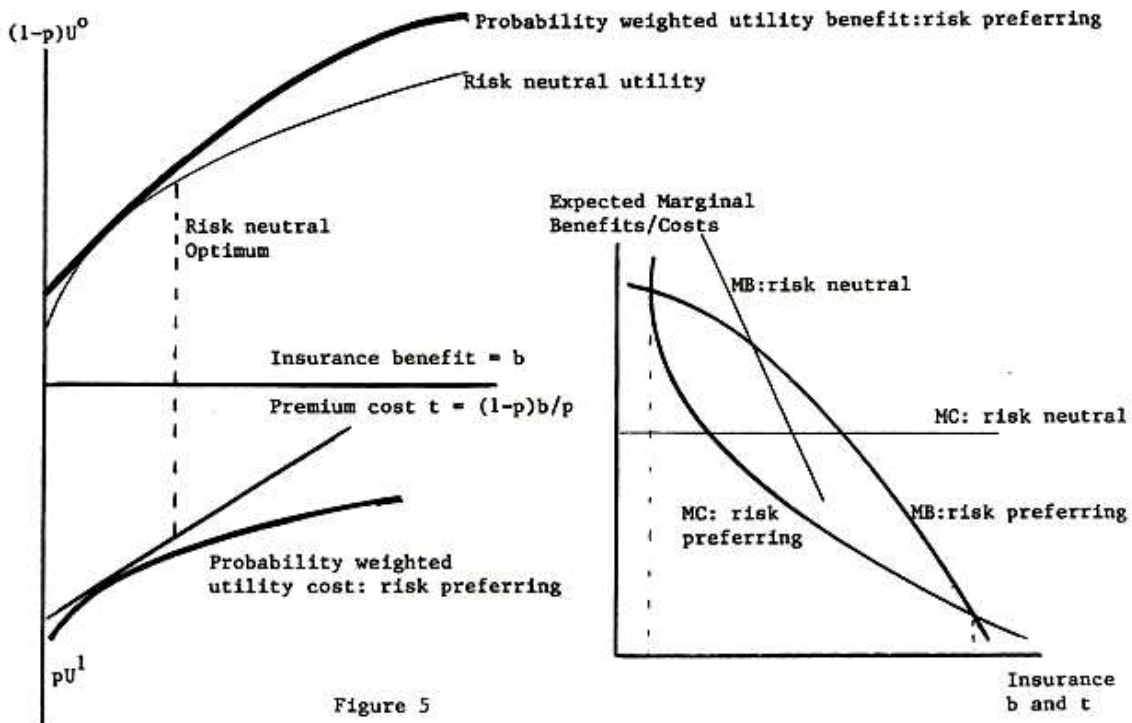


Figure 5

In the former case the unemployment consumption point moves up the vertical above the risk neutral  $y^0(*) = b(*)$ . In this region  $U^0(*) > U^1(*)$ ,  $s^0(*) > s^1(*)$  and  $U_Y^0(*) < U_Y^1(*)$ . Consistency with eq. (18) requires  $f'^0 > f'^1(*)$ , and, therefore, risk preference in the utility function,  $f'' > 0$ . Alternatively, consumption when unemployed can move below the risk neutral  $y^0(*)$ . In this region,  $s^0(*) < s^1(*)$  and, therefore,  $U_Y^0(*) > U_Y^1(*)$ . In this case, consistency with eq. (18) requires  $f'^0 < f'^1(*)$ . Thus if  $f'' > 0$  then  $V^0(*) < V^1(*)$ . Fig. 5 shows both cases. There the probability weighted underlying function of benefits and costs --  $V$  from the risk-neutral case -- has been subjected to a convex transformation through the point where  $U^0 = U^1$ . This increase in  $f''$  causes the probability weighted marginal utility cost to change from constant to declining, and the weighted marginal utility benefit to decline more rapidly. The possibilities for multiple solutions are clear.

## VI: MORAL HAZARD

But if the case for over provision or over insurance is so logical as the foregoing argument suggests, why

does it seem rarely observed? One reason may be that such optimal provision for adversity interacts with moral hazard. Should insurance proceed so far that bad fortune turns to good, the temptation to court the bad outcome may be irresistible.<sup>21</sup>

### Moral Hazard with Fair Pricing

It will shed light on the structure of the moral hazard problem here, to consider one agent who can insure himself against mishap at a perfectly fair price. Our analysis tells us he will always buy enough insurance so that in the event of misfortune his consumption of  $y^0(*) = s\hat{x}$ . With fair insurance this requires a premium payment of  $[(1-p)/p]s\hat{x}$ , so that  $y^1 = wz - [(1-p)/p]s\hat{x}$ , whence

$$(19) \quad \frac{y^1}{x^1} = s = \frac{wz - [(1-p)/p]s\hat{x}}{\hat{x} - z}$$

And therefore

$$(20) \quad z^* = s\hat{x}/p(w+s)$$

It follows that expected utility, optimized by purchase of fair insurance can be written:

$$(21) \quad V(p) = pU^1\left[\left\{\hat{x}\left(1 - \frac{s}{p(w+s)}\right)\right\}, \left\{\hat{x}\left(\frac{ws}{p(w+s)} - \frac{(1-p)s}{p}\right)\right\}\right] + (1-p)U^0(\hat{x}, s\hat{x})$$

Now to get at the moral hazard question let the agent be free to choose  $p$ . The values of  $\hat{x}$ ,  $s$ , and  $w$  are all taken as parameters, so that the last term  $U^0$  is constant. First order conditions for an optimum require

$$(22) \quad \frac{dV(p)}{dp} = U^1 - U^0 + p \frac{dU^1}{dp} = 0$$

$$(23) \quad p \frac{dU^1}{dp} = U_x^1 \frac{\hat{x}s}{p(w+s)} + U_y^1 \frac{\hat{x}s^2}{p(w+s)}$$

Then using the FOC  $U_x^1 = wU_y^1$  this yields

$$(24) \quad p \frac{dU^1}{dp} = U_y^1 \frac{\hat{x}s}{p} = U_x^1 \frac{\hat{x}s}{pw}$$

The underlying utility function is linear homogeneous, suppose it is  $U = x^\alpha y^{1-\alpha}$ ; then along income expansion paths  $s/w = (1-\alpha)/\alpha$ . Next, define  $s^{1-\alpha} = [\{(1-\alpha)/\alpha\}w]^{1-\alpha} = \beta$ . Then it follows that

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<sup>21</sup> This argument, however, does not tell strongly against saving/investing for retirement; in fact, it may provide explanation for large accumulation for old age additional to or alternative to bequest motives.

$$(25) \quad U^0 = \hat{x}\beta; \quad U^1 = \frac{\hat{x}[p(w+s)-s]}{p(w+s)}\beta; \quad U^0 - U^1 = \frac{\hat{x}s}{p(w+s)}\beta$$

$$(26) \quad p \frac{dU^1}{dp} = \frac{\hat{x}s}{pw} U_x^1 = \frac{\hat{x}(1-\alpha)}{p\alpha} U_x^1 = \frac{\hat{x}}{p} (1-\alpha) s^{1-\alpha} = \frac{\hat{x}}{p} (1-\alpha)\beta$$

Accordingly for  $dp > 0$

$$(27) \quad U^0 - U^1 + p \frac{dU^1}{dp} = \frac{\hat{x}}{p} \left[ -\frac{s}{w+s} + (1-\alpha) \right] \beta = \frac{\hat{x}}{p} \left[ -\frac{1}{\{\alpha/(1-\alpha)\} + 1} + (1-\alpha) \right] \beta = 0$$

Thus the rational and perfectly and fairly insured agent (with homothetic utility) is indifferent among outcomes<sup>22</sup>. Although he has no reason to take any action to avoid risks, neither has he any incentive embrace risk, provided he is not driven to a corner. In a space of  $U^0 - U^1$  his utility possibility curve is a straight 45° line. This establishes an entry point for considering the moral hazard problem under our assumptions.

The opportunity to insure fairly induces risk neutrality in his behavior, since unit costs (in terms of utility foregone in the good outcome) are constant (short of corner solutions) and thus this opportunity neither promotes nor suppresses moral hazard. It is doubtful that such unconcern about risks should be denominated "moral hazard" rather than normal economizing, since neither the buyer nor the maker of the insurance loses anything here. If the maker writes a fair insurance policy in any case and should have no complaint since he is compensated for otherwise unfavorable changes in  $p$ . And for this ideal buyer of insurance (with ideal insurance), if changing his risk profile can bring in money then he will simply economize with a choice of risk that maximizes this source income. Whether this would involve greater caution or less for him is irrelevant.

Once can see this from allowing resource expenditures "m" to influence  $p$ , so that  $p = p(m)$ , where  $m$  is an "outlay" required in both contingencies. Now in place of Equation (11) expected utility can be written

$$(28) \quad \text{Max}_{z, m, b} : W = p(m)U^1[\hat{x} - z, (wz - m - \{[1-p(m)]b/p(m)\})] + (1-p(m))U^0[\hat{x}, (b-m)]$$

First order conditions require as before

$$(29) \quad z : \quad -U_x^1 + wU_y^1 = 0$$

$$(30) \quad b : \quad -U_y^1 + U_y^0 = 0$$

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<sup>22</sup> This result is not new. Atkinson and Stiglitz (1980, p. 346) draw a linear utility possibility curve.

$$(31) \quad m : \quad pU'_y[-1 + b\{\frac{(p+1)}{p^2}\}p'] - (1-p)U'_y + p'[U^1 - U^0] = 0$$

From (30) it follows that  $b - m = s\hat{x}$ . In other words, fair insurance takes the consumption point (net of expenditure  $m$ ) up the vertical through  $\hat{x}$  to the intersection with the income expansion path with slope "s". Utilizing this implication we could proceed as before, solving (29) and (30) for  $z$  --- which gives  $z = (s\hat{x} + m) / p(w + s)$  --- and then folding this together with  $b - m = s\hat{x}$  into Equation (28) and asking, what now is the optimal value of  $m$  and thus of  $p$ . We can avoid this effort, however, by recalling that absent  $p$ ,  $m$  and  $p(m)$  the rational agent is indifferent among values of  $p$ . This must mean that the optimal  $m$  in (28) is either zero or negative. With no payoff to  $p'$  itself (as we have established) its value can only be instrumental. Thus if  $m$  is negative, our agent is compensated for taking risks. But since he is indifferent to risk itself because he is perfectly insured, he will seek the maximum in compensation. He wants to maximize  $-m > 0$ .

At this optimum the inverse function  $m = m^{-1}(p)$  will have zero first derivative. Moreover with  $-m$  at this maximum, the rational agent minimizes the amount of fairly priced insurance he will purchase. The positive compensation  $-m$ , lifts his endowment point at the bad contingency above point  $(\hat{x}, 0)$  to point  $(\hat{x}, -m)$ , so that to consume at  $s\hat{x}$  in "bad" times he wants to insure less.

This reasoning in no way undermines the qualitative conclusion suggested by our analysis, that an agent the size of an entire country should write self-insurance by stockpiling significant amounts for emergencies, especially if it can self insure at actuarially fair rates. The larger the size, the more plausible is fair self-insurance. However, this might occur over lengthy periods of time<sup>23</sup>, so that the process (aside from any other resources costs of stockpile accumulations and maintenance) should have to reckon with interest expense (implicit or explicit), and this could modify the neutralization of moral hazard just identified.

### **Moral Hazard Under More Realistic Insurance Pricing**

Introducing more realistic insurance-pricing assumptions, even with homothetic, risk neutral utility raises further interesting questions concerning the question of moral hazard. Suppose that rather than actuarial fairness, the price paid in the lucky contingency to obtain one unit of good  $-y$  in the unlucky contingency is as in Equation (13) above, i.e.  $[(1-p)/p]r + g$ . There is an interesting novelty here as Eq. (13) shows: the greater the risk of the bad contingency --- greater  $(1-p)$  --- the more of the income shortfall (i.e. of the distance  $s\hat{x}$ ) will be replaced, and the

closer will the solution  $U^{0*}$  approach the fair insurance outcome. The reason is that as  $p$  increases the constant load "g" recurs with greater likelihood and the effective actuarial cost of insurance increases, so less insurance is bought. Accordingly, to avoid the fixed cost, the insured entity desires low  $p$  i.e. greater risk, a sure moral hazard incentive.

This train of thought implies that pricing insurance as per Eq. (13)  $[\{(1-p)/p\}r + g]$  is a bad policy for the writer of insurance since it provides incentives to the insured party to minimize  $p$  --- aside from any other costs or benefits incident to choice of  $p$  germane to that insured party. Assumedly this effect is well established in the theory and practice of insurance writers, and has caused insurers to reject simple formulas like  $[\{(1-p)/p\}r + g]$  and for example to limit amounts of insurance purchasable, or to monitor risk histories and behaviors.

## VII CONCLUSION

The problem sketched out here is representative of a broad class of situations in which individuals or groups are at risk of being thrown back on their own resources and cut off from exchange with nature or with other persons or institutions. The main insight from the analysis is that rationally desired provision against the loss of a market for one's resources can easily be great enough to cause "over insurance" or "over protection." In fact this is the expected case. No matter how risk averse the subject, if he can make "fair odds" provision he will protect himself so much that he prefers the state "loss of market plus benefit" to the state "regular earnings minus savings". If this effect were more broadly recognized than it seems to be, the so called disincentive effect of unemployment benefits might seem less perverse and deplorable, the argument for dramatic transfers to the disabled might seem to require much less altruism, and the arguments for stockpiling taken more seriously.

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<sup>23</sup> The effect of time, interest, and discounting on the optimal stockpile is discussed in McGuire (2006).

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