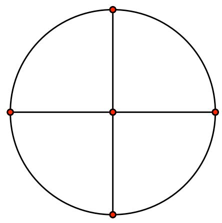
13.3 Evaluate Trigonometric Functions of Any Angle

Name ____ Date ___________

Today we extend the definitions of the 6 trigonometric functions so that they extend to angles defined as rotations.

1. The circle at right is a ***unit circle*** with center at the origin and radius 1. Label the coordinates of the four points where the x-axis and y-axis intersect the circle. Do you remember how to write the equation of the circle?



2. The diagram below shows the first quadrant of the unit circle.

   (a) Convert $\frac{\pi}{6}$ radians to degrees, then draw an angle in standard position whose terminal side determines an angle of $\frac{\pi}{6}$ radians.

   (b) Draw a vertical line to the x-axis to form a right triangle whose hypotenuse is a radius of the circle. Use special triangles to find:

   $\sin\frac{\pi}{6} =$ $\cos\frac{\pi}{6} =$ $\tan\frac{\pi}{6} =$

   (c) Find the x- and y-coordinates of the point where the hypotenuse of the triangle meets the unit circle. What do you notice?

3. Repeat Question 2 for angles of $\frac{\pi}{4}$ radians and $\frac{\pi}{3}$ radians.

$\theta = \frac{\pi}{4}$

$\theta = \frac{\pi}{3}$

Questions 1-3 suggest the following definitions for the trig functions in terms of x- and y-coordinates, for angles whose terminal sides are in Quadrant 1:

If (x, y) is on the unit circle:

$\sin\theta = y$ $\qquad$ $\csc\theta = \frac{1}{y},\ y \neq 0$

$\cos\theta = x$ $\qquad$ $\sec\theta = \frac{1}{x},\ x \neq 0$

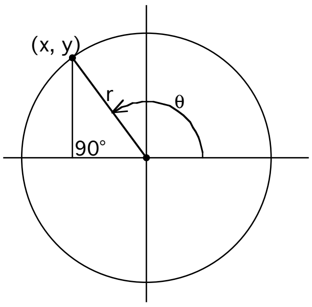
$\tan\theta = \frac{y}{x}, x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y},\ y \neq 0$

If (x, y) is *not* on the unit circle, then $r = \sqrt{x^2 + y^2}$ is the radius of the circle and the hypotenuse of the triangle. This leads to:

$\sin\theta = \frac{y}{r}$ $\qquad$ $\csc\theta = \frac{r}{y},\ y \neq 0$

$\cos\theta = \frac{x}{r}$ $\qquad$ $\sec\theta = \frac{r}{x},\ x \neq 0$

$\tan\theta = \frac{y}{x}, x \neq 0$ $\qquad$ $\cot\theta = \frac{x}{y},\ y \neq 0$



And now we take a big conceptual step – these definitions are applied to *all* angles, even those whose terminal sides are not in Quadrant 1, even those where there is no triangle formed.

4. Draw an angle θ in standard position whose terminal side contains the given point. Then use the appropriate definitions to evaluate the six trigonometric functions of θ.

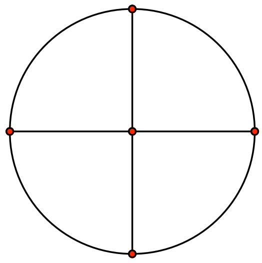
   (a) $(-0.6, 0.8)$

   (b) $(1, -3)$

   (c) $(0, -1)$

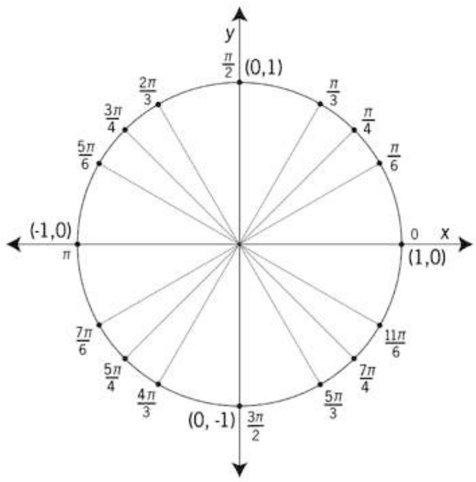
The circular function definitions for trig lead to patterns about where to expect positive or negative values for the different trig functions. Fill in the following charts with ***positive and negative signs***, where appropriate. Use the unit circles to the right for assistance.

Trig Functions	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin\theta$, $\csc\theta$				
$\cos\theta$, $\sec\theta$				
$\tan\theta$, $\cot\theta$				



5. Find the following values ***without*** a calculator. You will need to use:
   - The x- and y-coordinates you found in Questions 1-3
   - The circular function definitions of the 6 trig functions, and
   - The symmetries of the circle.

   A diagram of the unit circle is given for reference.



(a) $\sin\frac{7\pi}{4}$

(b) $\cos\frac{2\pi}{3}$

(c) $\tan\frac{7\pi}{6}$

(d) $\sin\pi$

(e) $\cos 150^\circ$

(f) $\csc\left(-30^\circ\right)$

(g) $\cot 450^\circ$

(h) $\sec\frac{17\pi}{4}$

(i) $\csc 3\pi$

(j) $\tan 510^\circ$

The ideas concerning symmetries of the circle that you used in the previous question can also be expressed using the concept of ***reference angles***.

A reference angle is defined as the ***acute angle between the terminal ray and the x-axis***. The measure of the reference angle is always positive. You locate the reference angle by drawing a segment ***perpendicular to the x-axis***.

6. Sketch the angle. Then find the measure of its reference angle.

(a) 320° (b) $\frac{7\pi}{9}$ (c) $-\frac{4\pi}{5}$

Can you figure out reference angle formulas for each quadrant? Give two versions for each, in degrees, and radians.

(a) Quadrant II (b) Quadrant III (c) Quadrant IV