A TWO-STAGE BATCH ARRIVAL QUEUEING SYSTEM WITH A RANDOM SETUP TIME UNDER MODIFIED BERNOULLI SCHEDULE VACATION*

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ABSTRACT
We consider a batch arrival two stages of service queueing system with a random setup time under Bernoulli vacation schedule, where the service if the first unit at the commencement of each busy period is preceded by a random setup time, on completion of which service starts. For this model we first obtain the steady state queue size distribution at a random epoch as well as at a departure epoch. Next we discuss some related vacation models. Finally, we obtain some system performance measures of this model.

Key words and phrases: $M^X/(G_1, G_2)/1$ queue, setup time, vacation time, Bernoulli schedule, queue size and busy period.

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RESUMEN
Nosotros consideramos una llegada del lote de dos fases de un sistema de colas de servicio con un tiempo de arreglo aleatorio y horario de vacación con distribución de Bernoulli, donde el servicio comienza si la primera unidad al principio de cada periodo ocupado es precedida por un tiempo de arreglo aleatorio, con cuya realización el servicio empieza. Para este modelo nosotros primero obtenemos la distribución de estado estacionario del tamaño de cola en un momento aleatorio, así como en un momento de comienzo. Después discutimos algunos modelos de vacación relacionados. Finalmente, obtenemos algunas medidas del sistema de comportamiento de este modelo.

1. INTRODUCTION

The queue with Bernoulli schedule vacation was first studied by Keilson and Servi [1986], where they introduced the concept of modified service time distribution. Recently, Madan [1987, 2000] studied Bernoulli vacation models for two stage heterogeneous service queueing system under certain modifications. Also he cited some interesting applications in many real life situations. Further, Ghafir and Silio [1993] recognized its applications in a Multiple Access Ring Network.

Recently, the $M^x/G/1$ queue with a random setup time, where the service of the first unit in each busy period is preceded by a random setup period was studied by Choudhury and Krishnamoorthy [2003], as proposed by Levy and Kleinrock [1986] as well as by Doshi [1985] under certain modifications. In fact some aspects of this model have also been studied by Doshi [1986], Takagi [1986] and Choudhury [1995, 2000].

Although some aspects of these types of models were studied by these authors, it seem that batch arrival queue with two stages of heterogeneous service will give us much more information on the number of batches instead of total number of individual units in deciding whether the server is activated or not. Thus in this paper we propose to study such a two stage batch arrival queue, where the concept of a random setup time is introduced under the Bernoulli vacation schedule.

The batch arrival queueing models under different vacation policies have been treated earlier by a good number of authors, for example see Baba [1986], Lee and Srinivasan [1989], Lee **et al**. [1994,1995], Choudhury [2002a, 2202b] and Madan and Abu-Dayyeh [2002], to mention a few. However our model is different from others and it is to some extent, generalized in nature, where in on completion of two successive

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stages of service, a unit may leave the system with probability (1-r) or the server goes for a vacation of random length with probability r ($0 \le r \le 1$) after completion of a *SET*.

In this paper, we first obtain the steady state queue size distribution at a random epoch as a generalization of the results obtained in Choudhury [2000]. Next, we obtain the queue size distribution at a departure epoch. Further, we demonstrate the existence of stochastic decomposition property of the queue size distribution which has been shown to decompose into the distributions of three independent random variables, one of which is the queue size of $M^x/(G_1, G_2)/1$queue without SET as discussed in Madan [2000, 2001]. The interpretation of the other two random variables will also be provided. Further, we discuss some related vacation models.

2. THE MATHEMATICAL MODEL

We consider a batch arrival queueing system with two stages of heterogeneous service on *FCFS* basis, where arrivals occur according to a compound Poisson process with the batch size random variable ‘X’. The server is turned off the system each time as soon as the system empties (turned off period). The system becomes operative only when one or a batch of customers arrive to the system. At this point of time the server does not offer proper service to the first waiting customer immediately; rather it undertakes an additional amount of time of random length called setup time (SET) (during which no proper work is done) in order to set the system in to operative mode before actual service begins (setup period). Assuming that the *SET* random variable follows a general law of distribution with distribution function (df) $S(x)$, *Laplace-Stieltjes transform* (*LST*) $S^*(\theta)$ and finite moments $E(S^k)$ ($k \ge 1$). The server then performs two stages of heterogeneous service in succession (busy period), the first stage service (*FSS*) followed by the second stage service (*SSS*). Further, it is assumed that the service time S_i (i =1, 2) of the *i* th stage service follows a general probability law with d.f $B_i(x)$, *LST* $B_i^*(\theta)$ and finite moments $E(B_i^k)$, $k \ge 1$, i = 1,2. As soon as the *SSS* of a unit gets completed, the server may go for a vacation of random length *V* (vacation period*)* with probability $r\,(0 \le r \le 1)$ or may continue to serve the next unit, if any, with probability (1 – r). Otherwise, it turns off the system. Next, we assume that the vacation time *V* of the server follows a general probability law with df $V(x)$, *LST* $V^*(\theta)$ and finite moments $E(V^k)$, k = 1, 2 and is independent of the *SET* S, service times B_i and the arrival process. Notationally, our model is denoted by $M^X(G_1,G_2)/V/1(BS)/SET$ queue, where V represents vacation time and BS represents Bernoulli schedule. Thus, the time required by a unit to complete a service cycle, which we may call as modified service time, is given by

$$B = B_1 + B_2 + V, \quad \text{with probability } r,$$

$$= B_1 + B_2, \text{ with probability } (1 - r),$$

so that $B^*(\theta)$, the LST of the modified service time for our model is given by

$$B^*(\theta) = (1-r)B_1^*(\theta)B_2^*(\theta) + rB_1^*(\theta)B_2^*(\theta)V^*(\theta)$$

and the first two moments of B are given by

$$E(B) = -\frac{d}{d\theta}B^*(\theta)\Big|_{\theta=0} = E(B_1) + E(B_2) + rE(V),$$

$$E(B^2) = (-1)^2\frac{d^2}{d\theta^2}B^*(\theta)\Big|_{\theta=0}$$

$$= E(B_1^2) + E(B_2^2) + rE(V^2) + 2[E(B_1)E(B_2) + rE(V)\{E(B_1) + E(B_2)\}].$$

3. QUEUE SIZE DISTRIBUTION AT A RANDOM EPOCH

In this section, we first set up the system state equations for the queue size distribution at a random epoch by treating the elapsed *FSS* time, *SSS* time and vacation time as supplementary variables. Then we solve the equations and derive the probability generating function (*PGF*) for it. We define

$\lambda =$ batch arrival rate,

X = size of the arriving batch (a random variable),

a_k = Prob [X = k],

$X(z) = \sum_{k=1}^{\infty} z^k a_k$, the *PGF* of X,

$E[X_{[k]}] = E[X(X-1) \ldots (X-k+1)]$ ($< \infty$), the k th factorial moment of X.

Further, it may be noted that V(x), S(x) and $B_i(x)$ being distribution functions, we have V(0) = 0, V(∞) = 1, S(0) = 0, S(∞) = 1, $B_i(0) = 0$ and $B_i(0) = 0$ for i = 1, 2 and that V(x), S(x) and $B_i(x)$ are continuous at *x* = 0, so that

$$v(x)dx = \frac{dV(x)}{1 - V(x)},$$

$$\eta(x)dx = \frac{dS(x)}{1 - S(x)},$$

$$\mu_i(x)dx = \frac{dB_i(x)}{1 - B_i(x)}, \ i = 1,2$$

are the first order differential functions (hazard rates) of V,S and B_i (i = 1,2) respectively.

Let $N_Q(t)$ be the queue size at time 't', $B_1^0(t)$ be the elapsed *FSS* time at time 't', $B_2^0(t)$ be the elapsed SSS time at time 't', $S^0(t)$ be the elapsed setup time at time 't' and $V^0(t)$ be the elapsed vacation time at time 't'. Further, we introduce the random variable

$Y(t) = 0$, if the server is turned off at time 't',

1, if the server is busy with *FSS* at time 't',

2, if the server is busy with *SSS* at time 't',

3, if the server is on vacation at time 't'.

4, if the server is on setup at time 't'.

Thus the supplementary variables $B_1^0(t)$, $B_2^0(t)$, $S^0(t)$ and $V^0(t)$ are introduced in order to obtain a bivariate Markov process $\{N_Q(t), \delta(t)\}$, where $\delta(t) = 0$ if $Y(t) = 0$, $\delta(t) = B_1^0(t)$ if $Y(t) = 1$, $\delta(t) = B_2^0(t)$ if $Y(t) = 2$, $\delta(t) = V^0(t)$ if $Y(t) = 3$ and $\delta(t) = S^0(t)$ if $Y(t) = 4$ and define the following limiting probabilities

$$P_{0,0} = \operatorname*{Lim}_{t\to\infty} \operatorname{Prob}\ [N_Q(t) = n,\ \delta(t) = 0\ ],$$

$$P_{1,n}(x)dx = \operatorname*{Lim}_{t\to\infty} \operatorname{Prob}\ [N_Q(t) = n,\ \delta(t) = B_1^0(t);\ \ x < B_1^0(t) \le x + dx\ ],\ \ x > 0,\ n \ge 1,$$

$$P_{2,n}(x)dx = \operatorname*{Lim}_{t\to\infty} \operatorname{Prob}\ [N_Q(t) = n,\ \delta(t) = B_2^0(t);\ \ x < B_2^0(t) \le x + dx\ ],\ \ x > 0,\ n \ge 1,$$

$$R_n(x)dx = \operatorname*{Lim}_{t\to\infty} \operatorname{Prob}\ [N_Q(t) = n,\ \delta(t) = V^0(t);\ \ x < V^0(t) \le x + dx\ ],\ \ x > 0,\ n \ge 0,$$

$$Q_n(x)dx = \operatorname*{Lim}_{t\to\infty} \operatorname{Prob}\ [N_Q(t) = n,\ \delta(t) = S^0(t);\ \ x < S^0(t) \le x + dx\ ],\ \ x > 0,\ n \ge 1.$$

Now the analysis of the limiting behavior of this queueing process at a random epoch can be performed with the help of Kolmogorov forward equations:

$$\frac{d}{dx}P_{1,n}(x)+[\lambda+\mu_1(x)]P_{1,n}(x)=\lambda\sum_{i=1}^{n}a_iP_{1,n-i}(x),\quad x>0,\ n\geq 1, \tag{3.1}$$

$$\frac{d}{dx}P_{2,n}(x)+[\lambda+\mu_2(x)]P_{2,n}(x)=\lambda\sum_{i=1}^{n}a_iP_{2,n-i}(x),\ x>0,\ n\geq 1, \tag{3.2}$$

$$\frac{d}{dx}Q_n(x)+[\lambda+\eta(x)]Q_n(x)=\lambda\sum_{i=1}^{n}a_iQ_{n-i}(x),\ x>0,\ n\geq 1, \tag{3.3}$$

$$\frac{d}{dx}R_n(x)+[\lambda+v(x)]R_n(x)=\lambda\sum_{i=1}^{n}a_iR_{n-i}(x),\ x>0,\ n\geq 1, \tag{3.4}$$

$$\frac{d}{dx}R_0(x)+[\lambda+v(x)]R_0(x)=0, \tag{3.5}$$

$$\lambda P_{0,0}=\int_0^\infty v(x)R_0(x)dx+(1-r)\int_0^\infty \mu_2(x)P_{2,1}(x)dx, \tag{3.6}$$

where $P_{1,0}(x)=0$, $P_{2,0}(x)=0$ and $Q_0(x)=0$ occurring in the above equations (3.1), (3.2) and (3.3) respectively.

These set of equations are to be solved under the following boundary conditions at x = 0 :

$$Q_n(0)=\lambda a_nP_{0,0};\ n\geq 1. \tag{3.7}$$

$$P_{1,n}(0)=(1-r)\int_0^\infty \mu_2(x)P_{2,n+1}(x)dx+\int_0^\infty v(x)R_n(x)dx+\int_0^\infty \eta(x)Q_n(x)dx;\ n\geq 1 \tag{3.8}$$

$$P_{2,n}(0)=\int_0^\infty \mu_1(x)P_{1,n}(x)dx,\ n\geq 1, \tag{3.9}$$

$$R_n(0)=r\int_0^\infty \mu_2(x)P_{2,n+1}(x)dx,\ \ n\geq 0, \tag{3.10}$$

and the normalizing condition

$$P_{0,0}+\sum_{n=1}^{\infty}\int_0^\infty Q_n(x)dx+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\int_0^\infty P_{i,n}(x)dx+\sum_{n=0}^{\infty}\int_0^\infty R_n(x)dx=1. \tag{3.11}$$

Next, we define the following *PGF*s:

$$P_i(x;z)=\sum_{n=1}^{\infty}z^nP_{i,n}(x),\ \ \text{for } i=1,2,\ x>0,\ \ |z|<1,$$

$$P_i(0;z)=\sum_{n=1}^{\infty}z^nP_{i,n}(0),\ \text{for } i=1,2,\ \ |z|<1,$$

$$Q(x;z)=\sum_{n=1}^{\infty} z^n Q_n(x),\ x>0,\ |z|<1,$$

$$Q(0;z)=\sum_{n=1}^{\infty} z^n Q_n(0),\ |z|<1,$$

$$R(x;z)=\sum_{n=0}^{\infty} z^n R_n(x),\ x>0,\ |z|<1,$$

$$R(0,z)=\sum_{n=0}^{N-1} z^n R_n(0),\ |z|<1.$$

Now proceeding in the usual manner with the equations (3.1) through (3.5), we get

$$P_1(x;z)=P_1(0;z)[1-B_i(x)]e^{-\lambda(1-X(z))x},\ x>0,\text{ for } i=1,2 \tag{3.12}$$

$$Q(x;z)=Q(0;z)[1-S(x)]e^{-\lambda(1-X(z))x},\ x>0, \tag{3.13}$$

$$R(x;z)=R(0;z)[1-V(x)]e^{-\lambda(1-X(z))x},\ x>0. \tag{3.14}$$

Multiplying equations (3.7) by appropriate powers of z and then taking summation over all possible values of 'n' we get

$$Q(0;z)=\lambda X(z)P_{0,0} \tag{3.15}$$

Hence from equations (3.13) and (3.15), we have

$$Q(x;z)=\lambda X(z)P_{0,0}[1-S(x)]\exp\{-\lambda(1-X(z))x\} \tag{3.16}$$

So that

$$Q(z)=\int_0^{\infty} Q(x;z)=\frac{P_{0,0}X(z)[1-S^*(\lambda-\lambda X(z))]}{[1-X(z)]}; \tag{3.17}$$

where $S^*(\lambda-\lambda X(z))]=\int_0^{\infty} e^{-\lambda(1-X(z))x}dS(x)$ is the z-transform of S. Similarly by multiplying equation (3.8) by appropriate powers of z and taking summation over all values of n and using (3.6) and (3.16), we get on simplification.

$$zP_1(0,z)=(1-r)B_2^*(\lambda-\lambda X(z))P_2(0,z)+\ zV^*(\lambda-\lambda X(z))R(0,z)+\ \lambda zP_{o,o}[X(z)S^*(\lambda-\lambda X(z))-1], \tag{3.18}$$

where $B_i^*(\lambda-\lambda X(z))=\int_0^{\infty} e^{-\lambda(1-X(z))x}dB_i(x)$ is the *z*-transform of B_i, for i =1, 2 and $V^*(\lambda - \lambda X(z)) = \int_0^{\infty} e^{-\lambda(1-X(z))x}dV(x)$ is the *z*-transform of V.

Proceeding in the similar manner with equations (3.9) and (3.10), we get

$$P_2(0,z)=P_1(0,z)B_1^*(\lambda-\lambda X(z)), \tag{3.19}$$

$$zR(0,z)=rP_2(0;z)B_2^*(\lambda-\lambda X(z)). \tag{3.20}$$

Now, utilizing equations (3.19) and (3.20) in equation (3.18), we get on simplification

$$P_1(0,z) = \frac{\lambda z P_{0,0}[1-X(z)S^*(\lambda-\lambda X(z))]}{[((1-r)+rV^*(\lambda-\lambda X(z))B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))]}. \quad (3.21)$$

Hence from equations (3.21) and (3.12) for i = 1, we get

$$P_1(z) = \int_0^\infty P_1(x,z)dx = \frac{zP_{0,0}[1-X(z)S^*(\lambda-\lambda X(z))][1-B_1^*(\lambda-\lambda X(z))]}{[1-X(z)][((1-r)+rV^*(\lambda-\lambda X(z)))B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z]}. \quad \ldots(3.22)$$

Similarly from equations (3.19), (3.21) and (3.12) for i =2, we get

$$P_2(z) = \int_0^\infty P_2(x,z)dx = \frac{zP_{0,0}[1-X(z)S_2^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))[1-B_2^*(\lambda-\lambda X(z))]}{[1-X(z)][\{(1-r)+rV^*(\lambda-\lambda X(z))\}B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z]}. \quad (3.23)$$

Finally, from equations (3.20), (3.19), (3.21) and (3.14), we have

$$R(z) = \int_0^\infty R(x,z)dx = \frac{r[1-X(z)S^*(\lambda-\lambda X(z))][1-V^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{[1-X(z)][\{(1-r)+rV^*(\lambda-\lambda X(z))\}B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z]}. \quad (3.24)$$

The unknown constant $P_{o,o}$ can be determined by using the normalizing condition (3.11), which is equivalent to $R_0 + P_1(1) + P_2(1) + Q(1) + R(1) = 1$. Thus we have

$$P_{o,o} = \frac{(1-\rho)}{C(S)}; \quad (3.25)$$

where $\rho = \lambda E(X)\{E(B_1) + E(B_2) + rE(V)$ is the utilization factor of this system and C(S) = [1+λE(S)]. So that the expected number of arrivals during the turned off period plus a random setup period *E(N)* (say).

$$E(N) = E(X)C(S) = E(X))[1 + \lambda E(S)].$$

Note that the equation (3.24) represents the steady state probability that the server is idle but available in the system. Also from equation (3.25), we have $\rho < 1$, which is the stability condition under which the steady state solution exist. Consequently, the system state probabilities can be obtained from equations (3.17), (3.21), (3.22),(3.24) and (3.25). Thus we get

Prob [the server is on setup period] = Q(1) = $\frac{\lambda E(S)(1-\rho)}{[1+\lambda E(S)]}$

Prob [the server is on vacation] = R(1) = $r\lambda E(X)E(V)$

Prob [the server is busy with FPS] = $P_1(1)$ = $\lambda E(X)E(B_1)$

Prob [the server is busy with SPS] = $P_2(1)$ = $\lambda E(X)E(B_2)$

Again by generalized idle period here we mean that turned off period plus a random setup period. Thus the system is idle if and only if either the server is on turned off or on a random setup period. Hence we have

Prob [The system is idle] = *Prob* [The server is on turned off period]

+ *Prob* [The server is on a random setup period]

= (1 - ρ).

Let $\psi(z) = P_{0,0} + P_1(z) + P_2(z) + Q(z) + zR(z)$ be the PGF of the queue size distribution at a random epoch, then

$$\psi(z) = \left(\frac{1 - X(z)S^*(\lambda - \lambda X(z))}{C(S)[1 - X(z)]}\right)\left(\frac{(1-\rho)(1-z)[(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{[(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z)) - z}\right). \qquad (3.26)$$

4. ANALYSIS OF THE QUEUE SIZE DISTRIBUTION

In this section, we derive the system state probabilities and analyze the *PGF* of the queue size distribution to provide its appropriate interpretation. Now after some algebric rearrangement with first term of the expression (3.26), we may write

$$\frac{[1 - X(z)S^*(\lambda - \lambda X(z))]}{C(S)[1 - X(z)]} = \frac{S^*(\lambda - \lambda X(z))}{[1 + \lambda E(S)]} + \left[\frac{\lambda E(S)}{[1 + \lambda E(S)]}\right]\left[\frac{1 - S^*(\lambda - \lambda X(z))}{E(S)(\lambda - \lambda X(z))}\right] = \xi(z) = \text{(say)}$$

For further analysis of this model let us define the following events:

T_0 = length of the turned off period

T_1 = length of the generalized idle period.

Clearly

$$E(T_0) = \frac{1}{\lambda}$$

and

$$E(T_1) = E(S) + E(T_0) = \frac{[1 + \lambda E(S)]}{\lambda}.$$

Now $\frac{E(T_0)}{E(T_1)} = \frac{1}{[1 + \lambda E(S)]}$ is the proportion of expected amount of time the server is on turned off period given that the system is idle. Hence by the theory of regeneration process long fraction of time the server is on turned off period given that the system is idle which occur with probability

Prob [The server is on turned off period / The system is idle]

$$= \frac{1}{[1 + \lambda E(S)]} = \alpha \text{ (say)}$$

Similarly it can be shown that

Prob [The server is on a random setup period / The system is idle]

$$= \frac{\lambda E(S)}{[1 + \lambda E(S)]} = (1 - \alpha).$$

Now utilizing these interpretation in equation (4.1), we may write

$$\xi(z) = \alpha I(z) + (1 - \alpha) I_R(z);$$

where $I(z)$ = The *PGF* of the number of units arrived during a random setup period.

$$= S^*(\lambda - \lambda X(z))$$

and $I_R(z)$ = The *PGF* of the number of units that arrived during the residual life of the random setup period.

$$= \frac{[1-S^*(\lambda-\lambda X(z))]}{E(S)(\lambda-\lambda X(z))}$$

Now, the stochastic decomposition property for this model can be demonstrated easily by showing

$$\psi(z) = \left(\frac{(1-\rho)(1-z)[(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{[(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z}\right)\left[\alpha I(z)+(1-\alpha)I_R(z)\right].$$

$$= \psi_0(z)\xi(z), \tag{4.2}$$

where $\psi_0(z)$, the first factor in the right hand side of (4.2), is the *PGF* of the stationary queue size distribution of an $M^X/(G_1, G_2)/1$ queue with vacation time under Bernoulli schedule and $\xi(z) = \left(\frac{1-X(z)S^*(\lambda-\lambda X(z)}{C(S)[1-X(z)}\right)$ is the *PGF* of the number of units present in the system during an idle period. More specifically, we may call $\xi(z)$ the additional queue size distribution caused by the generalized idle period.

We may note that the expression for $\psi_0(z)$,in the right hand side of (4.9) can also be obtained easily from the well known *Pollaczek-Khinchine formula*, by replacing the service time distribution by our modified service time distribution $B^*(\theta) = [(1-r)+rV^*(\theta)]B_1^*(\theta)B_2^*(\theta)$.

Let K(z) be the *PGF* of a batch of customers who arrived during our modified service time B, then

$$K(z) = [(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))$$

and

$$K'(1) = \lambda E(X)[E(B_1)+E(B_2)+rE(V)] = \rho.$$

Now utilizing $K(z)$ in *Pollaczek-Khinchine formula* (e.g. see Medhi [20], p-116), we may write

$$\psi_0(z) = \frac{(1-K'(1))(1-z)K(z)}{K(z)-z} = \left(\frac{(1-\rho)(1-z)[(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{[(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z}\right);$$

which is the first factor in the right hand side of equation (4.2).

In particular, if we take r = 0 (i.e. if there is no server vacation) then from equation (4.2), we get

$$\psi(z) = \left(\frac{(1-\rho)(1-z)[1-X(z)S^*(\lambda-\lambda X(z)]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{C(S)[1-X(z)][B_1^*(\lambda-\lambda X(z)B_2^*(\lambda-\lambda X(z))-z]}\right);$$

which is the *PGF* of queue size distribution at a random epoch of an $M^X/(G_1, G_2)/1$ queue with a random setup time. In such a model, the total service time required by an arriving unit to complete both stages of service is $B = B_1 + B_2$, so that $B^*(\theta) = B_1^*(\theta)B_2^*(\theta)$ and $\rho = \lambda E(X)[E(B_1) + E(B_2)] < 1$. Note that some aspects of the $M^X/G/1$ type of queue with a random setup time have been discussed by Choudhury [4].

Remark 4.1.

It is important to note here that the stationary queue size distribution at a random epoch of $M^X/(G_1,G_2)/V/1(BS)/SET$ queue given in equation (4.2) decomposes into the distributions of two independent random variables, viz.

(I) The stationary queue size distribution of an $M^X/(G_1,G_2)/V/1(BS)$ queue with a vacation time under Bernoulli schedule (a *SET* independent random variable)(represented by the first term) and

(II) The queue size distribution at a random epoch given that the server is idle due to number of arrivals during the turned off period plus a random setup period (a *SET* related random variable) (represented by the second factor).

5. RELATED VACATION MODEL

Now form the utility point of view the idle time, our model can also be considered as a case of multiple vacation model. Under the multiple vacation policy (*MVP*) the server keeps on taking vacations of random length till it finds at least one unit waiting in the system to start a busy period. Now putting $V \cong S$ and taking limit $\alpha \to 0$ in the above equation (4.2), we get

$$\lim_{\alpha\to 0}\psi(z) = \frac{(1-\rho)(1-z)[1-V^*(\lambda-\lambda X(z))][(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{E(V)(\lambda-\lambda X(z))[\{(1-r)+rV^*(\lambda-\lambda X(z))\}B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z]} = \psi_1(z) \text{ (say)} \qquad (5.1)$$

which is the *PGF* of the queue size distribution at a random epoch of an $M^X(G_1,G_2)/V/1(BS)$ queue, under *MVP.*

Next, if we assume batch arrivals with *MVP* but without the second stage service, then with $B_2^*(\lambda-\lambda X(z)) = 1$, we have $\rho = \lambda E(X)[E(B_1)+rE(V)]$ and equation (5.1) yields

$$\psi_1(z) = \frac{(1-\rho)(1-z)[1-V^*(\lambda-\lambda X(z))][(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))}{E(V)(\lambda-\lambda X(z))[\{(1-r)+rV^*(\lambda-\lambda X(z))\}B_1^*(\lambda-\lambda X(z))-z]} \qquad (5.2)$$

which is the PGF of the queue size distribution at a random epoch of an $M^X/(G_1,G_2)/V/1(BS)$ under *MVP*. Note that the above result was derived by Servy [1986] (also see Takagi [1991], p. 230) for single unit arrival case. Also, we note here that for r = 0 the equation (5.2) agrees with the equation (2.11) of Choudhury [2002].

6. QUEUE SIZE DISTRIBUTION AT A DEPARTURE EPOCH

Recently, Choudhury and Krishnamoorthy [2003] obtained the *PGF* of the queue size distribution at a departure epoch for M^X/G/1 queue with a random setup time by using the argument of embedded Markov chain. However, in this section an attempt has been made to obtain the *PGF* for our $M^X/(G_1, G_2)/V/1(BS)/SET$ queue. Following argument of *PASTA* we state that a departing customer will see 'j' customers in the queue just after a departure if and only if there were (j + 1) customers in the queue just before the departure. Now, denoting $\{\pi_j;\ j \geq 0\}$ as probability that there are 'j' customers in the queue at a departure epoch, we may write

$$\pi_j = k_0(1-r)\int_0^\infty \mu_2(x)P_{2,j+1}(x)dx + k_0\int_0^\infty v(x)R_j(x)dx,\ j \geq 0,$$

where k_0 is the normalizing constant.

Let $\pi(z)$ be the *PGF* of $\{\pi_j;\ j \geq 0\}$, then utilizing equations (3.19), (3.20), and (3.21) we get

$$\pi(z) = \sum_{j=0}^{\infty} z^j\pi_j = \frac{\lambda k_0 P_{0,0}[1-X(z)S^*(\lambda-\lambda X(z))][(1-r)+rV^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{[\{(1-r)+rV^*(\lambda-\lambda X(z))\}B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z]} \qquad (6.1)$$

Now, using the normalizing condition $\pi(1) = 1$, (5.2) yields

$$K_0 = \frac{(1-\rho)}{\lambda E(N)P_{0,0}}.$$

Thus we have

$$\pi(z) = \frac{(1-\rho)[1-X(z)S^*(\lambda-\lambda X(z))][(1-r)+rV^*(\lambda-\lambda X(z))]\,B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{E(N)[\{(1-r)+rV^*(\lambda-\lambda X(z))\}B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z]} \qquad (6.2)$$

Hence the relationship between $\psi(z)$ and $\pi(z)$ is given by

$$\pi(z) = \frac{[1-X(z)]}{E(X)(1-z)}\ \psi(z) = H(z)\pi(z)\xi(z), \qquad (6.3)$$

where $H(z) = \frac{[1-X(z)]}{E(X)(1-z)}$, is the *PGF* of the number of customers placed before an arbitrary customer (tagged customer) in a batch in which the tagged customer arrives. This number is given as backward recurrence time in the discrete time renewal process where renewal points generated by the arrival size random variable. This is due to the randomness nature of the arrival size random variable.

In particular, if we take *r* =1 in the above equation (6.2), we get

$$\pi(z) = \frac{(1-\rho)[1-X(z)S^*(\lambda-\lambda X(z))]V^*(\lambda-\lambda X(z))B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))}{E(N)[(\lambda-\lambda X(z))B_1^*(\lambda-\lambda X(z))B_2^*(\lambda-\lambda X(z))-z]} \qquad (6.4)$$

where now we have $\rho = \lambda E(X)[E(B_1)+E(B_2)+E(V)](<1)$.

Note that (5.4) is the *PGF* of the departure point queue size distribution of $M^X/(G_1,G_2)/1$ limited service queue with a single vacation and a random setup time. In such a model, if there is at least one unit in the system at the end of a vacation, the service is immediately started otherwise the server waits until a batch of customer arrives. Also we note that some aspects of this type of $M/G/1$ queueing system without a random setup time was considered by Takagi [1991] (e.g. see page-230). However, a model of similar nature for regular $M^X/G/1$ queue with limited service was also studied by Kuechen [1979].

Now setting *z*=0 in equation (5.2), we get

$$\pi(0) = \frac{(1-\rho)}{E(N)} = \pi_0;$$

which is the steady sate probability that no unit is waiting in the system at the departure point of time. Hence the relationship between π_0 and $P_{0,0}$ is given by

$$P_{0,0} = E(X)\pi_0 .$$

This exhibits an interesting phenomenon. It states that an observer is more likely to find the system empty than a departing customer leaves the system.

Remark 6.1.

From equation (5.4), we observe that the departure point queue size distribution of $M^X/(G_1, G_2)/V/1(BS)/SET$ queue is the convolution of three independent random variables: one (the first factor) is the number of units places before a tagged customer in a batch in which this tagged customer arrives. This is due to randomness property of the size of the arriving batch. The interpretations of the other two random variables are provided in *Remark.4.1*. The result obtained in this section is quite general and it covers many situations (e.g. see Takagi [1991]).

7. EXPECTED BUSY PERIOD

An interesting result which falls outside the preceding result is expected busy period. Hence in this section an attempt has been made to obtain the expected busy period. To obtain it we follow the argument of alternating renewal process, which seems to be simpler and elegant. We now define busy period as the length of the time interval that keeps the server busy without interruption. This continues up to the instant when the server becomes free again i.e. the system becomes empty again. We now define

T_1 = length of the generalized idle period

and

T_b = length of the busy period.

Now T_1 and T_b generalizes an alternating renewal process and therefore we may write

$$\frac{E(T_b)}{E(T_1)} = \frac{\rho}{(1-\rho)}. \tag{7.1}$$

Now since $E(T_1) = \frac{\{1+\lambda E(S)\}}{\lambda}$ (see section – 4), therefore from equation (7.1) we get

$$E(T_b) = \frac{E(X)[E(B_1)+E(B_2)]}{[1+\lambda E(S)](1-\rho)} + \frac{rE(X)E(V)}{[1+\lambda E(S)\}(1-\rho)} \tag{7.2}$$

Now if we take r = 0 and $E(B_2) = 0$ thus $\rho = \lambda E(X)E(B_1)$ the above equation (7.2) reduces to

$$E(T_b) = \frac{E(X)E(B_1)[1+\lambda E(S)]}{[1-\lambda E(X)E(B_1)]}$$

$$= \frac{E(X)E(B_1)}{(1-\rho)} + \frac{\rho E(S)}{(1-\rho)}; \tag{7.3}$$

which agrees with the result obtained by Choudhury and Krishnamoorthy [2003]. Note that for E(S) = 0 the above equation (7.3) agrees with the result obtained by Chaudhury [1995].

8. MEAN QUEUE SIZE

In this section we derive the queue size distribution at random epoch as well as at departure epoch of this model. Let L_{Q_1} be the mean queue size of this $M^X/(G_1,G_2)/V/1(BS)/SET$ queue at a random epoch. Then

$$L_{Q_1} = \frac{d\psi(z)}{dz} \text{ at } z=1$$

$$= \rho + = \frac{\lambda^2 E^2(X)\left[E(B_1^2)+E(B_2^2)+rE(V^2)\right]}{2(1-\rho)}$$

$$+ \frac{\lambda E(X(X-1))\left[E(B_1)+E(B_2)+rE(V)\right]}{2(1-\rho)}$$

$$+ \frac{\lambda^2 E^2(X)\left(E(B_1)E(B_2)+rE(V)\{E(B_1)+E(B_2)\}\right)}{(1-\rho)} + \lambda E(X)E(S)$$

$$+ \frac{\lambda E(X)E(S^2)}{2} + \frac{E(X(X-1)}{2E(X)} \tag{8.1}$$

Again let L_{Q_2} be the mean queue size of this $M^X/(G_1,G_2)/V_S/1(BS)/SET$ queue at departure epoch, then we have

$$L_{Q_2} = \frac{d\pi(z)}{dz} \text{ at } z=1$$

$$= \rho + = \frac{\lambda^2 E^2(X)\left[E(B_1^2)+E(B_2^2)+rE(V^2)\right]}{2(1-\rho)}$$

$$+\frac{\lambda E(X(X-1))\left[E(B_1)+E(B_2)+rE(V)\right]}{2(1-\rho)}$$

$$+\frac{\lambda^2 E^2(X)\left(E(B_1)E(B_2)+rE(V)\{E(B_1)+E(B_2)\}\right)}{(1-\rho)}$$

$$+\frac{\lambda E^2(X)[E(S)+\lambda E(S^2)]+E(X(X-1))\{1+\lambda E(S)\}}{E(N)} \tag{8.2}$$

Note that for r = 0, $E(B_2) = 0 = E(B_2^2)$ wet mean queue size of the $M^x/G/1$ queue with a random setup time. Also we note that for $E(S) = 0 = E(S^2)$ we get mean queue size of the $M^X/(G_1,G_2)/V/1(BS)$ queue. Such a model was considered by Madan [2001] for single unit arrival case.

Further, let W_Q denote the mean waiting time of an arbitrary customer for this $M^X(G_1,G_2)/V_S/1(BS)/SET$ queue. Then W_Q can be obtained from equation (8.1) by utilizing *Little's formula* $W_Q = \lambda E(X) L_Q$.

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