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ON THREE EQUIVALENCES CONCERNING PONOMAREV-SYSTEMS

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Abstract. Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X such that $\{st(x,\mathcal{P}_n)\}$ is a network at x in X for each $x \in X$. For each $n \in \mathbb{N}$, let $\mathcal{P}_n = \{P_\beta : \beta \in \Lambda_n\}$ and Λ_n be endowed the discrete topology. Put $M = \{b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Lambda_n :$ $\{P_{\beta_n}\}$ *forms a network at some point* x_b *in* $X\}$ and $f : M \longrightarrow X$ by choosing $f(b) = x_b$ for each $b \in M$. In this paper, we prove that f is a sequentially-quotient (resp. sequence-covering, compact-covering) mapping if and only if each $\mathcal{P}_n$ is a cs^*-cover (resp. fcs-cover, cfp-cover) of X. As a consequence of this result, we prove that f is a sequentially-quotient, s-mapping if and only if it is a sequence-covering, s-mapping, where "s" can not be omitted.

1. Introduction

A space is called a Baire's zero-dimensional space if it is a Tychonoff-product space of countable many discrete spaces. In [9], Ponomarev proved that each first countable space can be characterized as an open image of a subspace of a Baire's zero-dimensional space. More precisely, he obtained the following result.

Theorem 1.1. *Let X be a space with the topology $\tau = \{P_\beta : \beta \in \Lambda\}$. For each $n \in \mathbb{N}$, put $\Lambda_n = \Lambda$ and endow Λ_n the discrete topology. Put $Z = \Pi_{n\in\mathbb{N}}\Lambda_n$, which is a Baire's zero-dimensional space, and put $M = \{b = (\beta_n) \in Z : \{P_{\beta_n}\}$ forms a neighbourhood base at some point x_b in $X\}$. Define $f : M \longrightarrow X$ by choosing $f(b) = x_b$ for each $b \in M$. Then*

(1) *f is a mapping.*
(2) *f is continuous and onto.*
(3) *If X is first countable, then f is an open mapping.*

Recently, while generalizing the Ponomarev's methods, Lin ([6]) introduced *Ponomarev-systems* $\big(f, M, X, \{\mathcal{P}_n\}\big)$ as in the following definition.

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Definition 1.2.

(1) Let $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X, where $\mathcal{P}_x \subset (\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}$. $\mathcal{P}$ is called a network of X ([8]), if for each $x \in U$ with U open in X, there exists $P \in \mathcal{P}_x$ such that $x \in P \subset U$, where $\mathcal{P}_x$ is called a network at x in X.

(2) Let $\{\mathcal{P}_n\}$ be a sequence of covers of a space X. $\{\mathcal{P}_n\}$ is called a point-star network of X ([7]), if $\{st(x, \mathcal{P}_n)\}$ is a network at x in X for each $x \in X$, where $st(x, \mathcal{P}) = \bigcup\{P \in \mathcal{P} : x \in P\}$.

(3) Let $\{\mathcal{P}_n\}$ be a point-star network of a space X. For each $n \in \mathbb{N}$, put $\mathcal{P}_n = \{P_\beta : \beta \in \Lambda_n\}$ and endow Λ_n the discrete topology. Put $M = \{b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Lambda_n : \{P_{\beta_n}\}$ *forms a network at some point* x_b *in* $X\}$, then M, which is a subspace of the product space $\Pi_{n\in\mathbb{N}}\Lambda_n$, is a metric space and x_b is unique for each $b \in M$. Define $f : M \longrightarrow X$ by choosing $f(b) = x_b$, then f is a continuous and onto mapping. $\big(f, M, X, \{\mathcal{P}_n\}\big)$ is called a *Ponomarev*-system ([7, 10]).

In a *Ponomarev*-system $\big(f, M, X, \{\mathcal{P}_n\}\big)$, the following results have been obtained.

Theorem 1.3 ([6, 7, 10]). *Let $\big(f, M, X, \{\mathcal{P}_n\}\big)$ be a Ponomarev-system. Then the following hold.*

(1) *If each $\mathcal{P}_n$ is a point-finite* (*resp. point-countable*) *cover of X, then f is a compact mapping* (*resp. s-mapping*).

(2) *If each $\mathcal{P}_n$ is a cs^*-cover* (*resp. cfp-cover*) *of X, then f is a sequentially-quotient* (*resp. compact-covering*) *mapping.*

Take Theorem 1.3 into account, the following question naturally arises.

Question 1.4. Can implications (1) and (2) in Theorem 1.3 be reversed?

In this paper, we investigate the *Ponomarev*-system $\big(f, M, X, \{\mathcal{P}_n\}\big)$ to answer Question 1.4 affirmatively. We also prove that, in a *Ponomarev*-system $\big(f, M, X, \{\mathcal{P}_n\}\big)$, f is a sequence-covering mapping if and only if each $\mathcal{P}_n$ is an fcs-cover. As a consequence of these results, f is a sequentially-quotient, s-mapping if and only if it is a sequence-covering, s-mapping, where "s" can not be omitted.

Throughout this paper, all spaces are assumed to be regular and T_1, and all mappings are continuous and onto. $\mathbb{N}$ denotes the set of all natural numbers, $\{x_n\}$ denotes a sequence, where the n-th term is x_n. Let X be a space and let A be a subset of X. We call that a sequence $\{x_n\}$ converging to x in X is eventually in A if $\{x_n : n > k\} \bigcup \{x\} \subset A$ for some $k \in \mathbb{N}$. Let $\mathcal{P}$ be a family of subsets of X and let $x \in X$. $\bigcup\mathcal{P}$, $st(x, \mathcal{P})$ and $(\mathcal{P})_x$ denote the union $\bigcup\{P : P \in \mathcal{P}\}$, the union $\bigcup\{P \in \mathcal{P} : x \in P\}$ and the subfamily $\{P \in \mathcal{P} : x \in P\}$ of $\mathcal{P}$ respectively. For a sequence $\{\mathcal{P}_n : n \in \mathbb{N}\}$ of covers of a space X and a sequence $\{P_n : n \in \mathbb{N}\}$ of subsets of a space X, we abbreviate $\{\mathcal{P}_n : n \in \mathbb{N}\}$ and $\{P_n : n \in \mathbb{N}\}$ to $\{\mathcal{P}_n\}$ and $\{P_n\}$ respectively. A point $b = (\beta_n)_{n\in\mathbb{N}}$ of a Tychonoff-product space is abbreviated to (β_n), and the n-th coordinate β_n of b is also denoted by $(b)_n$.

2. The main results

Definition 2.1. Let $f : X \longrightarrow Y$ be a mapping.

(1) f is called a sequentially-quotient mapping ([1]) if for each convergent sequence S in Y, there exists a convergent sequence L in X such that $f(L)$ is a subsequence of S.

(2) f is called a sequence-covering mapping ([4]) if for each convergent sequence S converging to y in Y, there exists a compact subset K of X such that $f(K) = S\bigcup\{y\}$.

(3) f is called a compact-covering mapping ([8]) if for each compact subset L of Y, there exists a compact subset K of X such that $f(K) = L$.

Remark 2.2. (1) Compact-covering mapping $\Longrightarrow$sequence-covering mapping $\Longrightarrow$ (if the domain is metric) sequentially-quotient mapping ([6]).

(2) "sequence-covering mapping" in Definition 2.1 (2) was also called "pseudo-sequence-covering mapping" by Ikeda, Liu and Tanaka in [5].

Definition 2.3. Let (X, d) be a metric space, and let $f : X \longrightarrow Y$ be a mapping. f is called a π-mapping ([9]), if for each $y \in Y$ and for each neighbourhood U of y in Y, $d\big(f^{-1}(y), X - f^{-1}(U)\big) > 0$.

Remark 2.4. (1) For a *Ponomarev*-system $\big(f, M, X, \{\mathcal{P}_n\}\big)$, $f : M \longrightarrow X$ is a π-mapping ([7, 10]).

(2) Recall a mapping $f : X \longrightarrow Y$ is a compact mapping (resp. s-mapping), if $f^{-1}(y)$ is a compact (resp. separable) subset of X for each $y \in Y$. It is clear that each compact mapping from a metric space is an s- and π-mapping.

Definition 2.5. Let $\mathcal{P}$ be a cover of a space X.

(1) $\mathcal{P}$ is called a cs^*-cover of X ([6]) if for each convergent sequence S in X, there exists $P \in \mathcal{P}$ and a subsequence S' of S such that S' is eventually in P.

(2) $\mathcal{P}$ is called an fcs-cover of X ([3]) if for each sequence S converging to x in X, there exists a finite subfamily $\mathcal{P}'$ of $(\mathcal{P})_x$ such that S is eventually in $\bigcup\mathcal{P}'$.

(3) $\mathcal{P}$ is called a cfp-cover of X ([7]) if for each compact subset K, there exists a finite family $\{K_n : n \leq m\}$ of closed subsets of K and $\{P_n : n \leq m\} \subset \mathcal{P}$ such that $K = \bigcup\{K_n : n \leq m\}$ and each $K_n \subset P_n$.

Lemma 2.6. *Let $\big(f, M, X, \{P_n\}\big)$ be a Ponomarev-system and let $U = (\Pi_{n\in\mathbb{N}}\Gamma_n) \bigcap M$, where $\Gamma_n \subset \Lambda_n$ for each $n \in \mathbb{N}$. Then $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$ for each $k \in \mathbb{N}$.*

Proof. Let $b = (\beta_n) \in U$ and let $k \in \mathbb{N}$. Then $\{P_{\beta_n}\}$ forms a network at $f(b)$ in X and $\beta_k \in \Gamma_k$. So $f(b) \in P_{\beta_k} \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$. This proves that $f(U) \subset \bigcup\{P_\beta : \beta \in \Gamma_k\}$.

Theorem 2.7. *Let $\big(f, M, X, \{\mathcal{P}_n\}\big)$ be a Ponomarev-system. Then the following hold.*

(1) *f is a compact mapping (resp. s-mapping) if and only if $\mathcal{P}_m$ is point-finite (resp. point-countable) cover of X for each $m \in \mathbb{N}$.*

(2) *f is a sequentially-quotient mapping if and only if $\mathcal{P}_m$ is a cs^*-cover of X for each $m \in \mathbb{N}$.*

(3) *f is a compact-covering mapping if and only if $\mathcal{P}_m$ is a cfp-cover of X for each $m \in \mathbb{N}$.*

Proof. By Theorem 1.3, we only need to prove necessities of (1), (2) and (3). Let $m \in \mathbb{N}$.

(1) We only give a proof for the parenthetic part. If $\mathcal{P}_m$ is not point-countable, then, for some $x \in X$, there exists an uncountable subset Γ_m of Λ_m such that $\Gamma_m = \{\beta \in \Lambda_m : x \in P_\beta\}$. For each $\beta \in \Gamma_m$, put $U_\beta = ((\Pi_{n<m}\Lambda_n) \times \{\beta\} \times (\Pi_{n>m}\Lambda_n)) \bigcap M$. Then $\{U_\beta : \beta \in \Gamma_m\}$ covers $f^{-1}(x)$. If not, there exists $c = (\gamma_n) \in f^{-1}(x)$ and $c \notin U_\beta$ for each $\beta \in \Gamma_m$, so $\gamma_m \notin \Gamma_m$. Thus $x \notin P_{\gamma_m}$ from construction of Γ_m. But $x = f(c) \in P_{\gamma_m}$ from Lemma 2.6. This is a contradiction. Thus $\{U_\beta : \beta \in \Gamma_m\}$ is an uncountable open cover of $f^{-1}(x)$, but it has not any proper subcover. So $f^{-1}(x)$ is not separable, hence f is not an s-mapping.

(2) Let f be a sequentially-quotient mapping, and let $\{x_n\}$ be a sequence converging to x in X. Then there exists a sequence $\{b_k\}$ converging to b in M such that $f(b_k) = x_{n_k}$ for each $k \in \mathbb{N}$. Let $b = (\beta_n) \in (\Pi_{n\in\mathbb{N}}\Lambda_n) \bigcap M$. We claim that the subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is eventually in P_{β_m}. In fact, put $U = \big((\Pi_{n<m}\Lambda_n) \times \{\beta_m\} \times (\Pi_{n>m}\Lambda_n)\big) \bigcap M$, then U is an open neighbourhood of b in M. So sequence $\{b_k\}$ is eventually in U, hence sequence $\{x_{n_k}\}$ is eventually in $f(U)$. $f(U) \subset P_{\beta_m}$ from Lemma 2.6, so $\{x_{n_k}\}$ is eventually in P_{β_m}. Note that $\beta_m \in \Lambda_m$, so $P_{\beta_m} \in \mathcal{P}_m$. This proves that $\mathcal{P}_m$ is a cs^*-cover of X.

(3) Let f be a compact-covering mapping, and let C be a compact subset of X. Then there exists a compact subset K of M such that $f(K) = C$. For each $a \in K$, put $U_a = ((\Pi_{n<m}\Lambda_n) \times \{(a)_m\} \times (\Pi_{n>m}\Lambda_n)) \bigcap M$, where $(a)_m \in \Lambda_m$ is the m-th coordinate of a, then $U_a \bigcap K$ is an open (in subspace K) neighbourhood of a. So there exists an open (in subspace K) neighbourhood V_a of a such that $a \in V_a \subset Cl_K(V_a) \subset U_a \bigcap K$, where $Cl_K(V_a)$ is the closure of V_a in subspace K. Note that $\{V_a : a \in K\}$ is an open cover of subspace K and K is compact in M, so there exists a finite subset $\{a_1, a_2, \ldots, a_s\}$ of K such that $\{V_{a_i} : i = 1, 2, \ldots, s\}$ is a finite cover of K. Thus $\bigcup\big\{Cl_K(V_{a_i}) : i = 1, 2, \ldots, s\big\} = K$, and so $\bigcup\big\{f\big(Cl_K(V_{a_i})\big) : i = 1, 2, \ldots, s\big\} = f\big(\bigcup\big\{Cl_K(V_{a_i}) : i = 1, 2, \ldots, s\big\}\big) = f(K) = C$. For each $i = 1, 2, \ldots, s$, put $C_i = f\big(Cl_K(V_{a_i})\big)$. Since $Cl_K(V_{a_i})$ is compact in K, C_i is compact in C, so C_i is closed in C, and $C = \bigcup\{C_i : i = 1, 2, \ldots, s\}$. For each $i = 1, 2, \ldots, s$, $C_i = f(Cl_K(V_{a_i})) \subset f(U_{a_i} \bigcap K) \subset f(U_{a_i})$, and $f(U_{a_i}) \subset P_{(a_i)_m}$ from Lemma 2.6, so $C_i \subset P_{(a_i)_m}$. Note that $(a_i)_m \in \Lambda_m$, so $P_{(a_i)_m} \in \mathcal{P}_m$. This proves that $\mathcal{P}_m$ is a cfp-cover of X. □

By viewing the above theorem, we ask: in a *Ponomarev*-system $\big(f, M, X, \{\mathcal{P}_n\}\big)$, what is the sufficient and necessary condition such that f is a sequence-covering mapping? We give an answer to this question.

Theorem 2.8. *Let $\big(f, M, X, \{\mathcal{P}_n\}\big)$ be a Ponomarev-system. Then f is a sequence-covering mapping if and only if each $\mathcal{P}_n$ is an fcs-cover of X.*

Proof. *Sufficiency*: Let each $\mathcal{P}_n$ be an fcs-cover of X, and let $S = \{x_n\}$ be a sequence converging to x in X. For each $n \in \mathbb{N}$, since $\mathcal{P}_n$ is an fcs-cover, there exists a finite subfamily $\mathcal{F}_n$ of $(\mathcal{P}_n)_x$ such that S is eventually in $\bigcup \mathcal{F}_n$.

Note that $S - \bigcup \mathcal{F}_n$ is finite. There exists a finite subfamily $\mathcal{G}_n$ of $\mathcal{P}_n$ such that $S - \bigcup \mathcal{F}_n \subset \bigcup \mathcal{G}_n$. Put $\mathcal{F}_n \bigcup \mathcal{G}_n = \{P_{\beta_n} : \beta_n \in \Gamma_n\}$, where Γ_n is a finite subset of Λ_n. For each $\beta_n \in \Gamma_n$, if $P_{\beta_n} \in \mathcal{F}_n$, put $S_{\beta_n} = (S \bigcap P_{\beta_n}) \bigcup \{x\}$, otherwise, put $S_{\beta_n} = (S - \bigcup \mathcal{F}_n) \bigcap P_{\beta_n}$. It is easy to see that $S = \bigcup_{\beta_n \in \Gamma_n} S_{\beta_n}$ and $\{S_{\beta_n} : \beta_n \in \Gamma_n\}$ is a family of compact subsets of X.

Put $K = \{(\beta_n) \in \Pi_{n\in\mathbb{N}}\Gamma_n : \bigcap_{n\in\mathbb{N}} S_{\beta_n} \neq \emptyset\}$. Then

Claim 1: $K \subset M$ and $f(K) \subset S$.

Let $b = (\beta_n) \in K$, then $\bigcap_{n\in\mathbb{N}} S_{\beta_n} \neq \emptyset$. Pick $y \in \bigcap_{n\in\mathbb{N}} S_{\beta_n}$, then $y \in \bigcap_{n\in\mathbb{N}} P_{\beta_n}$. Note that $\{P_{\beta_n} : n \in \mathbb{N}\}$ forms a network at y in X if and only if $y \in \bigcap_{n\in\mathbb{N}} P_{\beta_n}$. So $b \in M$ and $f(b) = y \in S$. This proves That $K \subset M$ and $f(K) \subset S$.

Claim 2: $S \subset f(K)$.

Let $y \in S$. For each $n \in \mathbb{N}$, pick $\beta_n \in \Gamma_n$ such that $y \in S_{\beta_n}$. Put $b = (\beta_n)$, then $b \in K$ and $f(b) = y$. This proves that $S \subset f(K)$.

Claim 3: K is a compact subset of M.

Since $K \subset M$ and $\Pi_{n\in\mathbb{N}}\Gamma_n$ is a compact subset of $\Pi_{n\in\mathbb{N}}\Lambda_n$. We only need to prove that K is a closed subset of $\Pi_{n\in\mathbb{N}}\Gamma_n$. It is clear that $K \subset \Pi_{n\in\mathbb{N}}\Gamma_n$. Let $b = (\beta_n) \in \Pi_{n\in\mathbb{N}}\Gamma_n - K$. Then $\bigcap_{n\in\mathbb{N}} S_{\beta_n} = \emptyset$. There exists $n_0 \in \mathbb{N}$ such that $\bigcap_{n\leq n_0} S_{\beta_n} = \emptyset$. Put $W = \{(\gamma_n) \in \Pi_{n\in\mathbb{N}}\Gamma_n : \gamma_n = \beta_n \ for \ n \leq n_0\}$. Then W is open in $\Pi_{n\in\mathbb{N}}\Gamma_n$ and $b \in W$. It is easy to see that $W \bigcap K = \emptyset$. So K is a closed subset of $\Pi_{n\in\mathbb{N}}\Gamma_n$.

By the above three claims, f is a sequence-covering mapping.

Necessity: Let f be a sequence-covering mapping and let $m \in \mathbb{N}$. Whenever $\{x_n\}$ is a sequence converging to x in X, there exists a compact subset K of M such that $f(K) = \{x_n : n \in \mathbb{N}\} \bigcup \{x\}$. Since $f^{-1}(x) \bigcap K$ is a compact subset of M, there exists a finite subset $\{a_i : i = 1, 2, \ldots, s\}$ of $f^{-1}(x) \bigcap K$ and a finite open cover $\{U_i : i = 1, 2, \ldots, s\}$ of $f^{-1}(x) \bigcap K$, where for each $i = 1, 2, \ldots, s$, $U_i = ((\Pi_{n<m}\Lambda_n) \times \{(a_i)_m\} \times (\Pi_{n>m}\Lambda_n)) \bigcap M$ is an open neighbourhood of a_i, and $(a_i)_m \in \Lambda_m$ is the m-th coordinate of a_i. By Lemma 2.6, $x = f(a_i) \in f(U_i) \subset P_{(a_i)_m} \in (\mathcal{P}_m)_x$ for each $i = 1, 2, \ldots, s$. We only need to prove that sequence $\{x_n\}$ converging to x is eventually in $\bigcup\{P_{(a_i)_m} : i = 1, 2, \ldots, s\}$. If not, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \notin \bigcup\{P_{(a_i)_m} : i = 1, 2, \ldots, s\}$ for each $k \in \mathbb{N}$. That is, for each $k \in \mathbb{N}$ and each $i = 1, 2, \ldots, s$, $x_{n_k} \notin P_{(a_i)_m}$. For each $k \in \mathbb{N}$, we pick $b_k \in K$ such that $f(b_k) = x_{n_k}$. If for some $k \in \mathbb{N}$ and some $i = 1, 2, \ldots, s$, $b_k \in U_i$, then $x_{n_k} = f(b_k) \in f(U_i) \subset P_{(a_i)_m}$ from Lemma 2.6. This is a contradiction. So $b_k \notin U_i$ for each $k \in \mathbb{N}$ and each $i = 1, 2, \ldots, s$. Thus $\{b_k : k \in \mathbb{N}\} \subset K - \bigcup\{U_i : i = 1, 2, \ldots, s\}$. Note that $K - \bigcup\{U_i : i = 1, 2, \ldots, s\}$ is a compact metric subspace, there exists a sequence $\{b_{k_j}\}$ converging to a point $b \in K - \bigcup\{U_i : i = 1, 2, \ldots, s\}$. Thus $b \notin f^{-1}(x)$, so $f(b) \neq x$. On the other hand, $\{f(b_{k_j})\}$ converges to $f(b)$ by the continuity of f and $\{f(b_{k_j})\} = \{x_{n_{k_j}}\}$ converges to x, so $f(b) = x$. This is a contradiction. So sequence $\{x_n\}$ converging to x is eventually in $\bigcup\{P_{(a_i)_l} : i = 1, 2, \ldots, s\}$. □

3. Some consequences

cs^*-cover and fcs-cover are not equivalent in general, but there exist some relations between cs^*-cover and fcs-cover.

Proposition 3.1. *Let $\mathcal{P}$ be a cover of a space X. Then the following hold.*

(1) *If $\mathcal{P}$ is an fcs-cover of X, then $\mathcal{P}$ is a cs^*-cover of X.*

(2) *If $\mathcal{P}$ is a point-countable cs^*-cover of X, then $\mathcal{P}$ is an fcs-cover of X.*

Proof. (1) holds from Definition 2.5. We only need to prove (2).

Let $\mathcal{P}$ be a point-countable cs^*-cover of X. Let $S = \{x_n\}$ be a sequence converging to x in X. Since $\mathcal{P}$ is point-countable, put $(\mathcal{P})_x = \{P_n : n \in \mathbb{N}\}$. Then S is eventually in $\bigcup_{n \le k} P_n$ for some $k \in \mathbb{N}$. If not, then for any $k \in \mathbb{N}$, S is not eventually in $\bigcup_{n \le k} P_n$. So, for each $k \in \mathbb{N}$, there exists $x_{n_k} \in S - \bigcup_{n \le k} P_n$. We may assume $n_1 < n_2 < \cdots < n_{k-1} < n_k < n_{k+1} < \cdots$. Put $S' = \{x_{n_k} : k \in \mathbb{N}\}$, then S' is a sequence converging to x. Since $\mathcal{P}$ is a cs^*-cover, there exists $m \in \mathbb{N}$ and a subsequence S'' of S' such that S'' is eventually in P_m. Note that $P_m \in (\mathcal{P})_x$. This contradicts the construction of S'. □

Corollary 3.2. *Let $\big(f, M, X, \{\mathcal{P}_n\}\big)$ be a Ponomarev-system. Then the following are equivalent.*

(1) *f is a sequentially-quotient, s-mapping;*

(2) *f is a sequence-covering, s-mapping.*

Proof. Consider the following conditions.

(3) $\mathcal{P}_n$ is a point-countable cs^*-cover of X for each $n \in \mathbb{N}$;

(4) $\mathcal{P}_n$ is a point-countable fcs-cover of X for each $n \in \mathbb{N}$.

Then (1)$\Longleftrightarrow$(3) and (2)$\Longleftrightarrow$(4) from Theorem 2.7 and Theorem 2.8 respectively. (3)$\Longleftrightarrow$(4) from Proposition 3.1. So (1)$\Longleftrightarrow$(2). □

Can "s-" in Corollary 3.2 be omitted? We give a negative answer for this question. We call a family $\mathcal{D}$ of subsets of a set D is an almost disjoint family if $A \bigcap B$ is finite whenever $A, B \in \mathcal{D}$, $A \ne B$.

Example 3.3. There exists a space X, which has a point-star network $\{\mathcal{P}_n\}$ consisting of cs^*-covers of X, but $\mathcal{P}_n$ is not an fcs-cover of X for each $n \in \mathbb{N}$.

Proof. Let $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ endow usual subspace topology of real line $\mathbb{R}$. Let $n \in \mathbb{N}$, we construct $\mathcal{P}_n$ as follows.

Put $A_n = \{1/k : k > n\}$. Using Zorn's Lemma, there exists a family $\mathcal{A}_n$ of infinite subsets of A_n such that $\mathcal{A}_n$ is an almost disjoint family and maximal with respect to these properties. Then $\mathcal{A}_n$ must be infinite (in fact, $\mathcal{A}_n$ must be uncountable) and denote it by $\{P_\beta : \beta \in \Lambda_n\}$. Put $\mathcal{B}_n = \{P_\beta \bigcup \{0\} : \beta \in \Lambda_n\}$, and put $\mathcal{P}_n = \mathcal{B}_n \bigcup \{\{1/k\} : k = 1, 2, \ldots, n\}$. Thus $\mathcal{P}_n$ is constructed. We only need to prove the following three claims.

Claim 1: $\{\mathcal{P}_n\}$ is a point-star network of X.

Let $x \in U$ with U open in X. If $x = 0$, then there exists $m \in \mathbb{N}$ such that $A_m \subset U$. It is easy to check that $st(0, \mathcal{P}_m) = A_m \bigcup \{0\}$. So $0 \in st(0, \mathcal{P}_m) \subset U$. If

$x = 1/n$ for some $n \in \mathbb{N}$, then $st(1/n, \mathcal{P}_n) = \{1/n\}$. So $1/n \in st(1/n, \mathcal{P}_n) \subset U$. This proves that $\{\mathcal{P}_n\}$ is a point-star network of X.

Claim 2: For each $n \in \mathbb{N}$, $\mathcal{P}_n$ is a cs^*-cover of X.

Let $n \in \mathbb{N}$ and let $S = \{x_k\}$ be a sequence converging to x in X. Without loss of generalization, we can assume S is nontrivial, that is, the set $L = \{x_k : k \in \mathbb{N}\} \bigcap A_n$ is an infinite subset of A_n and the limit point $x = 0$. If $L \in \mathcal{A}_n$, it is clear that S has a subsequence is eventually in $L \bigcup \{0\} \in \mathcal{B}_n \subset \mathcal{P}_n$. If $L \notin \mathcal{A}_n$, then there exists $\beta \in \Lambda_n$ such that $L \bigcap P_\beta$ is infinite. Otherwise, $L \in \mathcal{A}_n$ by maximality of $\mathcal{A}_n$. Thus S has a subsequence is eventually in $P_\beta \bigcup \{0\} \in \mathcal{B}_n \subset \mathcal{P}_n$. So $\mathcal{P}_n$ is a cs^*-cover of X.

Claim 3: For each $n \in \mathbb{N}$, $\mathcal{P}_n$ is not an fcs-cover of X.

Let $n \in \mathbb{N}$. If $\mathcal{P}_n$ is an fcs-cover of X, then, for sequence $\{1/k\}$ converging to 0 in X, there exist $P_{\beta_1}, P_{\beta_2}, \ldots, P_{\beta_s} \in \mathcal{A}_n$ and some $m \in \mathbb{N}$ such that $A_m = \{1/k : k > m\} \subset \bigcup \{P_{\beta_i} : i = 1, 2, \ldots, s\}$. Since Λ_n is infinite, pick $\beta \in \Lambda_n - \{\beta_i : i = 1, 2, \ldots, s\}$. Then $A_m \bigcap P_\beta$ is infinite, and $A_m \bigcap P_\beta \subset \bigcup \{P_{\beta_i} : i = 1, 2, \ldots, s\}$. So there exists $i \in \{1, 2, \ldots, s\}$ such that $A_m \bigcap P_\beta \bigcap P_{\beta_i}$ is infinite. Thus $P_\beta \bigcap P_{\beta_i}$ is infinite. This contradicts that $\mathcal{A}_n$ is almost disjoint. So $\mathcal{P}_n$ is not an fcs-cover of X.

Thus we complete the proof of this example. □

Remark 3.4. Let X and $\{\mathcal{P}_n\}$ be given as in Example 3.3. Then, for *Ponomarev*-system $\big(f, M, X, \{\mathcal{P}_n\}\big)$, f is sequentially-quotient from Theorem 2.7 and Claim 2 in Example 3.3 (note: f is also a π-mapping from Remark 2.(1)), and f is not sequence-covering from Theorem 2.8 and Claim 3 in Example 3.3. So "s-" in Corollary 3.2 can not be omitted.

Remark 3.5. Recently, Lin proved that each sequentially-quotient, compact mapping from a metric space is sequence-covering, which answers [6, Question 3.4.8] (also, [2, Question 2.6]). Naturally, we ask: is each sequentially-quotient, π-mapping from a metric space sequence-covering? The answer is negative. In fact, let f be a mapping in Remark 3.4. Then f is a sequentially-quotient, π-mapping from a metric space M, but it is not sequence-covering.

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