Unit – 1

Testing of hypothesis

Small Samples

Testing of Hypothesis of Small Samples (Sample size n < 30)

(i) Student's t test

<table>
<tr><th>S.No Name</th><th>TYPE</th><th>formula</th><th>procedure</th><th></th></tr>
<tr>
<td>1. Student's t test(one sample)</td>
<td>(when SD of the sample is not given directly)</td>
<td>$t_{cal} = \frac{\bar{x} - \mu}{s^2 / \sqrt{n}}$ n – sample size
where $\bar{x}$ - sample mean $\bar{x} = \frac{\sum x}{n}$
μ - population mean s^2 - $\frac{1}{n-1} \sum (x - \bar{x})^2$</td>
<td>Given : n , $\bar{x}$,μ and x (sample data)
To set up : H_o & H_1
To find : t_{cal}
To find : t_{table} @ n-1 d.f with 1% or 5% level from t test table
If tcal < t table :- Accept H_o
If tcal > t table :- Reject H_o that is Accept H_1</td>
<td>Table :
<table>
<tr><td>x</td><td>$\bar{x}$</td><td>$(x - \bar{x})$</td><td>$(x - \bar{x})^2$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>$\sum (x - \bar{x})^2$</td></tr>
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<tr>
<td>2. Student's t test(one sample)</td>
<td>(when SD of the sample is given directly)</td>
<td>$t_{cal} = \frac{\bar{x} - \mu}{SD / \sqrt{n-1}}$ n – sample size
where $\bar{x}$ - sample mean ; $\bar{x} = \frac{\sum x}{n}$
μ - population mean SD – Standard deviation</td>
<td>Given : n , $\bar{x}$,μ and SD
To set up : H_o & H_1
To find : t_{cal}
To find : t_{table} @ n-1 d.f with 1% or 5% level from t test table
If tcal < t table :- Accept H_o
If tcal > t table :- Reject H_o that is Accept H_1</td>
<td></td>
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<table>
<tr>
<td>3. Student's t test(two sample)</td>
<td>(when SD of the sample is given directly)</td>
<td>$t_{cal} = \frac{\overline{x_1} - \overline{x_2}}{s\sqrt{1/n_1 + 1/n_2}}$ n_1& n_2 – sample size
$S^2 = \frac{1}{n_1 + n_2 - 2}\left[n_1 s_1^{\,2} + n_2 s_2^{\,2}\right]$ s_1 & s_2 Sample Standard deviation , where $\bar{x}_1 , \bar{x}_2$ - sample mean ;
$\overline{x_1} = \frac{\sum x_1}{n_1}$; $\overline{x_2} = \frac{\sum x_2}{n_2}$</td>
<td>Given : n , $\bar{x}$,μ and SD(s_1 & s_2)
To set up : H_o & H_1
To find : t_{cal}
To find : t_{table} @ n_1+n_2-2 d.f with 1% or 5% level from t test table
If tcal < t $_{table}$:- Accept H_o
If tcal > t $_{table}$:- Reject H_o that is Accept H_1</td>
<td></td>
</tr>
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<td>4. Student's t test(two sample)</td>
<td>(when SD of the sample is not given directly)</td>
<td>$t_{cal} = \frac{\overline{x_1} - \overline{x_2}}{s\sqrt{1/n_1 + 1/n_2}}$ n_1& n_2 – sample size
$S^2 = \frac{1}{n_1 + n_2 - 2}\left[\sum (x_1 - \overline{x_1})^2 + \sum (x_2 - \overline{x_2})^2\right]$
s_1 & s_2 Sample Standard deviation
where $\bar{x}_1 , \bar{x}_2$ - sample mean ; $\overline{x_1} = \frac{\sum x_1}{n_1}$;
$\overline{x_2} = \frac{\sum x_2}{n_2}$</td>
<td>Given : n , $\bar{x}$,μ and ; x_1 & x_2 (sample data)
To set up : H_o & H_1
To find : t_{cal}
To find : t_{table} @ n_1+n_2-2 d.f with 1% or 5% level from t test table
If tcal < t $_{table}$:- Accept H_o
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<td>Table :1
<table>
<tr><td>x_1</td><td>$\bar{x}_1$</td><td>$(x_1 - \overline{x_1})$</td><td>$(x_1 - \overline{x_1})^2$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>$\sum (x_1 - \overline{x_1})^2$</td></tr>
</table>
Table :2
<table>
<tr><td>x_2</td><td>$\bar{x}_2$</td><td>$(x_2 - \overline{x_2})$</td><td>$(x_2 - \overline{x_2})^2$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>$\sum (x_2 - \overline{x_2})^2$</td></tr>
</table>
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</table>

(ii) F- Test(H_o=(1)To test whether if any significant different between two estimate of population variance) or (2) (To test if Two samples have come from same population) or (3) To test if Two samples are drawn from same population)

<table>
<tr>
<td>**5. F – TEST (two sample)**</td>
<td>(To test whether if any significant different between two estimate of population variance)</td>
<td>F cal = $\frac{S_1^2}{S_2^2}$
Where $S_1^2 = \frac{\sum(x - \overline{x})^2}{n_1 - 1}$ n_1 - first sample size
$S_2^2 = \frac{\sum(x - \overline{x})^2}{n_2 - 1}$ n_2 – Second sample size
And $S_1^2 > S_2^2$
F cal = $\frac{Greater - Variance}{Smaller - Variance}$</td>
<td>Given : n_1 , n_2, $\sum(x_1 - \overline{x_1})^2$, $\sum(x_2 - \overline{x_2})^2$ (Sum of the squares of the deviations of the sample values from the sample mean)
To set up : H_o = $S_1^2 = S_2^2$
To set up : H_1 = $S_1^2 \neq S_2^2$
To find : F_{cal}
To find : t_{table} @ (n_1-1, n_2-1)d.f from F test table
If Fcal < F table :- **Accept H_o**
If Fcal >F table :- **Reject H_o** that is **Accept H_1**</td>
<td>Case (ii) if $\sum(x_1 - \overline{x_1})^2$ not given directly ($x_1$and x_2 are given directly)
Table :1
<table>
<tr><td>x_1</td><td>$\overline{x_1}$</td><td>$(x_1 - \overline{x_1})$</td><td>$(x_1 - \overline{x_1})^2$</td></tr>
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<tr><td></td><td></td><td></td><td>$\sum(x_1 - \overline{x_1})^2$</td></tr>
</table>
Table :2
<table>
<tr><td>x_2</td><td>$\overline{x_2}$</td><td>$(x_2 - \overline{x_2})$</td><td>$(x_2 - \overline{x_2})^2$</td></tr>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td></td><td>$\sum(x_2 - \overline{x_2})^2$</td></tr>
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(iii) **χ^2 test** (Suppose we are given a set of observed frequencies obtain under some experiment and we want to test if the experimental result support a particular hypothesis theory)

<table>
<tr>
<td>6.. χ^2 – (Chi – Square) One Sample</td>
<td>**χ^2** is used to test whether differences between Observed &</td>
<td>χ^2 **Cal** = $\sum \frac{(O - E)^2}{E}$
Where O – Observed Frequencies
E – Observed Frequencies

To set up : H_o = $O = E$ (Observed and Expected Frequencies are same)</td>
<td>Given : a set of observer frequencies
To set up : H_o = $O = E$
To set up : H_1 = $O \neq E$
To find : χ^2 cal
To find : χ^2 table @ (n_1-1)d.f with 1% or 5% level from t test table
If Fcal < F table :- **Accept H_o**
If Fcal >F table :- **Reject H_o** that is **Accept H_1**</td>
<td>Case (ii) if $\sum(x_1 - \overline{x_1})^2$ not given directly ($x_1$and x_2 are given directly) Table – 1
<table>
<tr><td>x_2</td><td>$\overline{x_2}$</td><td>$(x_2 - \overline{x_2})$</td><td>$(x_2 - \overline{x_2})^2$</td></tr>
<tr><td></td><td></td><td></td><td>Total</td></tr>
</table>
Table :2
<table>
<tr><td>O</td><td>E</td><td>$(O - E)$</td><td>$(O - E)^2$</td><td>$\frac{(O - E)^2}{E}$</td></tr>
<tr><td></td><td></td><td></td><td></td><td>Total</td></tr>
</table>
</td>
</tr>
</table>

(iv) χ^2 **test** (Suppose we are given a set of observed frequencies obtain under some experiment and we want to test if the experimental result support a particular hypothesis theory) **INDEPENDENCE OF ATTRIBUTES**

6.. **χ^2 –** (Chi – Square) More than one Attribute s

χ^2 is used to test whether differences between O & E

χ^2 **Cal** = $\sum \frac{(O - E)^2}{E}$ Where O , E – Observed & Excepted

Given :

a	b
c	d

Contingency table

a	b	a+b	Row Total
c	d	c+d	
a+c	b+d	N	
Column total			

Given : a set of observer frequencies

To set up : H₀ = $O = E$

To set up : H₁ = $O \neq E$

To find : χ^2_{cal}

To find : χ^2_{table} @ $(n_1\text{-}1)$d.f with 1% or 5% level from t test table

If χ^2 cal < χ^2 table :- **Accept H₀**

If χ^2 cal >χ^2 table :- **Reject H₀** that is **Accept H₁**

expected frequencies table

E(a) = $\frac{(a+b)(a+c)}{N}$	E(b) = $\frac{(a+b)(b+d)}{N}$
E(a) = $\frac{(a+c)(c+d)}{N}$	E(a) = $\frac{(b+b)(c+d)}{N}$

Case (ii) if $\sum (x_1 - \overline{x_1})^2$ not given directly ($x_1$and x_2 are given directly)

Table :1

x_2	$\overline{x_2}$	$(x_2 - \overline{x_2})$	$(x_2 - \overline{x_2})^2$
			Total

O	E	$(O - E)$	$(O - E)^2$	$\frac{(O - E)^2}{E}$

$$\sum \frac{(O - E)^2}{E}$$

Testing of Hypothesis – II

Large Samples

Testing of Hypothesis of Large Samples (Sample size n > 30)

S.No	Type	formula	procedure	
1. Large Samples	Single Proportion	$z_{cal} = \frac{p - P}{\sqrt{\frac{PQ}{n}}}$ where n – sample size p - sample Proportion , q = 1 - p P – Population proportion , Q = 1 - P	Given : sample information & sample size To find p & P To set up : H_o & H_1 To find : Z_{cal} To find : Z_{table} @ 1% or 5% level from t test table(Last row that is ∞ row) If Zcal < Z table :- **Accept** H_o If Zcal > Z table :- **Reject** H_o that is **Accept** H_1	Example problem : In a sample of 1000 people in Karnataka 540 are rice eater & the rest are wheat eater, can we assume that both rice & wheat are equally popular in this state at 1% level. p = 540 / 1000 (sample proportion) P = 1 / 2 (RICE -population proportion) Q = 1 – P
		H_o = **both rice & wheat are equally popular in this state ; P = 0.5** H_1 = **P ≠ 0.5 (Two tail test) both rice & wheat are not equally popular**		
2. 1. Large Samples	Difference of Proportion	$z_{cal} = \frac{\overline{p_1} - \overline{p_2}}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$ Where n_1& n_2 – sample size $p = \frac{[n_1 p_1 + n_2 p_2]}{n_1 + n_2}$; $q = 1 - p$	Given : sample information & size , P1 To find : p , q To set up : H_o & H_1 To find : z_{cal} To find : z_{table} @ 1% or 5% level from t test table If zcal < Z table :- **Accept** H_o If zcal > Z table :- **Reject** H_o that is **Accept** H_1	Example problem : Random sample of 400 men and 600 women were asked whether they would like to have a flyover near their residence. 200m & 325w were in favour to the proposal. Test the hypothesis that proportions of men and women in favour of the proposal are same at 5 % level. $p_1 = 200 / 400$; $p_2 = 325 / 600$ $n_1 = 400$, $n_2 = 600$
		H_o = **Assume that there is no significant difference between the option of men and women as far as proposal of flyover concerned($p_1 = p_2$)**		
3. 1. Large Samples	Single Mean	$z_{cal} = \frac{\overline{x} - \mu}{s/\sqrt{n}}$ or $z_{cal} = \frac{\overline{x} - \mu}{\sigma/\sqrt{n}}$ n – sample size S – sample S.D σ - population S.D $\overline{x}$ - sample mean μ - population Mean		Suppose we want to test whether the given sample of size n has been drawn from a populations with mean μ. We set up null hypothesis that there is no difference between $\overline{x}$ and μ Where $\overline{x}$ is the sample mean.

4. 1. **Large Samples**	Difference of mean	$z_{cal} = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\sigma_1^2/n_1 + \sigma_2^2/n_2}}$ n_1& n_2 – sample size s_1 & s_2 Sample Standard deviation where $\bar{x}_1, \bar{x}_2$ - sample mean ; $\overline{x_1} = \frac{\sum x_1}{n_1}$; $\overline{x_2} = \frac{\sum x_2}{n_2}$		Let $\overline{x_1}$ be the mean of a sample of size n_1 from a population with mean μ_1 and variance σ_1^2 Let $\overline{x_2}$ be the mean of a sample of size n_2 from a population with mean μ_2 and variance σ_2^2 To test whether there is any significant difference between $\overline{x_1}$ and $\overline{x_2}$ Ho : **the samples have been drawn from the same population** H1 : **the samples have been drawn from the different population**

UNIT II
Design of Experiments

Design of Experiments

By 'experiment' we mean the collection of data (which usually consists of a series of measurement of some feature of an object) for a scientific investigation according to a certain specified sampling procedures.

Consider the example of agricultural experiment which may be performed to verify the claim that a particular manure has got the effect of increasing the yield of paddy. Here the quantity of manure used and amount of yield of paddy are known as experimental variables. And factors such as rainfall, quality of soil and quality of seeds (will also affect the yield of paddy, which are not under study) are called extraneous variables.

Aim of design of experiments

In any statistical experiment we will have both experimental variables and extraneous variables. The aim of the design of experiments is to control extraneous variables and hence to minimize the enor so that the results of experiments could be attributed only to the experimental variables.

Basic principles of experimental design

1. Randomization 2.Replication 3.Local control.

Consider an agricultural experiment. To analyze the effect of a manure in the yield of paddy, we use the manure in some plots of same size(group of experimental units) is called experimental group and the some other group of plots in which the manure is used and which will provide a basis for comparison is called the control group.

By grouping, we mean combining sets of homogeneous plots into groups, so that different manures may be used in different groups. By **blocking,** we mean assigning the same number of plots in different blocks. . By **balancing,** we mean adjusting the procedures of grouping, blocking and assigning the manures in such a manner that a balanced configuration is obtained.

Replication

In order to study the effects of different manures on the yield are studied, each manure is used in more than one plot. In other words, we resort to replication which means repetition.

Basic designs of experiments.

I. Completely Randomized Design (ANOVA one way classification)

2. Randomized Block Design (ANOVA two way classification)

3. Latin Square Design (ANOVA three way classification)

ANOVA

ANOVA enables us to divide the total variation (represented by variance) in a group into parts which are accounted to different factors and a residual random variation which could be accounted for by any of these factors. The variation due to any specific factor is compared with the residual variation for significance, and hence the effect of the factors are concluded.

CRD

Consider an experiment of agriculture in which "h" treatments (manures) and "n" plots are available. To control the extraneous variables treatment "I" should be replicated on "n I" plots, treatment 2 should be replicated on "n2" plots and so on. To reduce the error we have to randomize this process that is which nl plots b'e used treatment I and so on. For this we number the plots (from I to n) and write the numbers on cards and shuffle well. Now, we select nl cards (as cards are selected at random the numbers will not be in order) on which treatments I will be used and so on. This process and design is called completely randomized design.

RBD

Consider an experiment of agriculture in which effects of "k" treatments on the yield of paddy used. For this we select "n" plots. If the quality of soil of these "n" plots is known, then these plots are divided into "h" blocks (each with one quality). Each of these "h" blocks are divided into "k" times (n = hk) and in each one of this "k" plots are applied the "k" treatments in a perfectly randomized manner such that that each treatment occurs only once in any of the block. This design is called randomized block design.

LSD

Consider an agricultural experiment, in which n2 plots are taken and arranged in the form of an n X n square, such that plots in each row will be homogeneous as far as possible with respect to one factor of classification, say, quality of soil and plots in each column will be homogeneous with respect to another factor of classification, say, seed quality. Then "n" treatments are given to these plots such that each treatment occurs only once in each tow and only once in each column. The various possible arrangements obtained in this manner are known as Latin squares of order "n" and the design is called Latin Square Design.

Note: In a 2 X 2 LSD, the degree of freedom for the residual variation is en -2) = 0 which is not possible. Therefore a 2 X 2 LSD is not possible.

Problems

1. The following table shows the lives in hours of four brands of electric lamps:

Brand

A: 1610 1610 1650 1680 1700 1720 1800

B: 1580 1640 1640 1700 1750

C: 1460 1550 1600 1620 1640 1660 1740 1820

D: 1510 1520 1530 1570 1600 1680

Perform an analysis of variance and test the homogeneity of the mean lives of the four brands of lamps.

Solution:

We are going to discuss mean lives of bulbs according to brands (only one factor). So, let us use ANOV A one way classification.

Step 1 : Null Hypothesis Ho: There is no significant difference in the mean life of bulbs according to brands.(J..I]= J..I2= J..I3= /14)

Step 2: Test Statistic (Calculating F ratio) As the data given is numerically larger, let us subtract all values given by 1640 and then divide them by 10 to make them simpler for calculation ease

-3	-6	-18	-13
-3	0	-9	-12
1	0	-4	-11
4	6	-2	-7
6	11	0	-4
8		2	4
16		10	
		18	
29	11	-3	-43

Total T = $\overline{\sum x = 29 + 11 - 3 - 43 = -6}$

Square of the above values are

9	36	324	169
9	0	81	144
1	0	16	121
16	36	4	49
36	121	0	16
64	0	4	16
256	0	100	0
0	0	324	0
391	193	853	515

Ix^2 =391+193+853+515=1952

Correction factor = $\frac{T^2}{N} = \frac{(\_6)2}{26}$ = 1.38

$$SST = LX^2 - \frac{T^2}{N} = 1952 \quad -1.38 = 1950.62$$

$$SSC = \frac{\{LAf}{n1} + \frac{\{LBf}{n2} + \frac{\{LC)2}{n3} + \frac{\{LDf}{n4} - \frac{T^2}{N}$$

$$= \frac{(29)2}{7} + \frac{(11f}{5} + \frac{(\_3)2}{8} + \frac{(-43)2}{6} - 1.38$$

$$= 120.14+24.2+1.125+308.17-1.38$$

$$= 452.25$$

SSE = SST - SSC = 1950.62 - 452.25 = 1498.37

ANOVA table is

SOURCE OF VARIATION	SUM OF SQUARES	DEGREES OF FREEDOM	MEAN SUM OF SQUARES	F-RATIO
BETWEEN BRANDS	SSC = 452.25	NO OF SAMPLES-1=4-1=3	$\frac{452.25}{3} = 150.75$	
WITHIN BRANDS	SSE = 1498.37	NO OF DATA-NO OF SAMPLES = 26-4 = 22	$\frac{1498.37}{22} = 68.11$	$\frac{150.75}{68.11} = 2.21$

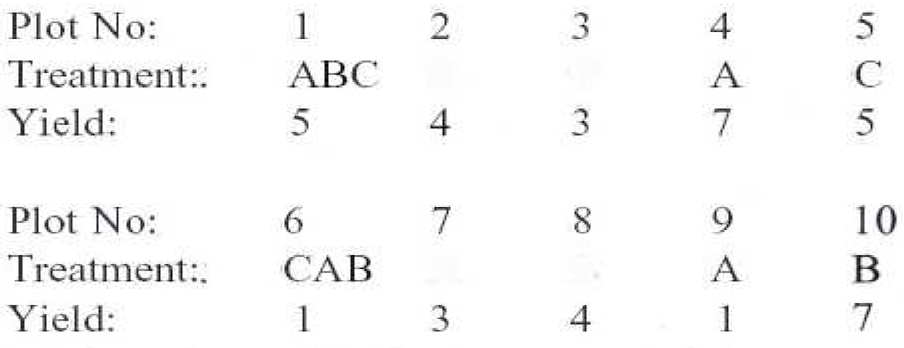
Step 3: Level of significance: Not given. So, let us take 5 % = 0.05 a

Step 4: Degrees of Freedom: (3,22)

Step 5: Table value : Fo.os(3,22)= 3.05

Step 6: Conclusion: Comparing the F - value calculated at step 2 with table value of step 5 Fcal < Flab,we accept the Null hypothesis. That is there is no difference in the mean life time of the lamps due to brands.

2. A **completely randomized design** experiment with 10 plots and 3 treatments gave the following results:



Plot No:	1	2	3	4	5
Treatment:.	ABC	[illegible]	[illegible]	A	C
Yield:	5	4	3	7	5

Plot No:	6	7	8	9	10
Treatment:.	CAB	[illegible]	[illegible]	A	B
Yield:	1	3	4	1	7

Analyze the results for treatment effects.

Solution:

All the plots are not applied with the same number of treatments and we are to analyze the treatment effects only, we apply One way classification.

Re arranging the data

Treatment A: 5 7

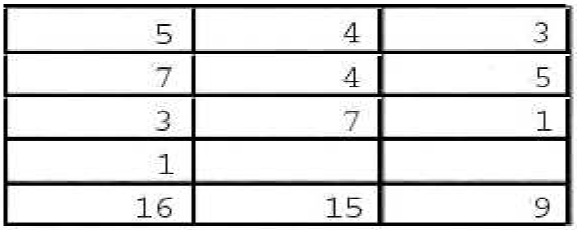
Treatment B: 4 4

Treatment C: 3 5

Step 1: Null Hypothesis

Ho: There is no difference in the yield between treatments.

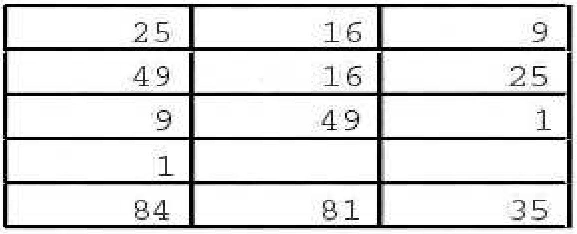
Step 2 : Test Statistic:



5	4	3
7	4	5
3	7	1
1		
16	15	9

Total = T = 16+15+9 = 40

The square of the values are



25	16	9
49	16	25
9	49	1
1		
84	81	35

LX^2 =84+81+35=200

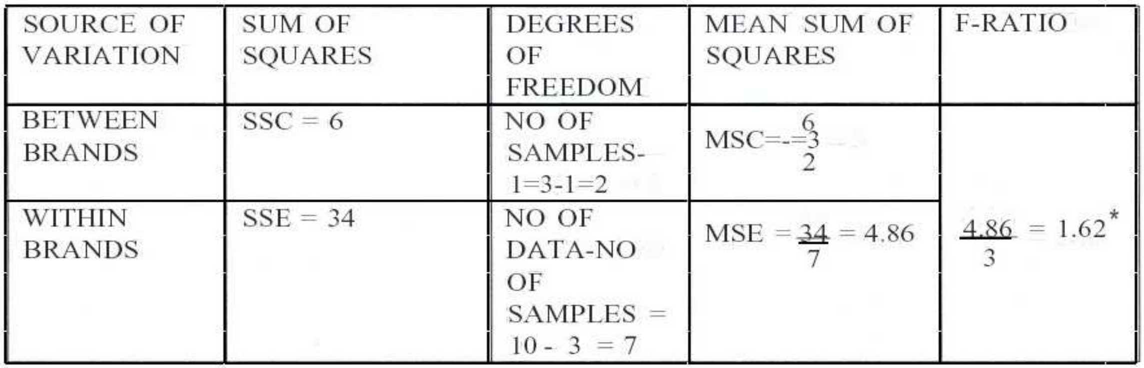
Correction factor = 160

SST= = 40

SSC= (16)2 + *(IS)2* + (9)2 -160 = 6

SSE = SST - SSC =40 - 6 = 34

The ANOVA table is



SOURCE OF VARIATION	SUM OF SQUARES	DEGREES OF FREEDOM	MEAN SUM OF SQUARES	F-RATIO
BETWEEN BRANDS	SSC = 6	NO OF SAMPLES-1=3-1=2	MSC=$\frac{6}{2}$=3	$\frac{4.86}{3}$ = 1.62*
WITHIN BRANDS	SSE = 34	NO OF DATA-NO OF SAMPLES = 10 - 3 = 7	MSE = $\frac{34}{7}$ = 4.86	

Step 3: Level of significance: Not given. So, let us take *S* % = *O.OS*

Step 4: Degrees of Freedom: (7,2)

Step 5: Table value : Fo.os(7,2)= *19.3S* •

Step 6: Conclusion: Comparing the F - value calculated at step 2 with table value of step *S* Fca1 < Flab,we accept the Null hypothesis. That is there is no difference in the mean yield between the treatments.

3. Three varieties of a crop are tested in a randomized block design with four replications, the layout being as given below: The yields are given in kilograms. Analyze for significance

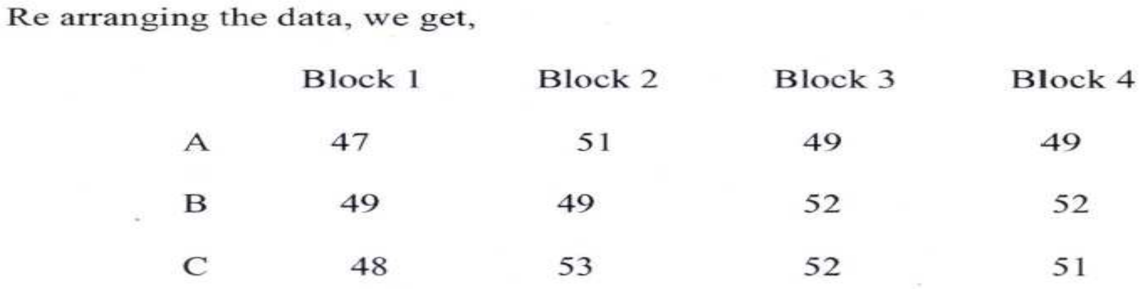
C48 *AS1* *BS2* A49

A47 B49 *CS2* *CS1*

B49 C53 A49 B50

Solution:

Each blocks are applied with same number of replications and from the question (RBD) we understand that it is two way classification.



Re arranging the data, we get,

	Block 1	Block 2	Block 3	Block 4
A	47	51	49	49
B	49	49	52	52
C	48	53	52	51

Step 1: **Null** Hypothesis Ho (I): There is no difference in the yield between treatments (rows).

Null Hypothesis Ho (II): There is no difference in the yield between blocks(columns).

Step 2: Test Statistic (Calculating F ratio)

As the data given is numerically larger, let us subtract all values given by 50 make them simpler for calculation ease.

-3	1	-1	-1	-4
-1	-1	2	0	0
-2	3	2	1	4
-6	3	3	0	0

Total T = - 6 +3+3+0 = 0

The square of the above values are

9	1	1	1
1	1	4	0
4	9	4	1
14	11	9	2

LX^2 =14+ 11 + 9 + 2 = 36

Correction factor = 0

SST = 36

SSE = SST - SSC-SSR = 36 - 18 - 8 = 10

The ANOVA table is

SOURCE OF VARIATION	SUM OF SQUARES	DEGREES OF FREEDOM	MEAN SUM OF SQUARES	F-RATIO
BETWEEN TREATMENTS (ROWS)	SSR= 8	NO OF ROWS -1 = r-l =3 -1 = 2	MSR=- $\frac{8}{2}$ =4	F_R= $\frac{MSR}{MSE}$ = '-$\frac{4}{1,67}$=2.4
BETWEEN BLOCKS (COLUMNS)	SSC = 18	NOOF COLUMNS -1 = c-l = 4 -1 = 3	MSC =$\frac{.!!}{3}$ = 6	Fe = $\frac{MSC}{MSE}$ = $\frac{6}{1,67}$=3.6
RESIDUAL ERROR	SSE = 10	(r-l)(c-l) = (2)(3) = 6	MSE= $\frac{\sim}{6}$=1.67	

Step 3: Level of significance: Not given. So, let us take 5 % = 0.05

Step 4: Degrees of Freedom: for rows (3,6) for columns (4,6)

Step 5: Table value : Foos(2,6) = 5.14 Foos(3,6) = 4.76

Step 6: Conclusion: Comparing the FR- value calculated at step 2 with table value of step 5 FR< Ftab,we accept the Null hypothesis I. That is there is no difference in the mean yield between the treatments(rows).

Comparing the Fe - value calculated at step 2 with table value of step 5 Fe< Ftab,we accept the Null hypothesis II. That is there is no difference in the mean yield between the blocks(columns).

4. Four experiments determine the moisture content of samples of a powder, each observer taking a sample from each of six consignments. The assessments are given below:

	Consignment					
	1	2	3	4	5	6
Observer						
1	9	10	9	10	11	11
2	12	11	9	11	10	10
3	11	10	10	12	11	10
4	12	13	11	14	12	10

Perform an analysis of variance on these data and discuss whether there is any significant difference between consignments or between observers.

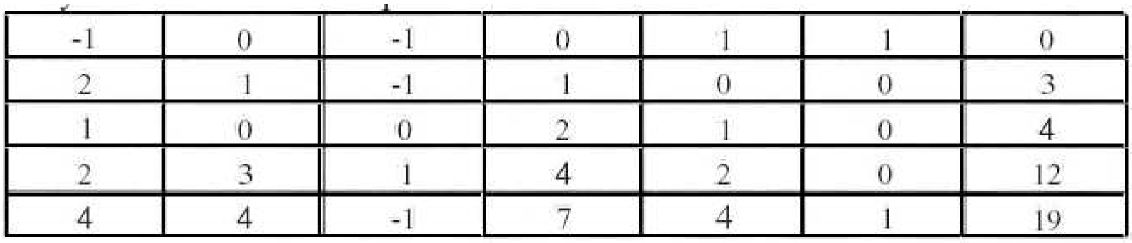
Solution:

From the data we understand that there are two factors. One is consignment and other is observer. So, we apply ANOV A two way classification.

Step 1: Null Hypothesis Ho (I): There is no difference in the yield between treatments (rows).

Null Hypothesis Ho (II): There is no difference in the yield between blocks(columns).

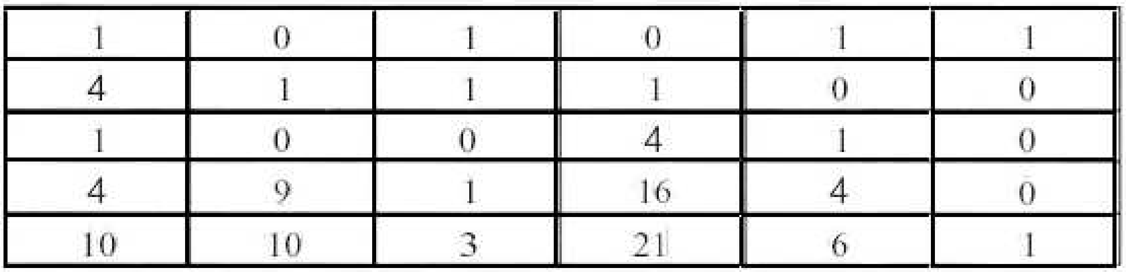
Step 2: Test Statistic (Calculating F ratio) As the data given is numerically larger, let us subtract all values given by 10 make them simpler for calculation ease.



-1	0	-1	0	1	1	0
2	1	-1	1	0	0	3
1	0	0	2	1	0	4
2	3	1	4	2	0	12
4	4	-1	7	4	1	19

Total = 4 + 4 + -1 + 7 + 4 +1 = 19

The square of the above table values are,



1	0	1	0	1	1
4	1	1	1	0	0
1	0	0	4	1	0
4	9	1	16	4	0
10	10	3	21	6	1

Ix^2 =10+10+3+21+6+1 = 51

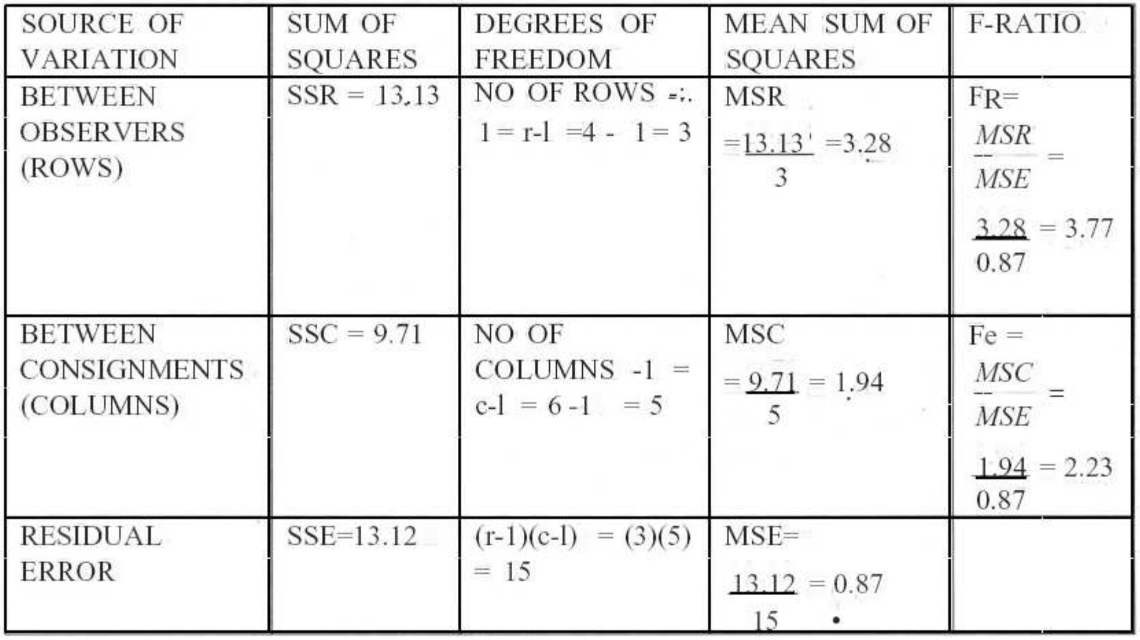
Correction factor = -15.04

SST= Ix - -=51-15.04 = 35.96

SSR = 15.04

SSE = SST - SSC-SSR = 35.96 - 13.13 - 9.71 = 13.12

The ANOVA table is



SOURCE OF VARIATION	SUM OF SQUARES	DEGREES OF FREEDOM	MEAN SUM OF SQUARES	F-RATIO
BETWEEN OBSERVERS (ROWS)	SSR = 13.13	NO OF ROWS -1 = r-l =4 - 1 = 3	MSR $=\frac{13.13}{3}=3.28$	FR= $\frac{MSR}{MSE}=$ $\frac{3.28}{0.87}=3.77$
BETWEEN CONSIGNMENTS (COLUMNS)	SSC = 9.71	NO OF COLUMNS -1 = c-l = 6 -1 = 5	MSC $=\frac{9.71}{5}=1.94$	Fe = $\frac{MSC}{MSE}=$ $\frac{1.94}{0.87}=2.23$
RESIDUAL ERROR	SSE=13.12	(r-1)(c-l) = (3)(5) = 15	MSE= $\frac{13.12}{15}=0.87$	

Step 3: Level of significance: Not given. So, let us take 5 % = 0.05

Step 4: Degrees of Freedom : for rows (3,15) for columns (5,15)

Step 5: Table value : Fo.os(3,15)= 3.29 Fo.os(5,15) = 5.05

Step 6: Conclusion: Comparing the FR- value calculated at step 2 with table value of step 5 FR>Ftab,we Reject the Null hypothesis 1. That is there is significant difference in the moisture content of the powder between the observers(rows).

Comparing the Fe - value calculated at step 2 with table value of step 5 *Fe*< Ftab, we accept the Null hypothesis II. That is there is no difference in the moisture content between the consignments(columns).

5. The following data resulted from an experiment to compare three burners BI, B2,B3. A Latin square design was used as the tests were made on 3 engines and were spread over 3 days.

	Engine 1	Engine 2	Engine 3
Day 1	B_1 - 16	B_2 – 17	B_3 – 20
Day 2	B_2 – 16	B_3 - 21	B_1 – 15
Day 3	B_3 – 15	B_1 – 12	B_2 – 13

Test the hypothesis that there is no difference between the burners.

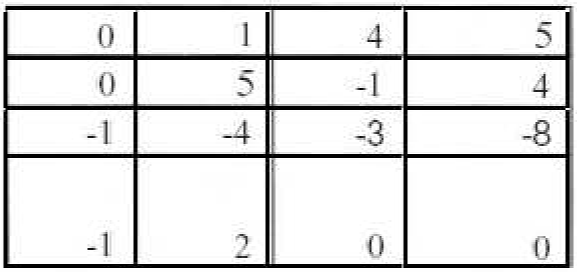
Step 1: Null Hypothesis Ho (I): There is no difference in the mean between days (rows).

Null Hypothesis Ho (II): There is no difference in the mean between Engines(columns).

Null Hypothesis Ho (III): There is no difference in the mean between Burners(letters).

Step 2: Test Statistic (Calculating F ratio)

As the data given is numerically larger, let us subtract all values given by 16 make them simpler for calculation ease.



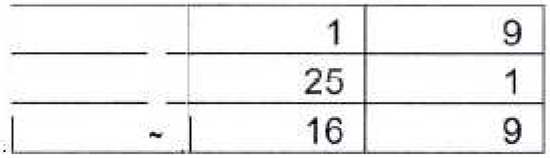
0	1	4	5
0	5	-1	4
-1	-4	-3	-8
-1	2	0	0

Total T = 1 B1 = 0 - 1 - 4 = - 5

B2 = 1 + 0 - 3 = - 2

B3 = 4 + 5 -1 = 8

The squared values are



	1	9
	25	1
~	16	9

Ix^2 = 69

Correction factor = 0.1111

SST = 68.88889

SSR = 8.3333 + 5.3333 + 21.3333 - 0.1111

= 34.8888

ssc= 0.3333 + 1.3333+0- 0.1111 = 1.5556

SSL = 8.3333+1.3333+21.3333 - 0.1111

= 30. 8888

SSE = SST - SSC- SSL - SSR = 68.8889 - 34.8888 - 1.5556 - 30.8888

= 1.5557

SOURCE OF VARIATION	SUM OF SQUARES	DEGREES OF FREEDOM	MEAN SUM OF SQUARES	F-RATIO
BETWEEN DAYS (ROWS)	SSR= 34.8888	NO OF ROWS-l = r-l =3 - 1 = 2	MSR $= \frac{34.8888}{2} = 17.4444$	$F_R = \frac{MSR}{MSE} = \frac{17.4444}{0.77785} =$ 22,43
BETWEEN ENGINES (COLUMNS)	SSC= 1.5556	NO OF COLUMNS - 1= c-l = 3- 1 = 2	MSC $= \frac{1.5556}{2} = 0.7778$	Fe = $\frac{MSE}{MSC} = \frac{0.77785}{0.7778} = 1.000$
BETWEEN BURNERS (LETTERS)	SSL= 30.8888	NO OF LETTERS - 1 = c-l = 3 - 1 =2	MSL $= \frac{30.8888}{2} = 15.4444$	$F_L = \frac{MSC}{MSE} = \frac{15.4444}{0.77785} = 19.86$
RESIDUAL ERROR	SSE= 1.5557	(n-l)(n-2) = (3-1)(3-2) = 2	MSE= $\frac{1.5557}{2} = 0.77785$	

Step 3: Level of significance: Not given. So, let us take 5 % = 0.05

Step 4: Degrees of Freedom: for rows (2,2) for columns (2,2) for letters (2,2)

Step 5: Table value : *Fo.os(2,2)* = 19

Step 6: Conclusion: Comparing the FR - value calculated at step 2 with table value of step 5 FR>Ftab, we Reject the Null hypothesis I. That is there is significant difference in the due to selection of days(rows).

Comparing the Fe - value calculated at step 2 with table value of step 5 Fe< Ftab, we accept the Null hypothesis II. That is there is no difference between the engines (columns). Comparing the FL - value calculated at step 2 with table value of step 5 Fe>Ftab, we reject the Null hypothesis III. That is there is difference between the numbers(letters).

MA 2266

Statistics and Numerical Methods

Unit – 1
Testing of hypothesis

Unit- 2
Design of Experiments

Unit – 3
Solution of Equation and Eigenvalue problems

Unit -4
Interpolation, Numerical differentiation and

Numerical integration

Unit – 5
Numerical Solution of ordinary Differential Equations

Unit – 3

Solution of Equation and Eigenvalue problems

Unit - 4
Interpolation, Numerical differentiation and Numerical integration

FORMULAE

1. **Lagrange's interpolation formula is**

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots\ldots\ldots(x_0-x_n)}\,y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\ldots\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots\ldots\ldots(x_1-x_n)}\,y_1$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)(x_n-x_3)\ldots\ldots\ldots(x_n-x_{n-1})}\,y_n$$

2. **Inverse of Lagrange's interpolation formula is**

$$x = f(y) = \frac{(y-y_1)(y-y_2)(y-y_3)\ldots\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\ldots\ldots\ldots(y_0-y_n)}\,x_0$$

$$+ \frac{(y-y_0)(y-y_2)(y-y_3)\ldots\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\ldots\ldots\ldots(y_1-y_n)}\,x_1$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)\ldots\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_2)(y_n-y_3)\ldots\ldots\ldots(y_n-y_{n-1})}\,x_n$$

3. **Newton's divided difference interpolation formula**

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + \dots\dots\dots$$

$$+ (x - x_0)(x - x_1)(x - x_2)\dots\dots (x - x_{n-1}) f(x_0, x_1, \dots. x_n)$$

4. **Newton's forward difference formula is**

$$P_n(x) = y_0 + \frac{u}{1!} \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0$$

$$+ \text{-------} + \frac{u(u-1)(u-2)\dots\dots.(u-(r-1))}{n!} \Delta^n y_0$$

where $u = \dfrac{x - x_0}{h}$

5. **Newton's backward difference formula is**

$$P_n(x) = y_n + \frac{v}{1!} \nabla y_n + \frac{v(v+1)}{2!} \nabla^2 y_n + \frac{v(v+1)(v+2)}{3!} \nabla^3 y_n$$

$$+ \text{-------} + \frac{v(v+1)(v+2)\dots\dots.(v+(r-1))}{n!} \nabla^n y_n$$

where $v = \dfrac{x - x_n}{h}$

6. **Newton's forward interpolation formula used only for equal intervals.**

INTRODUCTION

The estimation of values between well-known discrete points are called interpolation.

Interpolation is the process of finding the most appropriate estimate for missing data. For making the most probable estimate it requires the following assumptions.

1. **The frequency distribution is normal and is not marked by Sudden ups and downs.**
2. **The changes in the series are uniform with in a period.**

It is used to fill in the gaps in the statistical data for the sake of continuity of information

Many famous mathematicians have their names associated with procedures for interpolation: Gauss, Newton, Bessel, Stirling. The need to interpolate began with the earl studies of astronomy when the motion of heavenly bodies was to be determined from periodic observations. Interpolation technique is used in various disciplines like, business, economics, population studies, price determination etc.

INTERPOLATING FUNCTION

Let a set of tabular values of a function $y = f(x)$ where the explicit nature of the function is not known, then $f(x)$ is replaced by a simpler function $\varphi(x)$ agree with the set of tabulated points. Any other value may be calculated form $\varphi(x)$. This function $\varphi(x)$ is known as an interpolating function.

LAGRANGIAN POLYNOMIALS

The Lagrangian polynomial method is a very straight forward approach. The method perhaps is the simplest way to exhibit the existence of a polynomial for interpolation with unevenly spaced data. Data where the x-values are not equi-spaced often occur as the result of experimental observations or when historical data's are examined.

Suppose we have a table of data with ' n ' pairs of x and f(x) values. With x_i indexed by the variable ' i ' :

i	x	y = f(x)
0	x_0	y_0
1	x_1	y_1
2	x_2	y_2
3	x_3	y_3
:	:	:
:	:	:
N	x_n	y_n

Here we do not assume inform spacing between the x-values nor do we need the x-values arranged in a particular order. All the x-values must be distinct, however. The Lagrangian form for this is

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)\ldots\ldots\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)\ldots\ldots\ldots(x_0 - x_n)}\, y_0$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)\ldots\ldots\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)\ldots\ldots\ldots(x_1 - x_n)}\, y_1$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_2)\ldots\ldots\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_2)(x_n - x_3)\ldots\ldots\ldots(x_n - x_{n-1})}\, y_n$$

The arithmetic in this method is tedious although hand calculator are convenient for this type of computation. Writing a computer program that implements, the method is not hard to do. Both MATLAB and Mathematics can get interpolating polynomials of any degree. An interpolating polynomial, although passing through the points used in its construction does not, in general, give exactly correct values when used for interpolation. The reason is that the underlying relationship is often not a polynomial of the same degree. We are therefore interested in the error of interpolation

$$E(x) = (x - x_0) (x - x_1) \ldots\ldots.. (x - x_n) \frac{f^{(n+1)} (\xi)}{(n + 1) !}$$ **with ξ in the smallest interval { $x_0, x_1, \ldots., x_n$ }.**

The expression for error is interesting but is not always extremely useful. We can conclude, however, that if the function is "smooth", a low-degree polynomial should work satisfactorily. On the other hand, a "rough" function can be expressed to have larger errors when interpolated.

PROBLEM UNDER LAGRANGE'S INTERPOLATION FORMULA

Find the third degree polynomial f(x) satisfying the following data and compute f(4)

x	1	3	5	7
y	24	120	336	720

Solution:

$$y = f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} y_0$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} y_1$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_n$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} y_n$$

$$y = f(x) = \frac{(x - 3)(x - 5)(x - 7)}{(1 - 3)(1 - 5)(1 - 7)} 24$$

$$+ \frac{(x - 1)(x - 5)(x - 7)}{(3 - 1)(3 - 5)(3 - 7)} 120$$

$$+ \frac{(x - 1)(x - 3)(x - 7)}{(5 - 1)(5 - 3)(5 - 7)} 336$$

$$+ \frac{(x - 1)(x - 3)(x - 5)}{(7 - 1)(7 - 3)(7 - 5)} 720$$

$$y = f(x) = \frac{(x-3)(x-5)(x-7)}{-2} + \frac{15\,(x-1)(x-5)(x-7)}{2}$$

$$- 21\,(x-1)(x-3)(x-7) + 15\,(x-1)(x-3)(x-5)$$

$$y = f(x) = \frac{x^3 - 15x^2 + 71x - 105}{-2} + \frac{x^3 - 13x^2 + 47x - 35}{2}$$

$$- \quad 21\,(x^3 - 11x^2 + 31x - 21) + 15\,(x^3 - 9x^2 + 23x - 15)$$

$$= \left[-\frac{1}{2} + \frac{15}{2} - 21 + 15 \right] x^3 + \left[\frac{15}{2} - \frac{195}{2} + 231 - 135 \right] x^2$$

$$\left[-\frac{71}{2} + \frac{705}{2} - 605 + 345 \right] x + \left[\frac{105}{2} - \frac{525}{2} + 441 - 225 \right]$$

$f(x) = x^3 + 6x^2 + 11x + 6$ is the polynomial of degree three

Next to find the value of $f(4)$

Therefore $f(x) = x^3 + 6x^2 + 11x + 6$

$$f(4) = 4^3 + 6(4^2) + 11(4) + 6$$

$$= 64 + 96 + 44 + 6$$

$$= 210$$

INVERSE INTERPOLATION

The process of finding a value of x for the corresponding value of y is called inverse interpolation

In such a case, we will take y as independent variable and x as dependent variable and use Lagrange's interpolation formula

Taking y as independent variable

$$x = f(y) = \frac{(y - y_1)(y - y_2)(y - y_3)\ldots\ldots\ldots(y - y_n)}{(y_0 - y_1)(y_0 - y_2)(y_0 - y_3)\ldots\ldots\ldots(y_0 - y_n)}\, x_0$$

$$+ \frac{(y - y_0)(y - y_2)(y - y_3)\ldots\ldots\ldots(y - y_n)}{(y_1 - y_0)(y_1 - y_2)(y_1 - y_3)\ldots\ldots\ldots(y_1 - y_n)}\, x_1$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{y - y_0)(y - y_1)(y - y_2)\ldots\ldots\ldots(y - y_{n-1})}{(y_n - y_0)(y_n - y_2)(y_n - y_3)\ldots\ldots\ldots(y_n - y_{n-1})}\, x_n$$

The above formula is called inverse interpolation.

PROBLEM BASED ON INVERSE INTERPOLATION

Find the age corresponding to the annuity value 13.6 given the table

Age (x)	**30**	**35**	**40**	**45**	**50**
Annuity Value (y)	**15.9**	**14.9**	**14.1**	**13.3**	**12.5**

Solution:

$$x = f (y) = \frac{(y - y_1) (y - y_2) (y - y_3) (y - y_4)}{(y_0 - y_1) (y_0 - y_2) (y_0 - y_3) (y_0 - y_4)} x_0$$

$$+ \frac{(y - y_0) (y - y_2) (y - y_3) (y - y_4)}{(y_1 - y_0) (y_1 - y_2) (y_1 - y_3) (y_1 - y_4)} x_1$$

$$+ \frac{(y - y_0) (y - y_1) (y - y_3) (y - y_4)}{(y_2 - y_0) (y_2 - y_1) (y_2 - y_3) (y_2 - y_4)} x_2$$

$$+ \frac{(y - y_0) (y - y_1) (y - y_2) (y - y_4)}{(y_3 - y_0) (y_3 - y_1) (y_3 - y_2) (y_1 - y_4)} x_3$$

$$+ \frac{(y - y_0) (y - y_1) (y - y_2) (y - y_3)}{(y_4 - y_0) (y_4 - y_1) (y_4 - y_2) (y_4 - y_3)} x_4$$

$$x = f (y) = \frac{(y - 14.9) (y - 14.1) (y - 13.3) (y - 12.5)}{(15.9 - 14.9) (15.9 - 14.1) (15.9 - 13.3) (15.9 - 12.5)} 30$$

$$+ \frac{(y - 15.9) (y - 14.1) (y - 13.3) (y - 12.5)}{(14.9 - 15.9) (14.9 - 14.1) (14.9 - 13.3) (14.9 - 12.5)} 35$$

$$+ \frac{(y - 15.9) (y - 14.9) (y - 13.3) (y - 12.5)}{(14.1 - 15.9) (14.1 - 14.9) (14.1 - 13.3) (14.1 - 12.5)} 40$$

$$+ \frac{(y - 15.9) (y - 14.9) (y - 14.1) (y - 12.5)}{(13.3 - 15.9) (13.3 - 14.9) (13.3 - 14.1) (13.3 - 12.5)} 45$$

$$+ \frac{(y - 15.9)(y - 14.9)(y - 14.1)(y - 13.3)}{(12.5 - 15.9)(12.5 - 14.9)(12.5 - 14.1)(12.5 - 13.3)} \; 50$$

$$x = f\,(13.6) = \frac{(13.6 - 14.9)(13.6 - 14.1)(13.6 - 13.3)(13.6 - 12.5)}{(15.9 - 14.9)(15.9 - 14.1)(15.9 - 13.3)(15.9 - 12.5)} \; 30$$

$$+ \frac{(13.6 - 15.9)(13.6 - 14.1)(13.6 - 13.3)(13.6 - 12.5)}{(14.9 - 15.9)(14.9 - 14.1)(14.9 - 13.3)(14.9 - 12.5)} \; 35$$

$$+ \frac{(13.6 - 15.9)(13.6 - 14.9)(13.6 - 13.3)(13.6 - 12.5)}{(14.1 - 15.9)(14.1 - 14.9)(14.1 - 13.3)(14.1 - 12.5)} \; 40$$

$$+ \frac{(13.6 - 15.9)(13.6 - 14.9)(13.6 - 14.1)(13.6 - 12.5)}{(13.3 - 15.9)(13.3 - 14.9)(13.3 - 14.1)(13.3 - 12.5)} \; 45$$

$$+ \frac{(13.6 - 15.9)(13.6 - 14.9)(13.6 - 14.1)(13.6 - 13.3)}{(12.5 - 15.9)(12.5 - 14.9)(12.5 - 14.1)(12.5 - 13.3)} \; 50$$

$$= \frac{6.435}{15.912} - \frac{13.2825}{3.072} + \frac{39.468}{1.8432} + \frac{74.0025}{2.6624} - \frac{22.425}{10.4448}$$

$$= 0.4044 - 4.3237 + 21.4128 + 27.7954 - 2.1470$$

$$= 43.1419$$

DIVIDED DIFFERENCES

There are two disadvantages in using the Lagrangian polynomial or Neville's method for interpolation. First it involves more arithmetic operations than does the divided - difference method. Second, and more importantly, if we desire to add or substract appoint from the set used to construct the polynomial, we essentially have to start over in the computations. The divided – difference method avoids all of this computation.

Let the function $y = f(x)$ take the values $f(x_0), f(x_1), f(x_2), \ldots\ldots, f(x_n)$ corresponding to the values $x_0, x_1, x_2, \ldots\ldots x_n$ of the argument x where $x_1 - x_0, x_2 - x_1$ $x_3 - x_2, \ldots\ldots\ldots\ldots, x_n - x_{n-1}$ need not be necessarily equal.

The first divided difference of f (x) for the argument x_0, x_1 is defined as

$f(x_0, x_1)$ (or) $[x_0, x_1]$

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Similarly

$$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \ldots (1)$$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} \quad \text{and so on.}$$

Thus for defining a 1st divided difference, we need the functional values corresponding to two arguments.

The second divided difference of f(x) for three argument x_0, x_1, x_2 is defined as

$$f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

Similarly $\qquad \ldots\ldots\ldots (2)$

$$f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} \quad \text{and so on.}$$

The third divided difference of f(x) for four argument x_0, x_1, x_2, x_3 is defined as

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$$

Similarly ….. (3)

$$f(x_1, x_2, x_3, x_4) = \frac{f(x_2, x_3, x_4) - f(x_1, x_2, x_3)}{x_4 - x_1}$$ and so on.

Similarly for the forth divided difference of f(x) for five arguments are found

The quantities in (1) , (2) , (3) are called divided differences of order 1 , 2 , 3 respectively

x	f(x)	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference
x_0	$f(x_0)$				
		$f(x_0, x_1)$			
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$		
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$	
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$		$f(x_0, x_1, x_2, x_3, x_4)$
		$f(x_2, x_3)$		$f(x_1, x_2, x_3, x_4)$	
x_3	$f(x_3)$		$f(x_2, x_3, x_4)$		
		$f(x_3, x_4)$			
x_4	$f(x_4)$				

<u>NOTE</u>

In any difference column, the value of the divided difference is got by dividing the difference between the two adjacent values immediately to the left by the difference between the arguments opposite to the entries on the two diagonals through the divided difference.

PROPERTIES OF DIVIDED DIFFERENCE

THEOREM 1 ;

The divided differences are symmetrical in all their arguments, that is, the value of any difference is independent of the order of the argument.

THEOREM 2 :

The divided difference (or any order) of the sum or difference of two functions is equal to the sum or difference of the corresponding separate divided difference.

THEOREM 3 :

The divided difference of the product of a constant and a function is equal to the product of the constant and the divided difference of the function.

THEOREM 4 :

The n^{th} divided differences of a polynomial of the n^{th} degree are constant

PROBLEM BASED ON DIVIDED DIFFERENCE

1. **Form the divided difference table for the following data**

x	1	2	4	7	12
f(x)	22	30	82	106	206

Solution:

X	f(x)	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference
1	**22**				
		8			
2	**30**		**6**		
		26		**-1.6**	
4	**82**		**-3.6**		**0.19**
		8		**0.51**	
7	**106**		**1.5**		
		20			
12	**206**				

NEWTON'S DIVIDED DIFFERENCE FORMULA (OR) NEWTON'S INTERPOLATION FORMULA FOR UNEQUAL INTERVALS

Let y = f(x) takes the values $f(x_0)$, $f(x_1)$, …….. , $f(x_n)$ corresponding to the arguments $x_0, x_1, \ldots\ldots, x_n$

By definition,

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\rightarrow \quad f(x) = f(x_0) + (x - x_0)\, f(x, x_0) \ldots\ldots\ldots (1)$$

Similarly

$$f(x, x_0, x_1) = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\rightarrow \quad f(x, x_0) = f(x_0, x_1) + (x - x_1)\, f(x, x_0, x_1) \ldots\ldots (2)$$

Using the value of $f(x, x_0)$ in (1) , we have

$$f(x) = f(x_0) + (x - x_0)\,[f(x_0, x_1) + (x - x_1)\, f(x, x_0, x_1)]$$

$$= f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x, x_0, x_1) \ldots\ldots\ldots (3)$$

Again

$$f(x, x_0, x_1 x_2) = \frac{f(x, x_0, x_1) - f(x_0, x_1, x_2)}{x - x_2}$$

$$\rightarrow \quad f(x, x_0, x_1) = f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1 x_2) \text{....(4)}$$

Using the values of $f(x, x_0, x_1)$ in (3) we have

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) \\ &\quad + (x - x_0)(x - x_1)[f(x_0, x_1, x_2) + (x - x_2) f(x, x_0, x_1 x_2)] \\ &= f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &\quad + (x - x_0)(x - x_1)(x - x_2) f(x, x_0, x_1 x_2) \end{aligned}$$

Continuing in this manner, we get

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &\quad + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + (x - x_0)(x - x_1)(x - x_2) \ldots\ldots. (x - x_{n-1}) f(x_0, x_1, x_2, \ldots. x_n) \\ &\quad + (x - x_0)(x - x_1)(x - x_2) \ldots\ldots. (x - x_n) f(x, x_0, x_1, x_2, \ldots. x_n) \end{aligned}$$

If f (x) is a polynomial of degree ‘ n ‘ then

$$(x, x_0, x_1, x_2, \ldots. x_n) = 0 \text{ (since (n+1)}^{th} \text{ difference)}$$

Hence the above equation becomes

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ &\quad + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + (x - x_0)(x - x_1)(x - x_2) \ldots\ldots. (x - x_{n-1}) f(x_0, x_1, x_2, \ldots. x_n) \end{aligned}$$

The above equation is called Newton’s divided difference interpolation formula for unequal intervals.

PROBLEM BASED ON NEWTON'S DIVIDED DIFFERENCE FORMULA

Using Newton's divided difference formula, find $f(3)$ given $f(1) = -26$, $f(2) = 12$, $f(4) = 256$, $f(6)\ 844$

Solution:

We form the divided difference table since the intervals are unequal

x	u(x)	1st divided difference	2nd divided difference	3rd divided difference
1	- 26			
		38		
2	12		28	
		122		3
4	256		43	
		294		
6	844			

By Newton's divided difference interpolation formula

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2)\, f(x_0, x_1, x_2, x_3)$$

$$= -26 + (x - 1)(38) + (x - 1)(x - 2)(28) + (x - 1)(x - 2)(x - 4)(3)$$

$$= -26 + 38x - 38 + 28x^2 - 84x + 56 + 3x^3 - 21x^2 + 42x - 24$$

$$= 3x^3 + 7x^2 - 4x - 32$$

Next to find the value of $f(3)$

$$f(x) = 3x^3 + 7x^2 - 4x - 32$$

$$f(3) = 3(3)^3 + 7(3)^2 - 4(3) - 32$$

$$= 81 + 63 - 12 - 32$$

$$= 100$$

NEWTON FORWARD AN BACKWARD DIFFERENCE FORMULA

NEWTON'S FORWARD INTERPOLATION FORMULA FOR EQUAL INTERVALS

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots, y_n$ at the points $x_0, x_1, x_2, \ldots, x_n$ where $x_i = x_0 + ih$. Then Newton's forward interpolation polynomial is given by

$$P_n(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \text{-------} + \frac{u(u-1)(u-2)\ldots\ldots(u-(r-1))}{n!}\Delta^n y_0$$

$$\text{where } u = \frac{x - x_0}{h}$$

PROBLEM BASED ON NEWTON'S FORWARD INTERPOLATION FORMULA

A third degree polynomial passes through the point (0 , - 1) , (1 , 1) , (2 , 1) and (3 , - 2) using Newton's forward interpolation formula, find the polynomial and hence find the value at 1.5

Solution:

We form the difference table

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-1			
		2		
1	1		-2	
		0		-1
2	1		-3	
		-3		
3	-2			

Then Newton's forward interpolation polynomial is given by

$$P_n(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

where $u = \frac{x - x_0}{h}$

Here $x_0 = 0$ **,** $h = 1 - 0 = 1$ **(difference) and therefore** $u = x$

$$P_n(x) = -1 + \frac{x}{1!}(2) + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(-1)$$

$$= -1 + 2x - x(x-1) - \frac{x^2 - 3x + 2}{6}$$

$$= -x^2 + 3x - 1 - \frac{x^2 - 3x + 2}{6}$$

$$= \frac{1}{6}[-6x^2 + 18x - 6 - x^3 + 3x^2 - 2x]$$

$$= \frac{1}{6}[-x^3 - 3x^2 + 16x + 6]$$

$$= \frac{-1}{6}[x^3 + 3x^2 - 16x - 6]$$

Next to find y (1.5)

$$y(x) = \frac{-1}{6}[x^3 + 3x^2 - 16x - 6]$$

$$y(1.5) = \frac{-1}{6}[(1.5)^3 + 3(1.5)^2 - 16(1.5) - 6]$$

$$= 1.3125$$

NEWTON’S BACKWARD INTERPOLATION FORMULA FOR EQUAL INTERVALS

Let the function $y = f(x)$ take the values $y_0, y_1, y_2, \ldots, y_n$ at the points $x_0, x_1, x_2, \ldots, x_n$ where $x_i = x_0 + ih$. Then Newton’s backward interpolation polynomial is given by

$$P_n(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

$$+ \text{-------} + \frac{v(v+1)(v+2)\ldots\ldots(v+(r-1))}{n!}\nabla^n y_n$$

$$\text{where} \quad v = \frac{x - x_n}{h}$$

PROBLEM BASED ON NEWTON’S BACKWARD INTERPOLATION FORMULA

Using Newton’s backward interpolation formula, find the cubic polynomial and also compute f (4)

Solution :

We form the difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	1			
		1		
1	2		-2	
		- 1		12
2	1		10	
		9		
3	10			

Then Newton's backward interpolation polynomial is given by

$$P_n(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

where $$v = \frac{x - x_3}{h}$$

here $x_3 = 3$, $h = 1 - 0 = 1$ **(difference) and hence** $v = x - 3$

$$y(x) = 10 + \frac{(x-3)}{1!}(9) + \frac{(x-3)(x-2)}{2!}(10) + \frac{(x-3)(x-2)(x-1)}{3!}(12)$$

$$= 10 + 9(x-3) + 5(x-3)(x-2) + 2(x-3)(x-2)x-1)$$

$$= 10 + 9x - 27 + 5[x^2 - 5x + 6] + 2[x^3 - 6x^2 + 11x - 6]$$

$$= 10 + 9x - 27 + 5x^2 - 25x + 30 + 2x^3 - 12x^2 + 22x - 12$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Next to find y (4)

$$y(x) = 2x^3 - 7x^2 + 6x + 1$$

$$y(4) = 2(4)^3 - 7(4)^2 + 6(4) + 1$$

$$= 128 - 112 + 24 + 1$$

$$= 41$$

SHORT ANSWERS

1. **State Lagrange's interpolation formula**

   Lagrange's interpolation formula is

$$
\begin{aligned}
y = f(x) = {} & \frac{(x - x_1)(x - x_2)(x - x_3) \ldots\ldots\ldots (x - x_n)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \ldots\ldots\ldots (x_0 - x_n)} \; y_0 \\
& + \frac{(x - x_0)(x - x_2)(x - x_3) \ldots\ldots\ldots (x - x_n)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \ldots\ldots\ldots (x_1 - x_n)} \; y_1 \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& + \frac{(x - x_0)(x - x_1)(x - x_2) \ldots\ldots\ldots (x - x_{n-1})}{(x_n - x_0)(x_n - x_2)(x_n - x_3) \ldots\ldots\ldots (x_n - x_{n-1})} \; y_n
\end{aligned}
$$

2. **What is the assumption we make when Lagrange's formula is used?**

   Lagrange's interpolation formula can be used whether the values of x, the independent variable are equally spaced or not whether the difference of y become smaller or not

3. **What are the advantages of Lagrange's formula over Newton?**

The forward and backward interpolation formulae of Newton can be used only when the values of the independent variable x are equally spaced can also be used when the differences of the dependent variable y become smaller ultimately. But Lagrange's interpolation formula can be used whether the values of x, the independent variable are equally spaced or not and whether the difference of y become smaller or not.

4. **What is the disadvantage in practice in applying Lagrange's interpolation formula?**

Though Lagrange's formula is simple and easy to remember, its application is not speedy. It requires close attention to sign and there is always a chance of committing some error due to the number of positive and negative sign in the numerator and the denominator.

5. **What is inverse interpolation?**

Suppose we are given a table of values of x and y. direct interpolation is the process of finding the values of y corresponding to a value of x, not present in the table. Inverse interpolation is the process of finding the values of x corresponding to a value of y, not present in the table.

6. **Give the inverse of Lagrange's interpolation formula**

Inverse of Lagrange's interpolation formula is

$$x = f(y) = \frac{(y-y_1)(y-y_2)(y-y_3)\,\ldots\ldots\ldots\,(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\,\ldots\ldots\ldots\,(y_0-y_n)}\, x_0$$

$$+ \frac{(y-y_0)(y-y_2)(y-y_3)\,\ldots\ldots\ldots\,(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\,\ldots\ldots\ldots\,(y_1-y_n)}\, x_1$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_2)\,\ldots\ldots\ldots\,(y-y_{n-1})}{(y_n-y_0)(y_n-y_2)(y_n-y_3)\,\ldots\ldots\ldots\,(y_n-y_{n-1})}\, x_n$$

7. Define “ Divided difference’

Let the function $y = f(x)$ take the values $f(x_0), f(x_1), f(x_2), \ldots\ldots, f(x_n)$ corresponding to the values $x_0, x_1, x_2, \ldots\ldots x_n$ of the argument x where $x_1 - x_0, x_2 - x_1$ $x_3 - x_2, \ldots\ldots\ldots\ldots, x_n - x_{n-1}$ need not be necessarily equal.

The first divided difference of f (x) for the argument x_0, x_1 is defined as

$$f(x_0, x_1) \quad (\text{or}) \quad [x_0, x_1]$$

$$\left.\begin{aligned} f(x_0, x_1) &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\ &\text{Similarly} \\ f(x_1, x_2) &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ f(x_2, x_3) &= \frac{f(x_3) - f(x_2)}{x_3 - x_2} \quad \text{and so on.} \end{aligned}\right\} \quad \ldots (1)$$

Thus for defining a 1st divided difference, we need the functional values corresponding to two arguments.

The second divided difference of f(x) for three argument x_0, x_1, x_2 is defined as

$$\left.\begin{aligned} f(x_0, x_1, x_2) &= \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} \\ &\text{Similarly} \\ f(x_1, x_2, x_3) &= \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1} \quad \text{and so on.} \end{aligned}\right\} \quad \ldots\ldots\ldots(2)$$

The third divided difference of f(x) for four argument x_0, x_1, x_2, x_3 is defined as

$$\left.\begin{aligned} f(x_0, x_1, x_2, x_3) &= \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \\ \text{Similarly} & \\ f(x_1, x_2, x_3, x_4) &= \frac{f(x_2, x_3, x_4) - f(x_1, x_2, x_3)}{x_4 - x_1} \quad \text{and so on.} \end{aligned}\right\} \ldots\ldots (3)$$

Similarly for the forth divided difference of f(x) for five arguments are found

The quantities in (1) , (2) , (3) are called divided differences of order 1 , 2 , 3 respectively

8. **Give the Newton's divided difference interpolation formula**

$$\begin{aligned} f(x) = {} & f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ & + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + (x - x_0)(x - x_1)(x - x_2) \ldots\ldots\ldots (x - x_{n-1}) \; f(x_0, x_1, x_2, \ldots\ldots x_n) \end{aligned}$$

9. **State any two properties of divided difference**

    1. **The divided difference are symmetrical in all their arguments. That is the value of any difference is independent of the order of the arguments**
    2. **The divided differences of the sum of differences of two functions is equal to the sum or difference of the corresponding separate divided difference.**

10. Derive Newton's forward difference formula by using operator method.

Newton's forward difference formula is

$$P_n(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \text{-------} + \frac{u(u-1)(u-2)\ldots\ldots.(u-(r-1))}{n!}\Delta^n y_0$$

where $u = \frac{x - x_0}{h}$

11. Derive Newton's Backward difference formula by using operator method

Newton's backward difference formula is

$$P_n(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n$$

$$+ \text{-------} + \frac{v(v+1)(v+2)\ldots\ldots.(v+(r-1))}{n!}\nabla^n y_n$$

where $v = \frac{x - x_n}{h}$

12. When Newton's backward interpolation formula is used

The formula is used mainly to interpolate the values of y near the end of a set of tabular values and also for extrapolating the values of y a short distance ahead (to the right) of y_0

EXTRA SUMS AND ASSIGNMENTS

1. **Using Lagrange's interpolation formula, solve the problem below**

    1. **Find f (0) given**

x	**- 1**	**- 2**	**2**	**4**
y	**- 1**	**- 9**	**11**	**69**

    2. **Find f (x) given**

x	**0**	**1**	**4**	**5**
y	**4**	**3**	**24**	**39**

    3. **Find f (2) given**

x	**0**	**1**	**3**	**4**
y	**5**	**6**	**50**	**105**

2. **Using inverse interpolation formula solve the problem below**

    1. **Find the value of x when f (x) = 15 from the given data**

x	**5**	**6**	**9**	**11**
y	**12**	**13**	**14**	**16**

    2. **Find the value of x when f (x) = 100 from the given data**

x	**3**	**5**	**7**	**9**	**11**
y	**6**	**24**	**58**	**108**	**174**

    3. **Find the value of x when f (x) = 7 from the given data**

x	**1**	**3**	**4**
y	**4**	**12**	**19**

3. **Find the divided difference for the following problem below**

    1.

x	**2**	**5**	**10**
y	**5**	**29**	**109**

2.

x	5	15	22
y	7	36	160

3.

x	0	1	2	4	5
y	1	14	15	5	6

4. Using Newton's divided differences formula, solve the following problem

1. Find f (2)

x	4	5	7	10	11	13
y	48	100	294	900	1210	2028

2. Find f (x)

x	0	1	2	4	5	7
y	0	0	- 12	0	600	7308

3. Find f (5)

x	0	1	3	6
y	1	4	88	1309

5. Using Newton's forward interpolation formula, find the polynomial f(x) satisfying the following data.

1. Find y at x = 5

x	4	6	8	10
y	1	3	8	10

2. Find f (2)

x	0	5	10	15
y	14	379	1444	3584

3. Find f (4)

x	0	1	2	3
y	1	2	1	10

6. Using Newton’s backward interpolation formula, find the polynomial f(x) satisfying the following data.

1. Find y at x = 84

x	40	50	60	70	80	90
y	184	204	226	250	276	304

2. Find f (42)

x	20	25	30	35	40	45
y	354	332	291	260	231	204

3. Find f (9)

x	2	5	8	11
y	94.8	87.9	81.3	75.1

NUMERICAL DIFFERENTIATION AND INTEGRATION

1. **Newton's forward difference formula to find f ‘ (x), f ‘’ (x) , f ‘’’(x)**

$$f'(x) = \frac{1}{h}\Bigg[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0$$

$$+ \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\Bigg]$$

$$f''(x) = \frac{1}{h^2}\Bigg[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \ldots\ldots\Bigg]$$

$$f'''(x) = \frac{1}{h^3}\Bigg[\Delta^3 y_0 + \; + \frac{12u^2-18}{12}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\ldots\Bigg]$$

$$\text{where } u = \frac{x-x_0}{h}$$

2. **Newton's forward difference formula to find f ‘ (x), f ‘’ (x) , f ‘’’(x) when u = 0**

$$f'(x) = \frac{1}{h}\Bigg[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} \ldots\ldots\ldots\Bigg]$$

$$f''(x) = \frac{1}{h^2}\Bigg[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\Bigg]$$

$$f'''(x) = \frac{1}{h^3}\Bigg[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\Bigg]$$

3. **Newton’s backward difference formula to find f ‘ (x), f ‘’ (x) , f ‘’’(x)**

$$f'(x) = \frac{1}{h}\left[\nabla y_n + \frac{(2v+1)}{2!}\nabla^2 y_n + \frac{3v^2+6v+2}{3!}\nabla^3 y_n\right.$$

$$\left. + \frac{4v^3+18v^2+22v+6}{4!}\nabla^4 y_n + \ldots\ldots\ldots\ldots\right]$$

$$f''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2+18v+11}{12}\nabla^4 y_n + \ldots\ldots\right]$$

$$f'''(x) = \frac{1}{h^3}\left[\nabla^3 y_n + + \frac{12v^2+18}{12}\nabla^4 y_n + \ldots\ldots\ldots\ldots\ldots\right]$$

where $v = \dfrac{x - x_n}{h}$

4. **Newton’s forward difference formula to find f ‘ (x), f ‘’ (x) , f ‘’’(x) when v = 0**

$$f'(x) = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} \ldots\ldots\ldots\right]$$

$$f''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^3 y_n + \ldots\ldots\ldots\ldots\right]$$

$$f'''(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

5.

Rule	Degree of y (x)	No. of intervals	Errors	Order
Trapezoidal rule	One	any	$\| E \| < \dfrac{(b – a) h^2}{12} M$	h^2
Simpson’s 1/3 rule	Two	even	$\| E \| < \dfrac{(b – a) h^4}{180} M$	h^4

6. The trapezoidal rule is so called, because it approximates the integral by the sum of n trapezoidals

7. Trapezoidal rule

$$\int_a^b y\,dx = \frac{h}{2}\left[\text{(Sum of the first and last ordinate)} + 2\text{ (Sum of the remaining ordinates)} \right]$$

8. Simpons’s 1/3 rule

$$\int_a^b y\,dx = \frac{h}{3}\left[\text{(Sum of the first and last ordinate)} + 2\text{ (Sum of the odd ordinates)} + 4\text{ (Sum of the odd ordinates)} \right]$$

9. Simpson’s rule will give exact result, if the entire curve y = f(x) is itself a parabola.

10. Error in the trapezoidal formula is of the order h^2.

INTRODUCTION:

In this chapter we will discuss how to estimate the derivative or get the integral of a function by numerical methods. Numerical procedure differ from the analytical methods of calculus in that it can be applied to functions known only as a table of values as well as to a function stated explicitly. Although computers can be programmed to do the symbolic manipulation of formal integration techniques. Computers more often use the numerical methods to perform differentiation and integration.

To get the derivative, we first find the curve y = f(x) through the points and then differentiate and get its value at the required point. If the values of x are equally spaced, we get the interpolating polynomial due to Newton-Gregory. If the derivative is required at a point nearer to the starting value in the table, we use Newton's forward interpolation formula. If the derivative is at the end of the table, we use Newton's backward interpolation formula. If the value of derivative is required near the middle of the table value we use one of the central difference interpolation formula. In the case of unequal intervals, we can use Newton's divided difference formula or Lagrange's interpolation formula to get the derivative value.

DERIVATIVES FROM DIFFERENCE TABLES – DIVIDED DIFFERENCES AND FINITE DIFFERENCES

NEWTON'S FORWARD DIFFERENCE FORMULA

Our aim is to find the derivative of y = f (x) passing through the (n+ 1) points, at a point nearer to the starting value $x = x_0$.

Newton's forward difference interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \text{-------} + \frac{u(u-1)(u-2)\ldots\ldots(u-(r-1))}{n!} \Delta^n y_0 \ldots\ldots\ldots (1)$$

where $u = \dfrac{x - x_0}{h}$ and $y(x)$ is a polynomial of degree n in x

differentiating (1) w.r.to x, we get

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

$$= \frac{du}{dx} \frac{dy}{du}$$

$$= \frac{1}{h} \frac{dy}{du}$$

$$f'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \frac{4u^3 - 18u^2 + 22u - 6}{4!}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\right] \ldots\ldots\ldots (2)$$

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2 - 18u + 11}{12}\Delta^4 y_0 + \ldots\ldots\right] \ldots (3)$$

$$f'''(x) = \frac{1}{h^3}\left[\Delta^3 y_0 + \ + \frac{12u^2 - 18}{12}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\ldots\right] \ldots\ldots\ldots\ldots (4)$$

Newton’s forward difference formula to find f ‘ (x), f ‘’ (x) , f ‘’’(x) when u = 0

$$f'(x) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} \ldots\ldots\right] \ldots\ldots (5)$$

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^3 y_0 + \dots\dots\dots\right] \quad \dots\dots (6)$$

$$f'''(x) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \dots\dots\dots\dots\right] \quad \dots\dots (7)$$

Equation (5) , (6) and (7) give the values of 1st, 2nd, 3rd derivatives at the starting value $x = x_0$.

PROBLEM BASED ON NEWTON'S FORWARD DIFFERENCE FORMULA

Find the 1st, 2nd, 3rd derivative of f (x) at x = 1.5 if

x	1.5	2	2.5	3	3.5	4
y	3.3375	7	13.625	24	38.875	59

Solution:

We have to find the derivative at the point x = 1.5 which is the starting point of the given data. Therefore, we use Newton's forward interpolation formula.

Forward difference table:

x	f(x)	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference	5th divided difference
1.5	3.375					
		3.625				
2	7		3			
		6.625		.75		
2.5	13.625		3.75		0	
		10.375		.75		0
3	24		4.5		0	
		14.875		.75		
3.5	38.875		5.25			
		20.125				
4	59					

Here x = 1.5, x_0 = 1.5, h = 0.5 and therefore u = 0

Hence, Newton's forward difference formula to find f ' (x), f '' (x) , f '''(x) when u = 0 is

$$f'(x) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} \ldots\ldots\ldots\right]$$

$$f'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{3}{2} + \frac{0.75}{3}\right] = 4.75$$

$$f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]$$

$$f''(1.5) = \frac{1}{(0.5)^2}[3 - 0.75] = 9$$

$$f'''(x) = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\ldots\right]$$

$$f'''(1.5) = \frac{1}{(0.5)^3}[0.75] = 6$$

NEWTON’S BACKWARD DIFFERENCE FORMULA

Newton’s backward difference formula is

$$y (x) = y_n + \frac{v}{1 !} \nabla y_n + \frac{v (v + 1)}{2 !} \nabla^{2} y_n + \frac{v (v + 1) (v + 2)}{3 !} \nabla^{3} y_n$$

$$+ \text{-------} + \frac{v (v + 1) (v + 2) \ldots\ldots. (v + (r - 1))}{n !} \nabla^{n} y_n \quad \ldots\ldots(1)$$

$$\text{where} \quad v = \frac{x - x_n}{h}$$

differentiating (1) w.r.to x, we get

$$\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{dx}$$

$$= \frac{dy}{dv} \frac{1}{h}$$

$$= \frac{1}{h} \frac{dy}{dv}$$

$$f\,‘ (x) = \frac{1}{h} \left\{ \nabla y_n + \frac{(2v + 1)}{2 !} \nabla^{2} y_n + \frac{3v^2 + 6 v + 2}{3 !} \nabla^{3} y_n \right.$$

$$\left. + \frac{4 v^3 + 18 v^2 + 22 v + 6}{4 !} \nabla^{4} y_n + \ldots\ldots\ldots\ldots.. \right\} \ldots\ldots\ldots..\ldots(2)$$

$$f''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + (v+1)\nabla^3 y_n + \frac{6v^2 + 18v + 11}{12}\nabla^4 y_n + \ldots\ldots\right] \ldots\ldots(3)$$

$$f'''(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \quad + \frac{12v^2 + 18}{12}\nabla^4 y_n + \ldots\ldots\ldots\ldots\ldots\right] \ldots\ldots\ldots\ldots(4)$$

where $v = \dfrac{x - x_n}{h}$

Newton's forward difference formula to find f ' (x), f '' (x) , f '''(x) when v = 0 is

$$f'(x) = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} \ldots\ldots\ldots\right] \ldots\ldots(5)$$

$$f''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^3 y_n + \ldots\ldots\ldots\ldots\right] \ldots\ldots\ldots\ldots(6)$$

$$f'''(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\ldots\ldots\right] \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Equation (5) , (6) and (7) give the values of 1st, 2nd, 3rd derivatives at the starting value $x = x_n$.

PROBLEM BASED ON NEWTON’S BACKWARD DIFFERENCE FORMULA

Find the 1st, 2nd, 3rd derivative of f (x) at x = 4 if

x	1.5	2	2.5	3	3.5	4
y	3.3375	7	13.625	24	38.875	59

Solution:

We have to find the derivative at the point x = 1.5 which is the starting point of the given data. Therefore, we use Newton’s forward interpolation formula.

Backward difference table:

x	f(x)	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference	5th divided difference
1.5	3.375					
		3.625				
2	7		3			
		6.625		.75		
2.5	13.625		3.75		0	
		10.375		.75		0
3	24		4.5		0	
		14.875		.75		
3.5	38.875		5.25			
		20.125				
4	59					

Here x = 4, x_n = 4, h = 0.5 and therefore v = 0

Hence, Newton’s backward difference formula to find f ‘ (x), f ‘’ (x) , f ‘’’(x) when v = 0 is

$$f'(x) = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} \ldots\ldots\right]$$

$$f'(1.5) = \frac{1}{0.5}\left[20.125 + \frac{5.25}{2} + \frac{0.75}{3}\right] = 46$$

$$f''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^3 y_n + \ldots\ldots\ldots\ldots\right]$$

$$f''(1.5) = \frac{1}{(0.5)^2}[5.25 + 0.75] = 24$$

$$f'''(x) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\ldots\ldots\right]$$

$$f'''(1.5) = \frac{1}{(0.5)^3}[0.75] = 6$$

MAXIMA AND MINIMA OF A TABULATED FUNCTION

Newton's forward difference interpolation formula is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \text{-------} + \frac{u(u-1)(u-2)\ldots\ldots\ldots(u-(r-1))}{n!}\Delta^n y_0 \ldots\ldots\ldots (1)$$

differentiating (1) w.r.to x, we get

$$f'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \ldots\right] \ldots(2)$$

For maxima or minima y ' (u) = 0. hence equating the right hand side of (2) to zero and retaining only up to third differences, we obtain

$$\Delta y_0 + \frac{(2u - 1)}{2!} \Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!} \Delta^3 y_0 = 0$$

Substituting the values of $\Delta y_0 , \Delta^2 y_0 , \Delta^3 y_0$ from the difference table we solve this quadratic for u. then the corresponding values of x are given by $x = x_0 + uh$ at which y is maximum or minimum

NOTE:

If the interval of differencing is not constant (i.e., x's are not equally spaced), we get Newton's divided difference formula (or) Lagrange's interpolation formula for general x and then differencing it with respect to x, we can get the derivatives at any x in the range.

PROBLEM BASED ON MAXIMA AND MINIMA

Find the maximum and minimum value of y from the following table

x	0	1	2	3	4	5
y	0	.25	0	2.25	16	56.25

Solution:

Newton's forward difference interpolation formula is

$$y(x) = y_0 + \frac{u}{1!} \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0$$

$$+ \text{-------} + \frac{u(u-1)(u-2) \ldots\ldots. (u-(r-1))}{n!} \Delta^n y_0 \ldots\ldots.. (1)$$

differentiating (1) w.r.to x, we get

$$f'(x) = \frac{1}{h}\left\{\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \ldots\right. \quad \ldots(2)$$

Forward difference table:

x	f(x)	1st divided difference	2nd divided difference	3rd divided difference	4th divided difference	5th divided difference
0	0					
		0.25				
1	0.25		- 0.5			
		- 0.25		3		
2	0		2.50		6	
		2.25		9		0
3	2.25		11		6	
		13.75		15		
4	16		26.50			
		40.25				
5	56.25					

Here $x_0 = 0$, h = 1 then u = x

$$f'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \ldots\right]$$

$$= \left[0.25 + \frac{(2u-1)}{2!}(-0.50) + \frac{3u^2-6u+2}{3!}3 + \ldots\right]$$

$$= 4u^3 - 12u^2 + 8u$$

now f' (x) = 0 then $\rightarrow \; 4u^3 - 12u^2 + 8u = 0$

$\rightarrow \; 4u(u-2)(u-1) = 0$

$\rightarrow \; u = 0, 1, 2$

Also $y'(x) = 12u^2 - 24u + 8$

At $u = 0$ then $y'(x) = 8$ which is positive

At $u = 1$ then $y'(x) = -4$ which is negative

At $u = 2$ then $y'(x) = 8$ which is positive

Therefore y is maximum at u = 1 and minimum at u = 0 and 2

Therefore the maximum value y at u = 1 (i.e.,) x = 1 is

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \text{-------} + \frac{u(u-1)(u-2)\ldots\ldots.(u-(r-1))}{n!}\Delta^n y_0$$

$$y(1) = 0 + 0.25 = 0.25$$

Minimum at x = 0 and x = 2 is

$$y(0) = 0$$

$$y(2) = 0 + 2(0.25) + (-0.5) = 0$$

Hence the maximum value of y at x = 1 is 0.25 and the minimum value of y at x = 0 and x = 2 are 0 , 0 respectively

NUMERICAL INTEGRATION BY TRAPEZOIDAL AND SIMPSON'S 1/3

The process of evaluating a definite integral from a set of tabulated values of the integrand f(x) is called numerical integration. This process when applied to a function of single variable, is known as quadratic

TRAPEZOIDAL RULE

Take the curve thought (x_0 , y_0) and (x_1 , y_1) as a straight line (i.e.,) a polynomial of first order so that differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = \frac{h}{2}(y_0 + y_1)$$

Similarly

$$\int_{x_0+h}^{x_0+2h} f(x)dx = h\left[y_1 + \frac{1}{2}\Delta y_1\right] = \frac{h}{2}(y_1 + y_2)$$

……………………………………………………………………

……………………………………………………………………

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = h\left[y_{n-1} + \frac{1}{2}\Delta y_{n-1}\right] = \frac{h}{2}(y_{n-1} + y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots\ldots + y_{n-1})]$$

(i.e.,)

$$\int_a^b y\,dx = \frac{h}{2}\left[(\text{Sum of the first and last ordinate}) + 2(\text{Sum of the remaining ordinates})\right]$$

PROBLEM BASED ON TRAPEZOIDAL RULE

Evaluate

$$\int_{0}^{6} \frac{dx}{1+x^2}$$ **by Trapezoidal rule and verify by actual integration.**

Solution

Here $b - a = 6 - 0 = 6$. divide into 6 equal parts, hence $h = 1$

Hence the table is

x	0	1	2	3	4	5	6
y	1	0.5	0.5	0.2	0.058824	0.038462	0.27027

Hence the Trapezoidal rule is

$$\int_{a}^{b} y\,dx = \frac{h}{2}\,(\text{ Sum of the first and last ordinate })$$

$$+ 2\,(\text{ Sum of the remaining ordinates })$$

$$\int_{0}^{6} y\,dx = 0.5\{(1+ 0.27027) + 2(0.5+ 0.2+ 0.1 + 0.058824 + 0.038462)\}$$

$$= 1.41079950$$

Verification by Actual integration

$$\int_{0}^{6} \frac{dx}{1+x^2} = (\tan^{-1} x)^{6} = \tan^{-1} 6 - \tan^{-1} 0 = 1.40564765$$

SIMPSON'S ONE THIRD RULE

Take the curve thought (x_0 , y_0), (x_1 , y_1) and (x_2 , y_2) as a parabola (i.e.,) a polynomial of second order so that differences of order higher than second become zero, we get

$$\int_{x_0}^{x_0 + 2h} f(x)dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6} \Delta^2 y_0 \right] = \frac{h}{3}(y_0 + 4 y_1 + y_2)$$

Similarly

$$\int_{x_0 + 4h}^{x_0 + 2h} f(x)dx = 2h\left[y_2 + \Delta y_2 + \frac{1}{6} \Delta^2 y_2 \right] = \frac{h}{3}(y_2 + 4 y_3 + y_4)$$

……………………………………………………………………

……………………………………………………………………

$$\int_{x_0 +(n - 2)h}^{x_0 + nh} f(x)dx = 2h\left[y_{n-2} + \Delta y_{n-2} + \frac{1}{6} \Delta^2 y_{n-2} \right] = \frac{h}{3}(y_{n-2} + 4 y_{n-1} + y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0 + nh} f(x)dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \ldots\ldots + y_{n-1})$$

$$+ 2 (y_2 + y_4 + \ldots\ldots + y_{n-2})]$$

(i.e.,)

$$\int_{a}^{b} y\, dx = \frac{h}{3}\Big[\text{(Sum of the first and last ordinate)}$$

$$+ 4 \text{ (Sum of the odd ordinates) } + 2 \text{ (Sum of the odd ordinates)} \Big]$$

PROBLEM BASED ON SIMPSON'S 1/3RULE

Evaluate

$$\int_0^6 \frac{dx}{1+x^2}$$ **by Simpson's 1/3 rule and verify by actual integration.**

Solution

Here b – a = 6 – 0 = 6. divide into 6 equal parts, hence h = 1

Hence the table is

x	0	1	2	3	4	5	6
y	1	0.5	0.5	0.2	0.058824	0.038462	0.27027

Hence the Simpson's 1/3 rule is

$$\int_a^b y\,dx = \frac{h}{3}\left[(\text{ Sum of the first and last ordinate })\right.$$

$$\left.+ 4 (\text{ Sum of the odd ordinates }) + 2 (\text{ Sum of the odd ordinates})\right]$$

$$\int_0^6 y\,dx = 0.3333 \{ (1 + 0.27027) + 2 (0.2 + 0.58824)$$

$$+ 4 (0.5 + 0.1 + 24 + 0.038462) \}$$

$$= 1.36617433$$

Verification by Actual integration

$$\int_0^6 \frac{dx}{1+x^2} = (\tan^{-1} x)^6 = \tan^{-1} 6 - \tan^{-1} 0 = 1.40564765$$

Unit – 5

Numerical Solution of ordinary Differential Equations

INTRODUCTION

The subject of ordinary differential equations is an essential tool for modeling many physical situations: Spring – mass systems, resistor-Capacitor – inductance circuits, bending of beams , chemical reactions, pendulums and so on. These equations have also demonstrated their usefulness in ecology and economics.

An equation which involves the differential coefficients of one variable, is called the **ordinary differential equation**. More explicity a physical situation that concerns with the rate of change of one quantity with respect to another, gives rise to a differential equation.

The solution to a differential equation is the function that satisfied the differential equation and that also satisfies certain initial conditions on the function. In solving a differential equation analytically, we usually find a general solution containing arbitrary constants and then evaluate the arbitrary constants so that the expression agrees with the initial conditions.

A general equation of first order and first degree is

$$dy/dx = f(x,y) \qquad \ldots\ldots(1)$$

Many analytical techniques exist for finding the solution of such equations. Besides all these techniques, sometimes it happens that a problem cannot be solved at all or lead to solutions which are so difficult to obtain. In such cases the numerical technique is useful. In numerical methods we do not proceed in the hope of finding a relations between x and y, but we find the numerical values of the dependent variable for certain values of independent variable.

An analytical method for solving (1) leads to a relation

$$y = F(x) + C \quad \ldots\ldots.(2)$$

C being some independent constant, is called the general solution of (1). For a particular value of the constant C it represents a curve. Thus if with the differential equation we are also given a point say (x_0, y_0) , then by putting

(x_0, y_0) in (2) , we can find the value of that constant. This extra condition of the point is called a boundary or initial condition. Thus numerical solution of differential equation (1) . Such a solution is called the **numerical solution of the differential equation**.

This problem can be solved by any of the methods developed in this chapter, which will give the solution in one of the two forms given below.

(i) Single step methods or pointwise methods :

1. **Taylor's series method**
2. **Euler's method**
3. **Runge – Kutta method**

(ii) Multi step methods or step by step methods:

1. **Milne's predictor and corrector method**

TAYLOR'S SERIES METHOD:

The method gives a straightforward adaptation of classic calculus to develop the solution as an infinite series. It is a powerful single step method if we are able to find the successive derivatives easily. If f(x,y) involves some complicated algebraic structures then the calculation of higher derivatives become tedious and the method fails. This is the major drawback of this method. However the method will be very useful for finding the starting values for powerful methods like **Runge_Kutta method,Milne's method** etc.,

Consider the first order differential equation

$\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ ……..(1)

Let $x_1 = x_0 + h$ and let $y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + ...$

Once y_1 is known we can compute $y_1', y_1'',....$ from (1) , (2) etc.

Then y can be expanded in a **Taylor's series about** $x = x_1$ and we have

$$y(x_1 + h) = y(x_2) = y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + ... \quad ...(2)$$

continuing in this way we find the solution **y(x).**

Problem:

Using Taylor series method find y at x = 0.1 if $\frac{dy}{dx} = x^2 y - 1$**, y(0) = 1.**

Solution :

given y' = x^2 y – 1 and x$_0$ = 0, $y_0 = 1,$ **h=0.1**

Taylor series formula for y$_1$ is

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots\dots\dots \qquad \dots(1)$$

y' = x^2 y – 1 $\qquad y_0' = x^2{}_0 y_0 - 1 = 0 - 1 = -1$

$y'' = 2xy + x^2 y' \qquad y_0'' = 2x_0 y_0 + x^2{}_0 y_0' = 0 + 0 = 0$

$y''' = 2y + 4xy' + x^2 y'' \qquad y_0''' = 2(1) + 4(0)(-1) + (0)^2(0) = 2$

Therefore, (1) ⇒ $y_1 = 1 + \frac{0.1}{1!}(-1) + \frac{0.1^2}{2!}(0) + \frac{0.1^3}{3!}(2) + \dots\dots$

i.e., y (0.1) =1-0.1+0.00033-0.000025

=0.900305

SOLVING HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS USING TAYLOR'S SERIES METHOD:

Taylor's series method can be extended to higher order differential equations. The higher order differential equations can be expressed as a system of first order differential equations and the technique discussed for system of equations can be applied to solve the system.

PROBLEM:

By Taylor's series method find y(0.1) .given that y" = y + x y' and y(0)=1,
y'(0) = 0

Solution:

Given that

y” = y + x y’ and

y(0)=1,

y’(0) = 0

i.e, $y(x_0) = y_{0,} y'(x_0) = y'_0$

$y''_0 = y_0 + x_0 y_0' = 1$

differentiate with respect to x,

$y_0''' = y_0' + \{x_0 y_0'' + y_0'\}$ **= 0+{0 (1) + 0} = 0**

i.e., $y_0''' = 0$

By Taylor’s series , $y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + ...$

i.e, $y(0.1) = 1 + \frac{0.1}{1!}(0) + \frac{0.1^2}{2!} 1 + \frac{0.1^3}{3!}(0)$

= 1+(0.01)/2= 1.005

i.e., y(0.1) = 1.005

EULER’S AND MODIFIED EULER’S METHOD:

The Taylor series method may be awkward to apply if the derivatives become complicated and in this case the error is difficult to determine.

The error in a Taylor series will be small if the step size h is small. In fact, if we make h small enough , we may only need a few terms of the Taylor series expansion for good accuracy. The Euler method follows this idea to the exreme for first order differential equations it uses only the first two terms of the taylor series. It is one of the oldest methods suppose we are to find successively y_1 ,y_2 ,…,y_m ,…. where y_m is the value of y corresponding to x = x_m where x_m =x_0 + mh, m=1,2,… and h being small.

In this method, we use the property that in a small interval a curve is nearly a straight line. Thus at the point (x_0 , y_0) , we approximate the curve by the tangent at the point (x_0 , y_0).

The equation of the tangent at (x_0, y_0) is

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$= (x - x_0)\, f(x_0, y_0)$$

$$y = y_0 + (x - x_0)\, f(x_0, y_0)$$

Hence the value of y corresponding to $x=x_1$ is given by $y_1 = y_0 + (x_1 - x_0)\, f(x_0, y_0)$

$$y_1 = y_0 + h\, f(x_0, y_0) \qquad \text{........ (1)}$$

which gives the approximate value of y_1. Similarly, approximating the curve in the next interval (x_1, y_1)

by a line through (x_1, y_1) with slope $f(x_1, y_1)$, we get

$$y_2 = y_1 + h\, f(x_1, y_1)$$

In general it can be shown that $y_{n+1} = y_n + hf(x_n, y_n)$,n=0,1,2,3,…..

This **formula is called the Euler's algorithm.**

Thus in euler's method the actual curve of solution is approximated by a sequence of line segments. It can happen that the sequence of lines deviates from the curve of solution significantly.

EULER'S METHOD:

The euler's formula is given by

$y_{n+1} = y_n + hf(x_n, y_n)$ since n=0,1,2,…. and $h = x_1 - x_0$

Problem:

Using Euler's method, find y(0.2) and y(0.4) from the $\frac{dy}{dx} = x + y$, y(0) = 1.

Soln:

given $\frac{dy}{dx} = x + y$, y(0) = 1. i.e. $y(x_0) = y_0$

i.e, $x_0 = 0, y_0 = 1$

i.e., f(x,y) given

put n = 0, in $y_{n+1} = y_n + hf(x_n, y_n)$

$y_1 = y_0 + hf(x_{0,} y_0)$

Here $x_0 = 0$ Therefore h = 0.2 - 0 = 0.2

$y_1 = 1 + 0.2 f(0,1)$

$1 + 0.2(0 + 1) = 1.2$

i.e, **y(0.2) = 1.2**

To find y(0.4):

$y_2 = y_0 + 0.2 f(0.2,1.2) = 1 + 0.2(2.4) = 1.48$

i.e., $y_2 = 1.48$

i.e. y(0.4) = 1.48

MODIFIED EULER'S METHOD:

The modified euler's formula is

$$y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n))$$

where $h = x_1 - x_2$ and n = 0,1,2,3,……

Problem:

Using modified euler's method find y(0.1) if $\frac{dy}{dx} = x^2 + y^2$; y(0) =1.

Soln:

y(0) = 1 i.e ., $y(x_0) = y_0,\ x_0 = 0; y_0 = 1$

To find y(0.1):

Therefore, here $x_1 = 0.1$

i.e., f(x,y) = $\frac{dy}{dx} = x^2 + y^2$

Therefore, h = 0.1 – 0 = 0.1

By Euler's Modified formula ,

$$y_{n+1} = y_n + hf(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n))$$

where $h = x_1 - x_2$ and n = 0,1,2,3,……

put n=0 in the above formula, we get

$$y_1 = y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)$$

$= 1+0.1 \text{ f}(0+\frac{0.1}{2},1 + \frac{0.1}{2} f(0,1))$

$== 1+0.1 \text{ f}(0.05,1+0.05(0+1))$

$= 1+ 0.1 \text{ f}(0.05+1.05)$

$= 1+ 0.1 (0.05^2 + 1.05^2) =1.1105$

$y_1 = 1.1105$

RUNGE – KUTTA METHOD FOR SOLVING FIRST ORDER DIFFERENTIAL EQUATION:

The Taylor's series method of slving differential equations umerically is restricted by the work involved in finding the higher order derivatives. However there is a class methods known as Runge-Kutta methods which do not require the calculations of higher order derivatives and gives greater accuracy.

These methods have the following useful properties:

1. **To evaluate y$_{m+1}$, they need only information at the point (x_m, y_m).**
2. **They don't involve the derivatives of f (x, y) such as in Taylor's series method.**
3. **They agree with the Taylor's series solution upto the terms of h^r, where r differs from method to method and is known as the order of that Runge-Kutta method .**

Since Euler's method and modified forms satisfy all the three properties, they can be termed as Runge-Kutta methods of first and second order respectively.

In these methods the accuracy increases at the cost of calculations. Of this family of methods, the most widely used method is Runge-Kutta method of fourth order and so why the name of Runge-Kutta is used generally for this method. This method coincides with the Taylor's series solution upto terms of h^4.

The algorithm for this method is given below

$$k_1 = hf(x, y)$$

$$k_2 = hf(x + \frac{h}{2}, y + \frac{k_1}{2})$$

$$k_3 = hf(x + \frac{h}{2}, y + \frac{k_2}{2})$$

$$k_4 = hf(x + h, y + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $y_1 = y_0 + \Delta y$

Working Rule :

To solve $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

Calculate $k_1 = hf(x_0, y_0)$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

and $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ **when** $\Delta x = h$ **and** $y_1 = y_0 + \Delta y$

Now starting from point (x_1, y_1) **and repeat the process.**

Note 1.

The R-K method of second order is nothing but the modified Euler method.

Note 2:

One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Problem:

Using fourth order R-K method find y(0.1) and y(0.2) for the initial value problem

$\frac{dy}{dx} = x + y^2$ **,y(0)=1.**

Soln:

given f(x,y) = $\frac{dy}{dx} = x + y^2$ **,y(0)=1.** **here** **h =** $x_2 - x_1$ **= 0.2 - 0.1 = 0.1**

i.e., $x_0 = 0, y_0 = 1$

By the R-K Method,

The formula for this method is

$$k_1 = hf(x, y)$$

$$k_2 = hf(x + \frac{h}{2}, y + \frac{k_1}{2})$$

$$k_3 = hf(x + \frac{h}{2}, y + \frac{k_2}{2})$$

$$k_4 = hf(x + h, y + k_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $y_1 = y_0 + \Delta y$

The formula becomes

$$k_1 = 0.1 f(0,1) = 0.1$$

$$k_2 = 0.1 f(x_0 + \frac{0.1}{2}, y_0 + \frac{0.1}{2}) = 0.11525$$

$$k_3 = 0.1 f(x_0 + \frac{0.1}{2}, y_0 + \frac{0.11525}{2}) = 0.1168$$

$$k_4 = 0.1 f(x_0 + 0.1, y_0 + 0.1168) = 0.1347$$

$\Delta y_0 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ **=0.116**

and $y_1 = y_0 + \Delta y_0$ **=1+0.116=1.116**

i.e., $y_1 =$ **y(0.1) = 1.116**

i.e., $x_1 = 0.1, y_1 = 1.116$

Similarly we find the values of k,

$$k_1 = 0.1 f(0.1, 1.116) = 0.1346$$

$$k_2 = 0.1 f(x_1 + \frac{0.1}{2}, y_1 + \frac{0.1346}{2}) = 0.15501$$

$$k_3 = 0.1 f(x_1 + \frac{0.1}{2}, y_1 + \frac{0.15501}{2}) = 0.1576$$

$$k_4 = 0.1 f(x_1 + 0.1, y_1 + 0.1168) = 0.18205$$

$$\Delta y_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1346 + 2(0.15501) + 2(0.1576) + 0.18205) = \mathbf{0.1569}$$

$y_2 = y_1 + \Delta y_1$ **= 1.116+0.1569=1.273**

i.e., y(0.2) = $y_2 =$ **1.273**

MILNE'S PREDICTOR & CORRECTOR METHODS:

Runge – Kutta type methods are called single-step methods because they use only the informations from the last step computed. The methods of Milne's predictor- corrector . Adams-Bashforth predictor corrector formulae are multi-step methods.

In the previous method to calculate the value of y $_{n+1}$**, we need only the information at** (x_n, y_n) **and we do not care for the nature of the solution. But the methods of this section remove this objection. As obvious by the name, we first predict the value of y** $_{n+1}$ **by some formula, known as "predictor formula" and then recorrect this value by another formula, called "corrector formula". The values are called as predicted and corrected values respectively.**

MILNE'S PREDICTOR & CORRECTOR FORMULAE:

Milne's method is the most common multi step method that first predicts a value for y $_{n+1}$ **from the three past values of the derivative. It differs from the Adams method in that it integrates over more than one interval. The past values that we require may have been computed by a R-K method, or possibly by the Taylor-series method. In the Milne method, we suppose that four equispaced starting values of y are known, at the points x** $_n$ **,x** $_{n-1}$ **,x** $_{n-2}$ **and x** $_{n-3}$

Milne's predictor and corrector formula is

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \qquad \textbf{Predictor formula}$$

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}] \qquad \textbf{Corrector formula}$$

The formula for this method is as follows:

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n] \qquad \textbf{Predictor formula}$$

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$ **Corrector formula**

Problem:

Given $\frac{dy}{dx} = x^3 + y$ **; y(0) = 2 : y(0.2) = 2.073 ;y(0.4) = 2.452 and y(0.6) =** 3.023. Find y(0.8) by the Milne's Predictor and corrector method.

Soln:

Given $\frac{dy}{dx} = x^3 + y = y'$

Also, given that $\mathbf{x_0 = 0}$ **,** $\mathbf{y_0 = 2}$

$\mathbf{x_1 = 0.2}$, $\mathbf{y_1 = 2.073}$

$\mathbf{x_2 = 0.4}$, $\mathbf{y_2 = 2.452}$

$\mathbf{x_3 = 0.6}$, $\mathbf{y_3 = 3.023}$

$\mathbf{x_4 = 0.8}$, $\mathbf{y_4 = ?}$

$$h = x_1 - x_0 = 0.2 - 0 = 0.2$$

$$y' = x^3 + y$$

$$y_1' = x_1^3 + y_1 \ \mathbf{= 0.2}^3 + 2.073 = 2.081$$

$$y_2' = x_2^3 + y_2 \ \mathbf{= 0.4^3 + 2.452 = 2.516}$$

$$y_3' = x_3^3 + y_3 = 0.6^3 + 3.023 = 3.239$$

Our aim is to find that $\mathbf{y_4}$**. Therefore, we put the suffix** number n=3 in the predictor formula, we get

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$$

$$y_{4,P} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3] = 2 + \frac{4(0.2)}{3}[2(2.081) - 2.516 + 2(3.239)] = 4.1664$$

Similarly, we find the corrector value. we know that the formula for corrector is

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$$

our aim is to find that $\mathbf{y_4}$ **. Therefore we put the suffix** number n=3 in the corrector formula, we get

$$y_{4,C} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4] = 2.452 \left[\frac{0.2}{3}(2.516) + 4(3.239) + (0.8^3 + 4.1164)\right]$$

$$y_{4,C} = 3.7952$$ **-----------------------------**

SHORT QUESTIONS & ANSWERS

1. Write down the third order Taylor algorithm.

Soln: $y(x_1+h)=y(x_2)=y_2=y_1+\frac{h}{1!}y_1'+\frac{h^2}{2!}y_1''+\frac{h^3}{3!}y_1'''+..........$

2. Taylor series method will be very useful to give some ----------------------- for powerful numerical methods such as Runge Kutta method, Milne's method etc.

Soln: Initial starting values

3. Write down Euler algorithm to the differential equation $\frac{dy}{dx}=f(x,y)$ **.**

Soln: $y_{n+1}=y_n+hf(x_n,y_n)$ **since n = 0,1,2,…. and** $h=x_1-x_0$

4. State modified Euler algorithm.

Soln The modified euler's formula is

$$y_{n+1}=y_n+hf(x_n+\frac{h}{2},y_n+\frac{h}{2}f(x_n,y_n))$$

since n = 0,1,2,…. and $h=x_1-x_0$

5. Write down the Runge – Kutta formula of fourth order to solve $\frac{dy}{dx}=f(x,y),\ y(x_0)=y_0$

Soln:

Calculate $k_1=hf(x_0,y_0)$

$$k_2=hf(x_0+\frac{h}{2},y_0+\frac{k_1}{2})$$

$$k_3=hf(x_0+\frac{h}{2},y_0+\frac{k_2}{2})$$

$$k_4=hf(x_0+h,y_0+k_3)$$

and $\Delta y=\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

when Δx = h and $y_1=y_0+\Delta y$

6. Is Euler's method formula a particular case of second order Runge kutta method?

Soln:

Yes, Eulers modified formula is a particular case of second order 'Runge Kutta method.

7. Compare Taylor's series and R.K. method.

Soln: R-K methods do not require prior calculations of higher derivatives of y(x) as the Taylor method does.

Since the differential equations are using in applications often complicated, the calculations of derivatives may be difficult.

Also the R-K formulas involve the computation of f(x, y) at various positions instead of derivatives and this function occurs in the given equation.

8. the fourth order R-K methods are used widely in -------------- to differential equation.

Soln: Getting numerical solutions.

9. How many prior values are required to predict the next value in Adam's method?

Soln: Four prior values.

10. State the Milne's predictor and corrector formulae.

Soln: $y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_{n}]$ **Predictor formula**

$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_{n} + y'_{n+1}]$ **Corrector formula**