SYDNEY GRAMMAR SCHOOL

2011 Trial Examination

FORM VI
MATHEMATICS EXTENSION 1

Wednesday 10th August 2011

General Instructions

- Reading time — 5 minutes
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 7 per boy
- Candidature — 126 boys

Examiner

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QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Simplify $\dfrac{(n+1)!}{n!}$. **1**

(b) Find $\displaystyle\int \frac{1}{9+x^2}\,dx$. **1**

(c) When the polynomial $P(x) = x^3 + 3x^2 + ax - 10$ is divided by $x - 2$, the remainder is 24. Find a. **2**

(d) Differentiate $y = \sin^{-1}(x^3)$. **2**

(e) Suppose that α, β and γ are the roots of the equation $x^3 - 3x^2 - 4x + 12 = 0$.

(i) Write down the value of $\alpha\beta + \alpha\gamma + \beta\gamma$. **1**

(ii) Hence find the value of $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. **1**

(f) (i) Without the use of calculus, sketch the polynomial $y = x(x+1)(x-4)$ showing all the intercepts with the axes. **2**

(ii) Hence, or otherwise, solve the inequation $\dfrac{x(x+1)}{x-4} \geq 0$. **2**

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\sin^{-1}(\sin \frac{2\pi}{3})$. **1**

(b) Find $\displaystyle\lim_{x\to\infty} \frac{3-x}{2x+3}$. **1**

(c) The point A is $(2,\ -4)$ and the point B is $(5,\ 2)$. The point P divides the interval AB externally in the ratio $4:1$. Find the coordinates of P. **2**

(d) Find the gradient of the tangent to the curve $y = \tan^{-1}(\sin x)$ at $x = \pi$. **2**

(e) A ball is projected vertically upwards from the ground. After t seconds, the height of the ball is given by $h = 45t - 5t^2$ metres.

(i) At what time does the ball returns to the ground? **1**

(ii) When is the ball instantaneously at rest? **1**

(iii) What is the greatest height attained by the ball? **1**

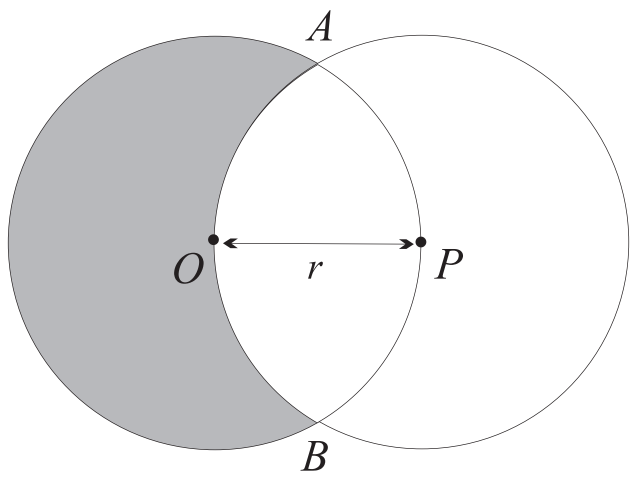
(f) (i) Sketch the graph of the function $y = \left|x^2 - 4\right|$. **2**

(ii) At what points is $f(x) = \left|x^2 - 4\right|$ not differentiable? **1**

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) State the domain and range of $f(x) = 2\cos^{-1}\frac{x}{4}$. **2**

(b)



In the diagram above, two circles of equal radius r units are drawn such that their centres O and P are r units apart. The two circles intersect at A and B.

(i) Show that the quadrilateral $AOBP$ is a rhombus. **1**

(ii) Show that $\angle AOB = 120°$. **1**

(iii) Find the area of the shaded region in terms of r. **2**

(c) The function $f(x) = x\log x + x - 1{\cdot}1$ has a zero near $x = 1$. Take $x = 1$ as a first approximation and use Newton's method once to obtain a closer approximation to this zero. **3**

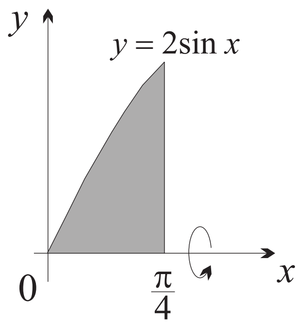
(d) Find the term independent of x in the expansion of $\left(4x^3 - \dfrac{1}{x}\right)^{12}$. **3**

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Given that α is an acute angle and $\cos\alpha = \frac{3}{4}$, find the exact value of $\tan\frac{\alpha}{2}$. **2**

(b) Using the substitution $u = 4x + 1$, evaluate $\displaystyle\int_0^1 \frac{4x}{(4x+1)^2}\,dx$. **3**

(c)



The diagram above shows the region bounded by the curve $y = 2\sin x$, the x-axis and the line $x = \frac{\pi}{4}$. Find the exact volume of the solid generated when the shaded region is rotated about the x-axis. **3**

(d) A particle is moving in a straight line according to the equation

$$x = \sqrt{3}\cos 3t - \sin 3t\,,$$

where x metres is its displacement from the origin after t seconds.

(i) Show that the particle is moving in simple harmonic motion. **2**

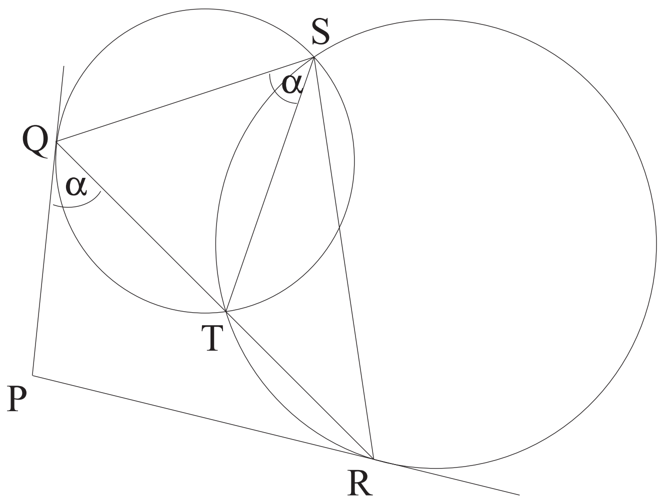
(ii) Find the time at which the particle first passes through the origin. **2**

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Prove by mathematical induction that for all positive integer values of n, **4**

$$\frac{1}{3}\times\frac{1}{1}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{7}\times\frac{1}{5}+\cdots+\frac{1}{(2n+1)}\times\frac{1}{(2n-1)}=\frac{n}{2n+1}\,.$$

(b)

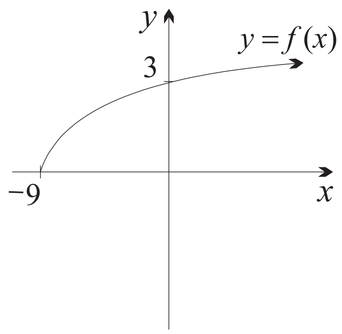


In the diagram above PQ and PR are tangents to the circles SQT and STR respectively, and the points Q, T and R are collinear.

(i) Given that $\angle QST = \alpha$, state a reason why $\angle PQT = \alpha$. **1**

(ii) Prove that $PQSR$ is a cyclic quadrilateral. **2**

(c)



The diagram above shows a sketch of $y = f(x)$ where $f(x) = \sqrt{x+9}$.

(i) Copy the diagram. On the same set of axes, sketch the graph of the inverse function $y = f^{-1}(x)$, clearly marking the x and y-intercepts. **1**

(ii) What is the domain of $f^{-1}(x)$? **1**

(iii) Find an expression for $f^{-1}(x)$. **1**

(iv) Given that the graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at the point P, find the x-coordinate of P. **2**

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) When an object falls from rest at $t = 0$ through a resisting liquid, the rate of change of its velocity at time t is given by $\dfrac{dv}{dt} = -k(v - 600)$, where k is a positive constant.

(i) Show that $v = 600 + Pe^{-kt}$ is a solution to the differential equation for some constant P. **1**

(ii) If the velocity of the object at $t = 3\,\text{s}$ is $25\,\text{ms}^{-1}$, find P and k. **2**

(iii) Find the velocity of the object at $t = 10\,\text{s}$. Give your answer correct to one decimal place. **1**

(iv) What is the limiting value of v as $t \to \infty$? **1**

(b) Let $(2x + y)^{12} = \sum_{k=0}^{12} T_k$ where $T_k = {}^{12}\text{C}_k \times (2x)^{12-k} \times y^k$.

(i) Show that $\dfrac{T_{k+1}}{T_k} = \dfrac{y(12 - k)}{2x(k + 1)}$. **1**

(ii) Suppose that $x = 4$ and $y = 5$ in the expansion of $(2x + y)^{12}$. Show that there are two consecutive terms that are equal, and greater in value than any of the other terms. **2**

(c) (i) Find the general solutions of the equation **3**

$$2\cos 3x \sin 4x + 2\cos 3x - \sin 4x - 1 = 0\,.$$

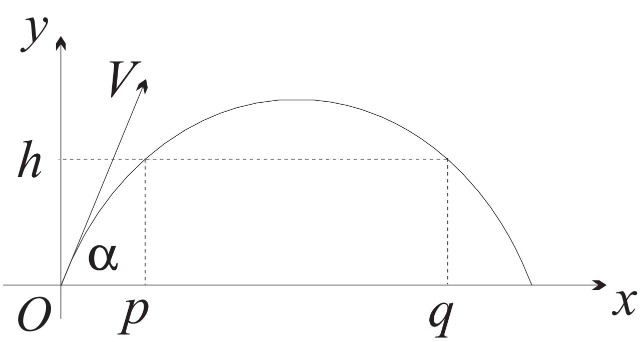
(ii) Hence write down all the solutions in the domain $0 \le x \le \pi$. **1**

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Using the identity $(1+x)^{2n} = (1+x)^n(1+x)^n$, show that **2**

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2 .$$

(b)



A particle is projected from a point O at an angle of elevation α with level ground at an initial velocity $V\,\text{ms}^{-1}$, as in the diagram above.

The particle just clears two vertical poles of height h metres at horizontal distances of p and q metres from O. Take acceleration due to gravity as $10\,\text{ms}^{-2}$ and ignore air resistance. You may assume the equations of motion:

$$x = Vt\cos\alpha$$
$$y = Vt\sin\alpha - 5t^2$$

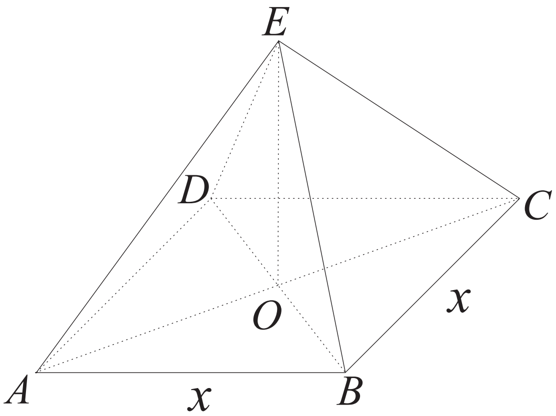
(i) Find an expression for V^2 in terms of α, p and h. **2**

(ii) Hence show that $\tan\alpha = \dfrac{h(p+q)}{pq}$. **2**

Question Seven continues on the next page

QUESTION SEVEN (Continued)

(c)



A square pyramid has its apex vertically above the centre of the base. The square base has side length x and the volume of the pyramid is V. The area of each triangular face is $\frac{S}{4}$ for some constant S.

(i) Show that $S^2 = x^4 + \dfrac{36V^2}{x^2}$. **2**

(ii) Prove that if V is constant and x is variable, then S has its minimum value when **2**

$$x^3 = (3\sqrt{2})V.$$

(iii) When S is at its minimum, show that each triangular face is equilateral. **2**

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} \, x^{n+1}, \;\; n \neq -1; \;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \;\; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} \, e^{ax}, \;\; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \;\; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \;\; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \;\; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \;\; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \;\; a > 0, \;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

$$\text{NOTE: } \ln x = \log_e x, \; x > 0$$