3.1 BASIC GRAPHS

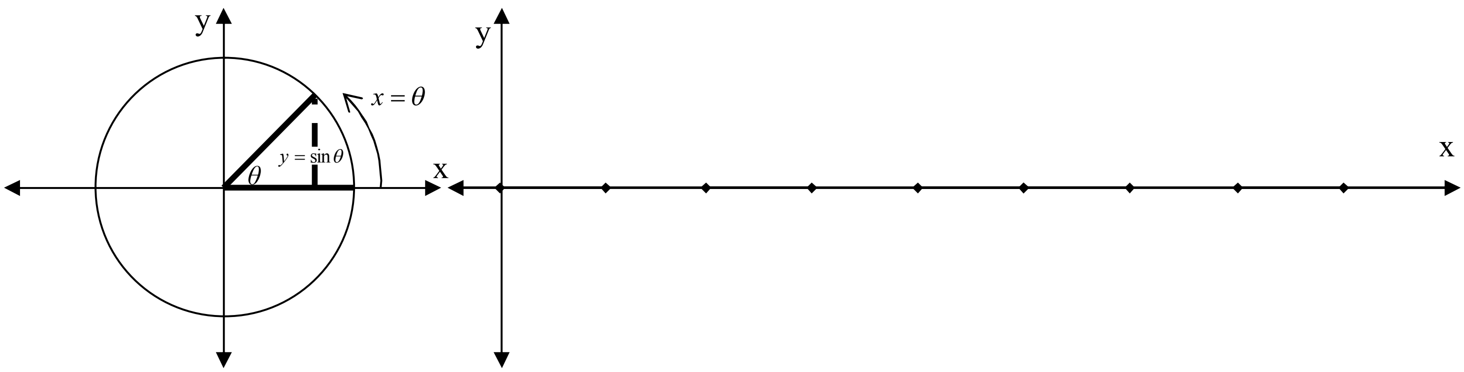
Graph of the sine function

The graph of the sine function is the graph of the set of all ordered pairs of real numbers (x, y) that satisfy $y = \sin x$ ($x \in \mathbf{R}$). A one-to-one correspondence occurs between the unit circle and the graph of the sine curve. The radian measure of a point on the unit circle represents the x-coordinate of the sine curve, while the perpendicular distance from the point on the unit circle to the horizontal axis represents the y-coordinate of the sine curve.

Example #1: Graph the cosine function. State the domain, range, period, and amplitude.

θ	0	π/4	π/2	3π/4	π	5π/4	3π/2	7π/4	2π
sin θ									

Note: Always show the critical points of the graph of the sine curve, including the intercepts, maximum and minimum values.



The domain of the sine function is ___________________. The range of the sine function is ___________________.

The graph of the sine curve completes one cycle from 0 to 2π. Therefore, the **period** of the sine graph is ______.

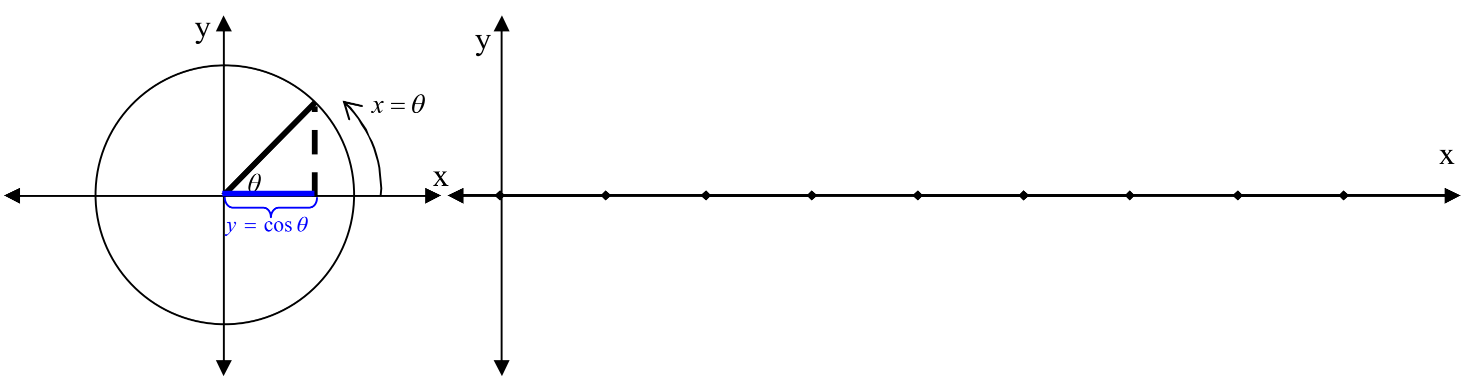
Graph of the cosine function

The graph of the cosine function is the graph of the set of all ordered pairs of real numbers (x, y) that satisfy $y = \cos x$ ($x \in \mathbf{R}$). A one-to-one correspondence also occurs between the unit circle and the graph of the cosine curve. The radian measure of a point on the unit circle represents the x-coordinate of the cosine curve, while the adjacent side length from the reference triangle formed from the point on the unit circle to the horizontal axis represents the y-coordinate of the sine curve.

Example #2: Graph the cosine function. State the domain, range, period, and amplitude.

θ	0	π/4	π/2	3π/4	π	5π/4	3π/2	7π/4	2π
cos θ									

Note: Always show the critical points of the graph of the cosine curve, including the intercepts, maximum and minimum values.



The domain of the cosine function is ___________________. The range of the cosine function is ___________________.

The graph of the cosine curve completes one cycle from 0 to 2π. Therefore, the **period** of the cosine graph is ______.

Graph of the Tangent Function

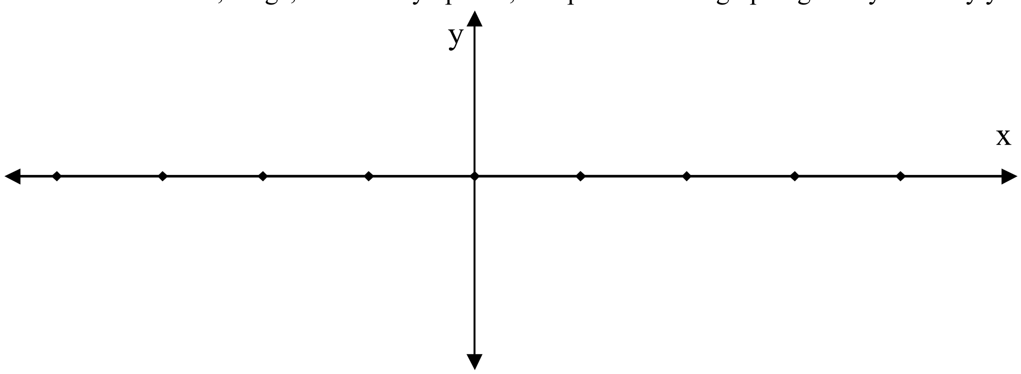
Recall that the tangent function has the identity $\tan x = \sin x / \cos x$, so the tangent is undefined when $\cos x = 0$. The cosine function is 0 at $x = -\pi/2, \pi/2, 3\pi/2, \ldots$ As a result, the graph of the tangent function has vertical asymptotes when $\cos x = 0$. Also, recall that the tangent function is odd, so the graph of $y = \tan x$ is symmetric with respect to the origin.

x	$-\pi/2$	-1.5	-1	$-\pi/4$	0	$\pi/4$	1	1.5	$\pi/2$
tan x									

As x → $-\pi/2$, $\tan x \to$

As x → $\pi/2$, $\tan x \to$

Example #3: Sketch the graph of $y = \tan x$ in the interval $-2\pi \le x \le 2\pi$. Be sure to show all vertical asymptotes and intercepts. State the domain, range, vertical asymptotes, and period. Use a graphing utility to verify your graph.



The domain of the tangent function is: ________________________________. The range of the tangent function is:

The vertical asymptotes of the tangent graph: _______________________________. The period of the tangent function is:

Graph of the Cotangent Function

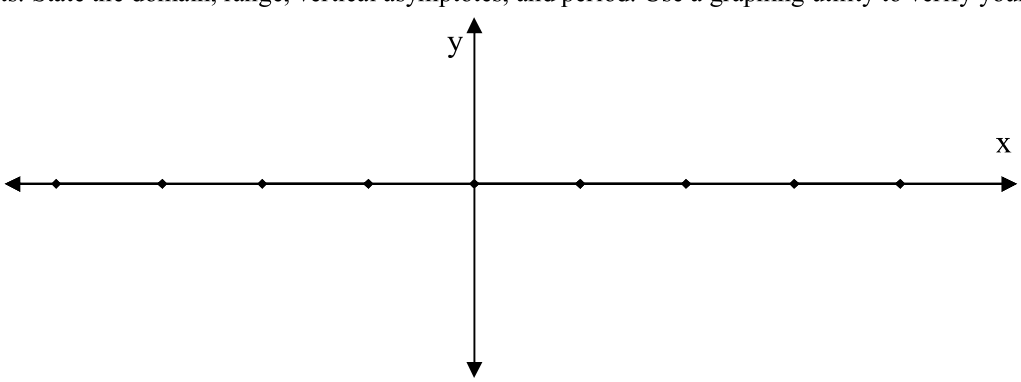
The graph of the cotangent function is similar to the graph of the tangent function. Recall the trigonometric identity $\cot x = \cos x / \sin x$. So the graph of the cotangent function is undefined when $\sin x = 0$. The sine function is 0 at $x = -\pi, 0, \pi, 2\pi, \ldots$ As a result, the graph of the cotangent function has vertical asymptotes when $\sin x = 0$. Like the tangent function, the cotangent function is odd, so it is also symmetric with respect to the origin.

x	0	0.1	0.2	$\pi/4$	$\pi/2$	$3\pi/4$	3	3.1	π
cot x									

As x → 0, $\cot x \to$

As x → π, $\cot x \to$

Example #4: Sketch the graph of $y = \cot x$ in the interval $-2\pi \le x \le 2\pi$. Be sure to show all vertical asymptotes and intercepts. State the domain, range, vertical asymptotes, and period. Use a graphing utility to verify your graph.



The domain of the cotangent function is: ________________________________. The range of the cotangent function is:

The vertical asymptotes of the cotangent graph: _______________________________. The period of the cotangent function is:

Note: The graphs of the tangent and cotangent functions have no amplitude, unlike the sine and cosine curves.

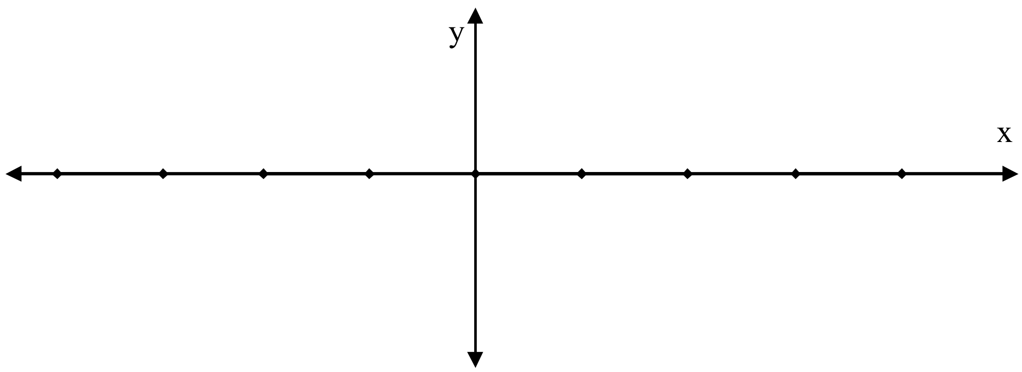
Graphs of Reciprocal Functions

The graphs of the two remaining trigonometric functions can be obtained from the graphs of the sine and cosine functions.

Recall the reciprocal identities: $\csc x = 1/\sin x$ and $\sec x = 1/\cos x$, where the cosecant function is undefined when $\sin x = 0$ and the secant function is undefined when $\cos x = 0$. So the vertical asymptotes for the cosecant graph are $x = 0, \pi, 2\pi...$ and for the secant graph are $x = \pi/2, 3\pi/2, ...$

To obtain points for the reciprocal functions, take reciprocals of the y-coordinates of the corresponding trigonometric function. For instance, the graph of $\sin x$ at (π/6, 1/2) corresponds to the point (π/6, 2) on the graph of $\csc x$.

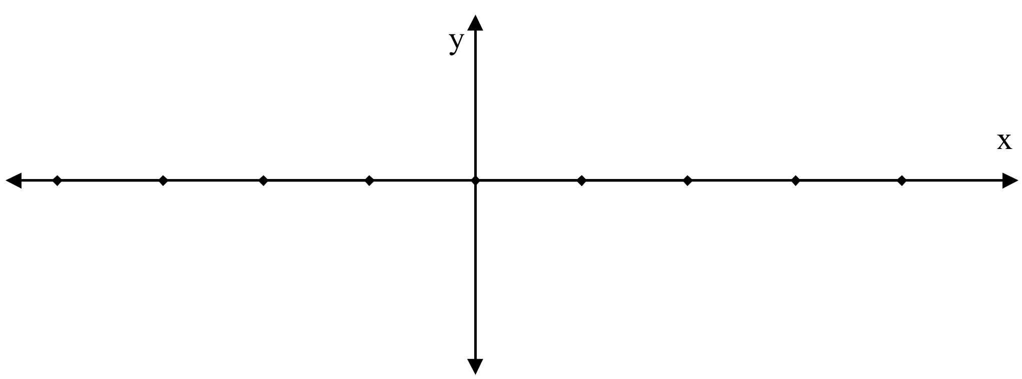
Example #5: Sketch the graph of the cosecant function: $y = \csc x$ by sketching its reciprocal function first on the same graph in the interval [-2π, 2π]. Show the vertical asymptotes and local maximums or minimums. State the domain, range, and period.



The domain of the cosecant function is: ________________________________. The range of the cosecant function is:

The vertical asymptotes of the cosecant graph: ______________________________. The period of the cosecant function is:

Example #6: Sketch the graph of the secant function: $y = \sec x$ by sketching its reciprocal function first on the same graph in the interval [-2π, 2π]. Show the vertical asymptotes and local maximums or minimums. State the domain, range, and period.



The domain of the secant function is: ________________________________. The range of the secant function is:

The vertical asymptotes of the secant graph: ______________________________. The period of the secant function is:

Note: The graphs of the cosecant and secant functions have no amplitude, unlike the sine and cosine curves.

Use the table to summarize the graphs, domains, ranges, and periods of the six basic trigonometric functions.

$y = \sin x$	$y = \cos x$	$y = \tan x$
$y = \csc x$	$y = \sec x$	$y = \cot x$

3.2 GRAPHS OF SINE AND COSINE FUNCTIONS

Basic Sine and Cosine Curves

The graph of a sine function is a **sine curve**. Likewise, the graph of a cosine function is a **cosine curve**.
Recall that the sine and cosine values on the unit circle have a one-to-one correspondence to their graphs.
One cycle of the sine and cosine curve represents one period of the function (from 0 to 2π).

Example#1: Sketch the graphs of the basic sine and cosine functions. Be sure to note the five key points in one period of the graph: the intercepts, maximum points, and minimum points. State the domain and range of each function.

Sine Graph Cosine Graph

Guidelines for Graphing Sine and Cosine Functions

Step 1: Find the period of the sine or cosine function.

Step 2: Divide the period interval into four equal parts using the following procedure:

- Find the midpoint of the period by averaging the x-values of the endpoints of the cycle.
- Find the first-quarter and third-quarter points by averaging the endpoints of the intervals from above.

Step 3: Plot the five critical points of the sine or cosine curve. Draw the smooth curve over the given interval.

Exploration: Use a graphing utility to graph $y = a\sin x$, where a = 0.5, -2, and 3. Sketch your results. How does the value of a affect the shape of the graph? What happens to the graph if a is negative? (Be sure your calculator is in radian mode!)

Example #2: Find the amplitude of each function. Graph each function in the interval: $0 \le x \le 2\pi$

(a) $y = -3\cos x$

(b) $y = \dfrac{2}{3}\sin x$

Exploration: Use a graphing utility to graph $y = \sin bx$, where $b = 0.5$, 2, and π. Sketch your results. How does the value of b affect the graph?

Example #3: Find the period and amplitude of each function. Graph each function over the given interval.

(a) $y = 2\cos 2x;\ -\pi \le x \le 2\pi$

(b) $y = -\frac{3}{4}\sin\frac{\pi x}{2};\ -2 \le x \le 6$

Graphs of Sine and Cosine Functions

The graphs of $y = a\sin(bx)$ and $y = a\cos(bx)$ have the following characteristics (assume b > 0):

Amplitude =

Period =

If $|a| > 1,$ the sine or cosine curve is vertically stretched.

If $0 < |a| < 1,$ the sine or cosine curve is vertically shrunk.

If $a < 1,$ the sine or cosine curve is reflected over the x-axis.

If $b > 1,$ the basic sine or cosine curve is horizontally compressed.

If $0 < b < 1,$ the basic sine or cosine curve is horizontally stretched.

Exploration: Use a graphing utility to graph $y = d + \sin x$, where d = -2, 1, and 3. Sketch your results. How does the value of d affect the graph?

Example #4: Sketch the graph of $y = 1 - 2\sin(3x)$ over the interval $-\frac{\pi}{2} < x < \pi$ by hand. Be sure to label the five key points within one period, and state the domain, range, amplitude, and period of the graph.

Period and Frequency

Period:

Frequency:

For any periodic phenomenon, if P is the period and f is the frequency, then:

Example #5: A tuning fork produces an oscillating vibration at the tip of one prong that follows the model of a cosine wave. Given a frequency of 262 Hz (cycles/sec), an amplitude of 0.065 cm to the left of the rest position when t = 0, find a and b so that $y = a\cos bt$ is an approximate model for this motion.

3.3 MORE GRAPHS OF SINE AND COSINE FUNCTIONS

Exploration: Use a graphing utility to graph $y = \sin(x + c)$, where $c = -\frac{\pi}{4}, 0, \frac{\pi}{4}$. Sketch each graph. How does the value of c affect the graph?

Properties of Sine and Cosine Curves

For $b > 0$, the **amplitude** of $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$ represents half the distance between the maximum and minimum values of the function. Amplitude = Range =

The **period** of the sine and cosine function is given as the length of one complete cycle of the sine or cosine curve:
Period = Domain =

The **phase shift** is the horizontal translation of the sine or cosine curve. Phase Shift =
Note: Set $bx + c = 0$ and $bx + c = 2\pi$ to determine the left-hand and right-hand endpoints of one cycle.

Note: If b is negative, then the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ are used to rewrite the function.

Steps for Graphing Sine and Cosine Functions of the form: $y = k + a\sin(bx + c)$ and $y = k + a\cos(bx + c)$

Step 1: Determine the vertical shift using the k-value and draw the line $y = k$.
Step 2: Find the interval of one period of the function.
- Set $bx + c = 0$ and $bx + c = 2\pi$ to determine the left-hand and right-hand endpoints of one cycle.

Step 3: Divide the period interval into four equal parts.
- Find the midpoint of the period by averaging the x-values of the endpoints of the cycle.
- Find the first-quarter and third-quarter points by averaging the endpoints of the intervals from above.

Step 4: Determine the amplitude of the function. Plot the five critical points of the sine or cosine curve.
Step 5: Extend the graph to fit the indicated interval. Draw the smooth curve over the given interval.

Example #1: Find the amplitude, period, and phase shift of each function. Graph each function over the given interval. Be sure to plot and label the critical points.

(a) $y = \frac{3}{2}\sin(2x - \pi);\ 0 \le x \le 2\pi$

(b) $y = \cos\left(\frac{x}{2} + \frac{\pi}{4}\right)$; $-\frac{\pi}{2} \le x \le \frac{7\pi}{2}$

(c) $y = 3\cos\left(2x + \frac{5\pi}{3}\right)$; $-\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$

Example #2: Sketch the graph of $y = -2\sin(3x + \pi) - 1$ by hand. Be sure to label the five key points within one period, and state the critical characteristics of the graph.

Simple Harmonic Motion

An the motion of an object that oscillates up and down forever with the same amplitude and frequency can be described as **simple harmonic motion**. These functions are used extensively to model real-world phenomena such as modeling water waves, electric current, light waves and resonance.

Example #3: An alternating current generator produces an electric current (measured in amperes) that is described by the equation: $I = 35\sin(40\pi t - 10\pi)$, where t is time in seconds.

Find the amplitude, period, frequency, and phase shift for the current. Graph the equation on the interval $0 \le t \le 0.3$.

3.6 GRAPHS OF OTHER TRIGONOMETRIC FUNCTIONS

Graphs of Reciprocal Functions

Recall the reciprocal identities: $\csc x = 1/\sin x$ and $\sec x = 1/\cos x$, where the cosecant function is undefined when $\sin x = 0$ and the secant function is undefined when $\cos x = 0$. So the vertical asymptotes for the cosecant graph are $x = \pi n$ and for the secant graph are $x = \pi/2 + n\pi$, where n is an integer.

To sketch the graph of a cosecant or secant function, you will need to make a sketch of its reciprocal function and use it as a guide. Note that the "hills" and "valleys" are interchanged. For example, a hill (or maximum point) on the sine curve corresponds to a valley (a local minimum) on the cosecant curve, and a valley (or minimum point) on the sine curve corresponds to a hill (a local maximum) on the cosecant curve.

Example #1: Sketch the graph of $y = \csc x$ and $y = \sec x$ by sketching its reciprocal function first on the same graph. Show the vertical asymptotes and local maximums or minimums. State the domain, range, and period.

Graphing $y = a\csc(bx + c)$ and $y = a\sec(bx + c)$

Graphing the more general forms of the cosecant and secant functions follows the same process for graphing $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$. Since the period is the same as the sine and cosine graphs, we find the period and phase shift of the cosecant and secant graphs by solving $bx + c = 0$ and $bx + c = 2\pi$.

Step #1: Sketch the corresponding graph of $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$.
Step #2: Include the vertical asymptotes. Note that x-intercepts of the corresponding sine and cosine functions become vertical asymptotes of the cosecant and secant functions.
Step #3: Plot the local maximum and minimum values for the cosecant and secant functions (which occurs at the minimum and maximum values of the sine and cosine curve).
Step #4: Draw the appropriate curves of the cosecant or secant graph.

Example #2: Graph each function. State the domain, range, period, vertical asymptotes, local minimums and maximums.

(a) $2\csc\left(4x + \frac{\pi}{2}\right)$; $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(b) $-\sec\left(\frac{x}{3} - \pi\right)$; $-3\pi \le x \le 9\pi$

Graph of the Tangent and Cotangent Functions

Recall that the tangent and cotangent functions have the identities $\tan x = \sin x/\cos x$ and $\cot x = \cos x/\sin x$, so the tangent is undefined when $\cos x = 0$ while the cotangent is undefined when $\sin x = 0$. **Since the period of the tangent and cotangent functions are both π**, then the vertical asymptotes of the tangent function occur at $x = \pi/2 + n\pi$, while the vertical asymptotes of the cotangent function occur at $x = n\pi$.

Note that the sketch of the **tangent** function intersects the x-axis at _________________________ of its vertical asymptotes

and _____________________ without bound between the asymptotes.

Also, the sketch of the **cotangent** function intersects the x-axis at ___________________________ of its vertical asymptotes

and _____________________ without bound between the asymptotes.

Example #3: Sketch the graph of $y = \tan x$ and $y = \cot x$ over two periods. Be sure to label all vertical asymptotes and intercepts. State the domain, range, and period of each function.

Note: Two consecutive asymptotes can be found in the graph of the function $y = a\tan(bx - c)$ by solving the equations $bx - c = -\pi/2$ and $bx - c = \pi/2$. The midpoint between two consecutive asymptotes is an x-intercept of the graph. Recall that the period of $y = a\tan(bx + c)$ is π/b and there is no amplitude.

Exploration: Use a graphing utility to graph the function $y = a\tan(x)$, where a = -2, 0.5, 3. Sketch your results and label your vertical asymptotes and x-intercepts. Determine how the value of a affects the graph of the tangent function.

Graphing the general form of $y = a\tan(bx + c)$

Step 1: Determine the vertical asymptotes of the tangent function by setting $bx + c = -\pi/2$ and $bx + c = \pi/2$ and solving for x.
Step 2: Determine the x-intercept(s) of the tangent function by finding the midpoint(s) of the vertical asymptotes.
Step 3: Graph increasing curves between the vertical asymptotes without bound that intersect the x-intercepts.

Example #4: Graph each function. State the domain, range, period, vertical asymptotes, and x-intercepts.

(a) $2\tan(2x - \pi);\ -\dfrac{3\pi}{4} \le x \le \dfrac{3\pi}{4}$

(b) $-\tan\left(\dfrac{2\pi x}{3} + \dfrac{\pi}{6}\right);\ -1 \le x \le 2$

Graphing the general form of $y = a\cot(bx + c)$

Step 1: Determine the vertical asymptotes of the cotangent function by setting $bx + c = 0$ and $bx + c = \pi$ and solving for x.
Step 2: Determine the x-intercept(s) of the cotangent function by finding the midpoint(s) of the vertical asymptotes.
Step 3: Graph decreasing curves between the vertical asymptotes without bound that intersect the x-intercepts.

Example #5: Graph each function. State the domain, range, period, vertical asymptotes, and x-intercepts.

(a) $\dfrac{1}{4}\cot 4x;\ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

(b) $-\cot\left(\dfrac{x}{6} + \dfrac{2\pi}{3}\right);\ -4\pi \le x \le 4\pi$