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ESL

Lattice Electromagnetic Theory via Differential Forms

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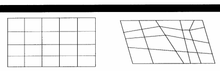
***Summary* - The theory of differential forms and exterior calculus are being employed for a rigorous representation of Maxwell's equations for finite-difference and finite-element time-domain simulations on irregular, unstructured lattices (grids). Differential forms and their discrete counterparts (co-chains) provide a natural bridge between the continuum and the discrete versions of the theory, and allow for a natural factorization of the equations into a metric part and a purely topological (i.e., invariant under homeomorphisms) part. The various sources of inconsistencies that commonly plague discretization schemes in irregular, unstructured grids can be more easily identified, classified, and avoided.**

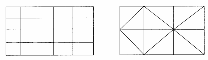
Motivation

- To obtain a discrete, coordinate-free representation of Maxwell's equations on a lattice which preserves the topological structure of the continuum theory at an exact level.
- Conservation laws and most theorems of the continuum are automatically satisfied, independently of the metric structure of the lattice (irregular or not).
- Metric approximations (interpolation, quadratures) are localized in a minimal number of operations.
- *Classify* the most common sources of inconsistency in the usual discretization schemes (FDTD, FEM, FVTD etc.).

Factorization of Maxwell's equations



Homeomorphic lattices with distinct metric structures.



Non-homeomorphic lattices

- Lattice equations invariant under homeomorphisms.
- Such invariance divides lattices into equivalence classes.
- All metric information is contained in the constitutive equations.

Maxwell's Equations in Forms Language

- Dynamic equations.

$$dE = i\omega B \quad (2a)$$

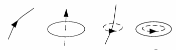
$$dH = -i\omega D + J_E \quad (2b)$$

$$dB = 0 \quad (2c)$$

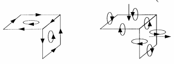
$$dD = \rho_E \quad (2d)$$

  - E, H: 1-forms, associated with edges.
  - B, D: 2-forms, associated with faces.
- Constitutive equations are defined through Hodge operators $\star$, $D = \star_\epsilon E, B = \star_\mu H$ [isomorphism between space of k-forms and space of $(n-k)$-forms].

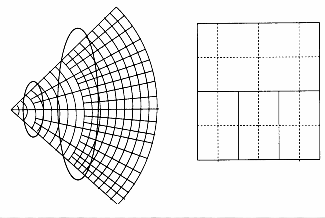
Dual Lattices and Orientation



- E, B: ordinary forms with internal orientation (forces). H, D: twisted forms with external orientation (sources).



- Primary grid: internal. Dual grid: external. Different behavior under coordinate reflection.



Discrete Hodge Operators

- Procedural equivalents of $[\star_\epsilon]$ and $[\star_\mu]$ on vector calculus language are calculated through interpolation, quadratures. At this point, *approximations* are necessarily invoked.
- Interpolation of the dynamic variables on simplicial lattice $\langle s_i^1, E\rangle$, $\langle s_i^2, B\rangle$, $\langle \tilde{s}_i^1, H\rangle$, $\langle \tilde{s}_i^2, D\rangle$: *Whitney forms.*
- $k=1$ case: $\Omega_{s_i^1} \xrightarrow{g_E} \tau_{s_i^1} = \zeta_{i,0}\nabla\zeta_{i,1} - \zeta_{i,1}\nabla\zeta_{i,0}$
- Useful to avoid spurious modes in the FEM. Dual lattice: not simplicial anymore.